Econ 802

Answers to First Midterm

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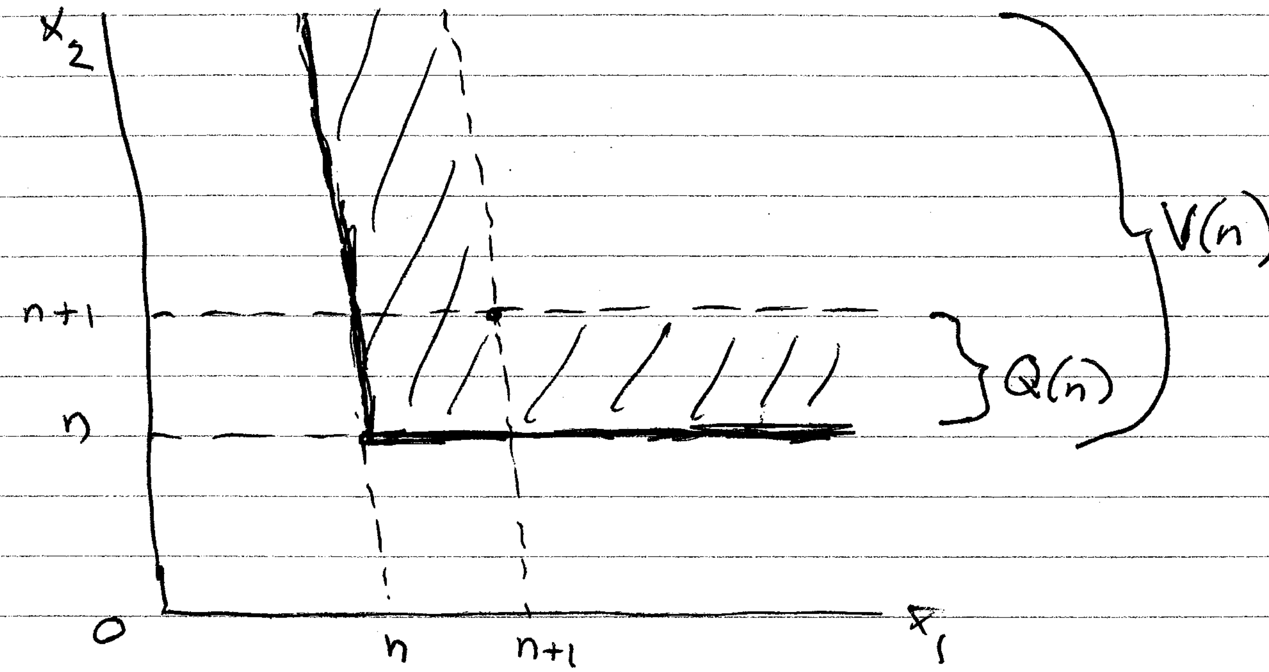
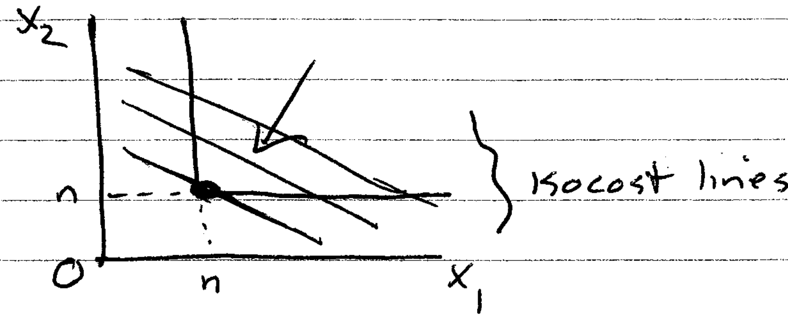
1. (a)



The isoquant set $Q(n)$ consists of the shaded points, including the lower boundary but not the upper boundary. Assuming free disposal the input requirement set $V(n)$ includes all $x \in Q(n)$ plus all points with $x_1 \geq n+1$ and $x_2 \geq n+1$.

(b) A profit maximizing firm would only operate at the corners in part (a) where $y = x_1 = x_2 = n$ for some integer n, because otherwise the firm is spending more than necessary on inputs. Profit can be written as $py - w_1 x_1 - w_2 x_2 = pn - w_1 n - w_2 n = n(p - w_1 - w_2)$. This problem has no solution if $p - w_1 - w_2 > 0$ because there is no upper bound on n. If $p - w_1 - w_2 = 0$ then any integer is a solution, and if $p - w_1 - w_2 < 0$ then the solution is $n = 0$.

(c) Yes, there is always a solution as long as $w_1 > 0$ $w_2 > 0$. It is clear from the graph in part (a) that the lowest possible isocost line for $y = n$ is reached at the corner where $x_1 = x_2 = n$.



