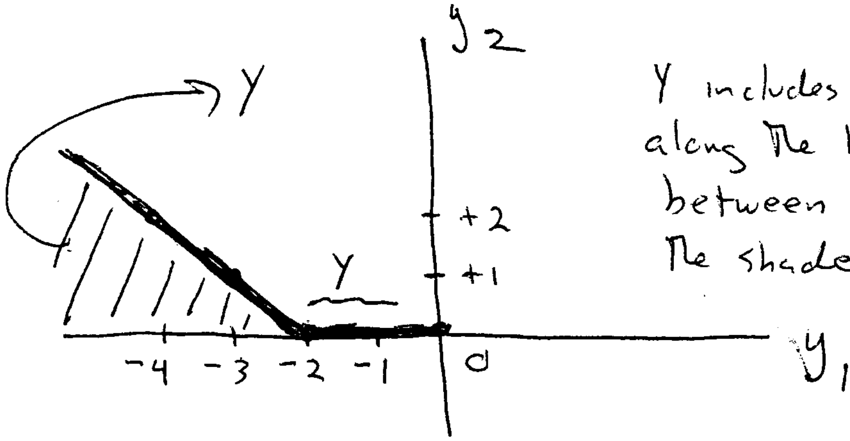
Econ 802

Answers to Midterm #1

Greg Dow

October 2005

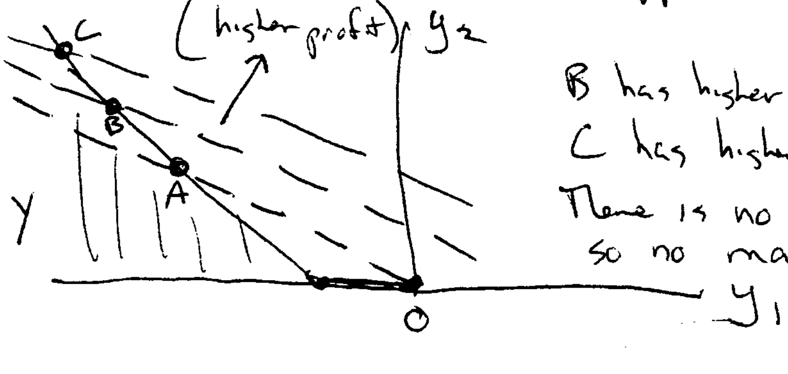
1\. (a)



Y includes the line segment along the horizontal axis between -2 and 0, plus the shaded area (including its upper and lower boundaries)

Y is not convex because $(0,0)$ and $(-3,1)$ are both in the set but the line segment between these points is not in Y. Because Y is not convex, it cannot be strictly convex.

(b) The upper boundary of Y for $y_1 \le -2$ has slope -1. The origin is in Y, and any isoprofit line flatter than -1 that goes through the origin will eventually intersect the upper boundary of Y. Thus if $-\frac{p_1}{p_2} > -1$ or $\frac{p_1}{p_2} < 1$, it is possible to get zero profit in two ways - by choosing $(0,0)$ or by choosing the intersection point on the upper boundary. But it is always possible to get to a higher isoprofit line by going further out along the upper boundary of Y, like this:



B has higher profit than A
C has higher profit than B etc.
There is no limit to this process, so no maximum profit.

On the other hand, if $\frac{P_1}{P_2} = 1$, the isoprofit lines are <u>parallel</u> to the upper boundary of Y and the best the firm can do is to choose the origin: This gives <u>zero</u> profit and every other point in Y gives less profit (is on a lower isoprofit line). The same is true if $\frac{P_1}{P_2} > 1$ (the isoprofit lines are <u>steeper</u> than the upper boundary of Y). In both of these cases, the profit max problem has a solution (the <u>origin</u>)

(c) To have DRS, we would need $f(tx) < t f(x)$ for all x and $t > 1$ where f is the production function. But this is false when $0 < x < 1$ and $t < 2$ (letting inputs now be positive numbers). In this case $f(x) = 0$ and $f(tx) = 0$ also. Thus we don't have DRS. For similar reasons we don't have IRS either (the horizontal segment of Y rules this out). You might think we have CRS because $f(tx) = t f(x) = 0$ for $0 < x < 1$ and $t < 2$. But consider $x = 3$, which gives $f(3) = 1$. Doubling input ($t = 2$) gives $f(tx) = f(6) = 4 \neq t f(x) = 2f(3) = 2$. So we don't have CRS either. The conclusion about returns to scale depends on what x we choose.

2. (a) Let x^0 be optimal at (p^0, w^0) so that

$$\pi(p^0, w^0) = p^0 f(x^0) - w^0 x^0 \geq p^0 f(x) - w^0 x \quad \text{for all } x \geq 0$$

Then notice that for any $t > 0$ we have

$$t\left[p^0 f(x^0) - w^0 x^0\right] \geq t\left[p^0 f(x) - w^0 x\right] \quad \text{for all } x \geq 0$$

$$\Rightarrow (tp^0) f(x^0) - (tw^0) x^0 \geq (tp^0) f(x) - (tw^0) x \quad \text{for all } x \geq 0$$

This implies that x^0 also maximizes profit at prices (tp^0, tw^0)

Therefore

$$\begin{aligned} \pi(tp^0, tw^0) &= (tp^0) f(x^0) - (tw^0) x^0 \\ &= t\left[p^0 f(x^0) - w^0 x^0\right] \\ &= t\, \pi(p^0, w^0) \Rightarrow \text{homog of degree 1.} \end{aligned}$$

(b) From Hotelling, $\pi_p(p,w) = y(p,w)$ and
$\pi_i(p,w) = -x_i(p,w)$ for $i = 1 \cdots n$.

Therefore, $p\,\pi_p(p,w) + \sum_{i=1}^{n} w_i \pi_i(p,w)$

$$= p\,y(p,w) - \sum_{i=1}^{n} w_i x_i(p,w) = \pi(p,w)$$

Thus the expression is equal to maximum profit by Hotelling's Lemma. This is not surprising – in (a) we proved that $\pi(p,w)$ is linearly homogeneous and any linearly homogeneous function $h(x)$ satisfies $h(x) = \sum_{i=1}^{n} \frac{\partial h(x)}{\partial x_i} \cdot x_i$.

(c) There are several ways to approach this question. For example you might take the identity $\pi_L(p) \equiv \pi_S(p, z(p))$ and differentiate with respect to p to get $\frac{\partial \pi_L(p)}{\partial p} = \frac{\partial \pi_S[p, z(p)]}{\partial p} + \frac{\partial \pi_S[p, z(p)]}{\partial z} z'(p)$

Then use Hotelling to show that $\frac{\partial \pi_L(p)}{\partial p} = y_L(p)$

and $\frac{\partial \pi_S[p, z(p)]}{\partial p} = y_S[p, z(p)]$

But $y_L(p) = y_S[p, z(p)]$
(the outputs are the same if the firm has $z(p)$ in the short run)
Hence $\frac{\partial \pi_S[p, z(p)]}{\partial z} z'(p) = 0$; since in general $z'(p) \neq 0$
(we expect price to affect demand for z)
Then we should expect $\frac{\partial \pi_S[p, z(p)]}{\partial z} = 0$.

A faster way to get this result is to notice that
$$\pi_L(p) = \pi_S[p, z(p)] \geq \pi_S[p, z] \text{ for all } z$$
(you can never do better than $\pi_S[p, z(p)]$ in the short run at any z)
So $\pi_S(p, z)$ is maximized with respect to z at $z(p)$. The FOC for maximization is $\frac{\partial \pi_S[p, z(p)]}{\partial z} = 0$.

3 (a) $f(x)$ is concave because it is linear. The definition of a concave function requires that if $x'' = tx + (1-t)x'$ where $0 \le t \le 1$, we must have $f(x'') \ge tf(x) + (1-t)f(x')$.

Here we have $f(x'') = ax'' = atx + a(1-t)x'$

$$= tax + (1-t)ax'$$
$$= t f(x) + (1-t) f(x')$$

so the definition of concavity is satisfied.

The function is not strictly concave because this would require $f(x'') > t f(x) + (1-t) f(x')$ when $0 < t < 1$, and as we just saw, we get = instead.

The Hessian matrix consists entirely of zeroes:

$$\frac{\partial^2 f(x)}{\partial x^2} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & & & & \\ 0 & \cdots & & \cdots & 0 \end{bmatrix}$$

since all second derivatives of a linear function must be zero.

The necessary SOC holds because $h' \frac{\partial^2 f(x)}{\partial x^2} h \le 0$ for all h. But the sufficient SOC does not hold because we cannot find any vector h for which the quadratic form is strictly negative.

(b) Yes, a solution always exists. For any $y > 0$, there is some x^0 that will produce it (for example, set $x_1 = \frac{y}{a_1}$ and $x_i = 0$ for all $i \ge 2$). Clearly the minimum cost of producing y cannot exceed wx^0. Thus we can limit attention to x vectors such that

(i) $wx \le wx^0$

(ii) $f(x) = ax \ge y$ (assuming free disposal of output)

(iii) $x \ge 0$

Conditions (i) and (iii) ensure that the relevant set is bounded, and the weak inequalities ensure that it is closed.

Since the expression wx to be minimized is a continuous function of x, there must be some x^* that minimizes cost.

(c) $L = -wx + \lambda[ax - y] + \mu x$ (setting it up as a maximization problem)

$$\frac{\partial L}{\partial x_i} = -w_i + \lambda a_i + \mu_i = 0 \; ; \; \mu_i \geq 0, \; x_i \geq 0, \; \mu_i x_i = 0$$
(all $i = 1 \dots n$)

$$\frac{\partial L}{\partial \lambda} = ax - y = 0.$$

If $x_i^* > 0$ then $\mu_i = 0$ and $w_i = \lambda a_i \Rightarrow \frac{w_i}{a_i} = \lambda$.

for all $j \neq i$ we have $\mu_j \geq 0 \Rightarrow -w_j + \lambda a_j \leq 0$

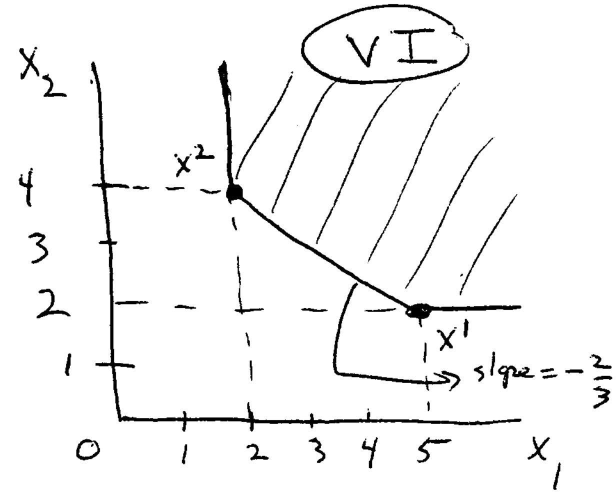
$$\Rightarrow \lambda a_j \leq w_j \Rightarrow \lambda \leq \frac{w_j}{a_j}, \text{ so } \frac{w_i}{a_i} \leq \frac{w_j}{a_j} \text{ for all } j = 1..n$$

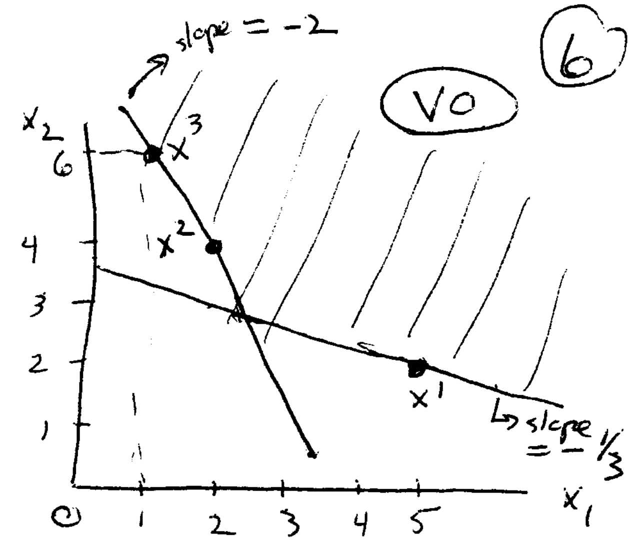
Thus if the firm chooses $x_i^* > 0$, the ratio $\frac{w_i}{a_i}$ of price to productivity for that input must be at least as small as for any other input $j \neq i$.

4 (a) Yes. Cost in period 1 is $w^1 x^1 = 5 + 6 = 11$ and if the firm had used x^2 instead, cost would have been $w^1 x^2 = 2 + 12 = 14$ which is higher.

Similarly, actual cost in period 2 was $w^2 x^2 = 16 + 16 = 32$. If the firm had used x^1 instead, cost would have been $w^2 x^1 = 40 + 8 = 48$ which is higher

(b)





The smallest set VI must include x^1 and x^2, plus the line segment between them (by convexity). It must also include all points to the northeast of these points (monotonicity)

The largest set VO includes all x that are on or above the isocost line through x^1 with slope $-1/3$ and on or above the isocost line through x^2 with slope -2. Including any more points would violate cost min in period $t=1$, $t=2$ or both. Note that VO is both convex and monotonic as required.

(c) The vector $(1,6)$ definitely rules out VI because this point is not in VI. To see whether it rules out VO also, we need to see whether it is on or above the isocost line through x^2. The prices for this isocost line are $w^2 = (8,4)$. So the vector $(1,6)$ would cost $8 + 24 = 32$. This is exactly the same as $w^2x^2 = 32$, therefore x^2 and $x^3 = (1,6)$ are on the same isocost line. Thus x^3 is in VO and this does not rule out VO as a possible $V(y)$ set. Also, we can conclude that the true isocost line through x^3 in period $t = 3$ must be steeper than the one in period $t = 2$ (or identical). If it were flatter, then x^2 would be cheaper than x^3 in period 3 which contradicts cost min. Thus $w^3 = (w_1^3, w_2^3)$ must have $\frac{w_1^3}{w_2^3} \geq +2$ or $-\frac{w_1^3}{w_2^3} \leq -2$.

5. (a) $$e(x) = \frac{\sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} \cdot x_i}{f(x)} = \frac{\alpha x_1^{\alpha-1} \cdot x_1 + \beta x_2^{\beta-1} \cdot x_2}{x_1^{\alpha} + x_2^{\beta}}$$

$$= \frac{\alpha x_1^{\alpha} + \beta x_2^{\beta}}{x_1^{\alpha} + x_2^{\beta}}$$

If $e(x) > 1$ for all x, this must be true as $x_2 \to 0$, which implies $\alpha > 1$. Similarly, it must be true as $x_1 \to 0$, which implies $\beta > 1$.

If $e(x) < 1$ for all x, the same reasoning gives $\alpha < 1$ and $\beta < 1$.

(b) $\min \; w_1 x_1 + w_2 x_2$ subject to $x_1^{\alpha} + x_2^{\alpha} = y$

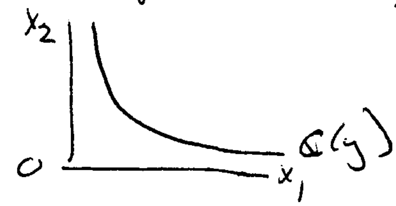
$$L = w_1 x_1 + w_2 x_2 - \lambda [x_1^{\alpha} + x_2^{\alpha} - y]$$

$$\left.\begin{aligned} \frac{\partial L}{\partial x_1} &= w_1 - \lambda \alpha x_1^{\alpha-1} = 0 \\ \frac{\partial L}{\partial x_2} &= w_2 - \lambda \alpha x_2^{\alpha-1} = 0 \end{aligned}\right\} \Rightarrow \frac{w_1}{w_2} = \left(\frac{x_1}{x_2}\right)^{\alpha-1}$$

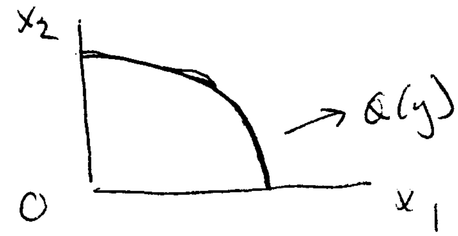
$$\frac{\partial L}{\partial \lambda} = x_1^{\alpha} + x_2^{\alpha} - y = 0$$

Note that $$TRS = -\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}} = \frac{-\alpha x_1^{\alpha-1}}{\alpha x_2^{\alpha-1}} = -\left(\frac{x_1}{x_2}\right)^{\alpha-1}$$

If $\alpha < 1$, the slope of the isoquant gets flatter as $\frac{x_1}{x_2}$ increases, like this:



If $\alpha > 1$, the slope of the isoquant gets steeper as $\frac{x_1}{x_2}$ increases like this:



8

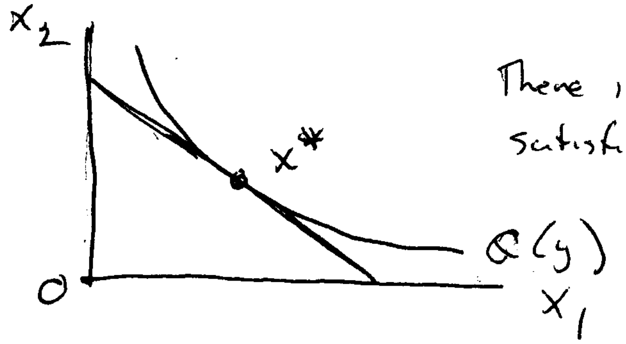
If $\alpha = 1$, the production function is linear and so is the isoquant.

To get a quasi-concave production function, we need a <u>convex</u> input requirement set $V(y)$, which implies $\alpha \leq 1$.

Since we want a <u>sufficient</u> SOC, we want <u>strict</u> quasi-concavity which requires strictly ~~quasi~~ convex $V(y)$.

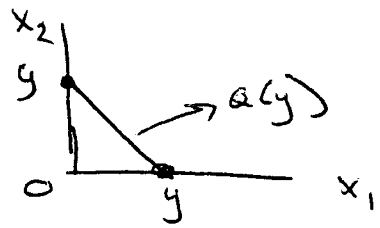
This is true if and only if $\boxed{\alpha < 1.}$

In this case we get a graph like this:



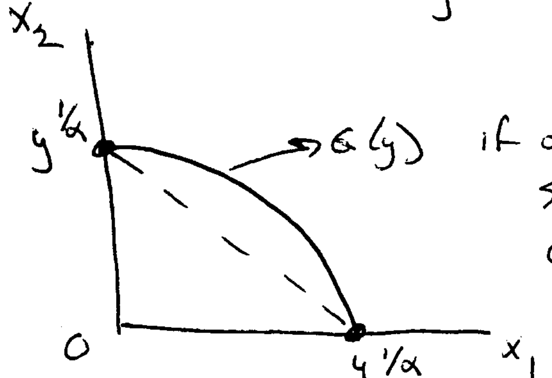
There is a unique x^* that satisfies the FOC and this solves the cost min problem.

(c) If $\alpha \geq 1$, the sufficient SOC in (b) does not hold. We either get



if $\alpha = 1$

or



if $\alpha > 1$. In either case, boundary solutions are optimal. Since the dashed line has slope $= -1$, the firm uses $x_1 = y^{1/\alpha}$, $x_2 = 0$ if $w_2 > w_1$; $x_1 = 0$, $x_2 = y^{1/\alpha}$ if $w_2 < w_1$; and it is indifferent if $w_2 = w_1$.

So $c(w, y) = w_1 y^{1/\alpha}$ if $w_1 \leq w_2$
$\qquad = w_2 y^{1/\alpha}$ if $w_1 \geq w_2$

More compactly, $c(w, y) = y^{1/\alpha} \cdot \min\{w_1, w_2\}$

This is <u>not</u> differentiable at the point $w_1 = w_2$.