2. (a) We want to maximize $p[\ln(x_1+1) + \ln(x_2+1)] - w_1 x_1 - w_2 x_2$
This can be broken into two separate problems:
Choose x_1 to max $p\ln(x_1+1) - w_1 x_1$
and choose x_2 to max $p\ln(x_2+1) - w_2 x_2$

Consider the first problem. The derivative with respect to x_1 is $\frac{p}{x_1+1} - w_1$. If this is ≤ 0 at $x_1 = 0$ then it is < 0 for any $x_1 > 0$ and the (unique) solution is $x_1^* = 0$.
If the derivative is > 0 at $x_1 = 0$ then clearly there is some (unique) $x_1^* > 0$ at which $\frac{p}{x_1^*+1} - w_1 = 0$, and this is the solution.

Since the same argument applies to x_2, we conclude there is always a solution $x^* = (x_1^*, x_2^*)$ and it is unique.

(b) Write $p\ln(x_1+1) + p\ln(x_2+1) - w_1 x_1 - w_2 x_2 + \mu_1 x_1 + \mu_2 x_2$

FOC are
$$\frac{p}{x_1+1} - w_1 + \mu_1 = 0 \qquad x_1 \geq 0,\ \mu_1 \geq 0\ \mu_1 x_1 = 0$$
$$\frac{p}{x_2+1} - w_2 + \mu_2 = 0 \qquad x_2 \geq 0\ \mu_2 \geq 0,\ \mu_2 x_2 = 0$$

If $x_1 > 0$ then $\mu_1 x_1 = 0$ implies $\mu_1 = 0$ and

$$\frac{p}{x_1+1} = w_1 \Rightarrow \frac{p}{w_1} = x_1 + 1 = \frac{p}{w_1} - 1 = x_1 > 0$$

So we must have $\boxed{p > w_1}$

If $x_1 = 0$ Then $\mu_1 \geq 0$ implies $\frac{p}{x_1+1} - w_1 \leq 0 \Rightarrow \boxed{p \leq w_1}$

Likewise, $x_2 > 0$ iff $\boxed{p > w_2}$ and $x_2 = 0$ iff $\boxed{p \leq w_2}$

(c) The argument in (a) shows that this is really a combination of two separate one-variable optimization problems, and it is easy to check that a sufficient SOC holds in each case. But let's use the more standard approach of computing the Hessian matrix of the production function. This can be written as

$$\frac{\partial^2 f(x^*)}{\partial x^2} = \begin{bmatrix} f_{11}(x^*) & f_{12}(x^*) \\ f_{21}(x^*) & f_{22}(x^*) \end{bmatrix}$$

where
$$\left.\begin{aligned} f_{11} &= -(x_1+1)^{-2} < 0 \\ f_{22} &= -(x_2+1)^{-2} < 0 \\ \text{and } f_{12} &= f_{21} = 0 \end{aligned}\right\}$$
so diagonal elements are <u>negative</u> and the overall determinant is $f_{11}f_{22} - f_{12}f_{22}$ $= f_{11}f_{22} > 0.$

So the Hessian is negative definite at x^*. This is <u>sufficient</u> for x^* to solve the problem (of course it also satisfies the necessary condition).

3. (a) Suppose x^* solves the profit max problem at prices (p^*, w^*). This implies $p^* f(x^*) - w^* x^* \geq p^* f(x) - w^* x$ for all $x \geq 0$

If all prices are multiplied by $t > 0$, we have

$$(tp^*) f(x^*) - (tw^*) x^* \geq (tp^*) f(x) - (tw^*) x \text{ for all } x \geq 0.$$

Therefore x^* also solves the problem at the new prices. Since (p^*, w^*) was arbitrary, it must be true that $x(tp, tw) = x(p, w)$ for all $(p, w) > 0$ and $t > 0$.

(b) Suppose x^* solves the cost min problem at prices w^*. This implies that $w^* x^* \leq w^* x$ for all $x \geq 0$ such that $f(x) = y$. If we multiply all input prices by $t > 0$, we have $(tw^*) x^* \leq (tw^*) x$ for all $x \geq 0$ such that $f(x) = y$. Therefore x^* also solves the problem at the new prices tw^*. Since w^* was arbitrary, it must be true that $x(tw, y) = x(w, y)$ for all $w > 0$ and $t > 0$.

(c) We can write $\pi(p, w) = pf[x(p, w)] - wx(p, w)$
and $\pi(tp, tw) = (tp) f[x(tp, tw)] - (tw) x(tp, tw)$
But we already know from (a) that the optimal input bundle does not change. So

$$\pi(tp, tw) = (tp) f[x(p, w)] - (tw) x(p, w)$$
$$= t[pf[x(p, w)] - wx(p, w)]$$
$$= t\pi(p, w)$$

So the profit function is homogeneous of degree one.

Likewise, the cost function is $c(w, y) = wx(w, y)$
and $c(tw, y) = (tw) x(tw, y)$
But we know from part (b) that the optimal input bundle does not change. So

$$c(tw, y) = (tw) x(w, y)$$
$$= tc(w, y)$$

So the cost function is homogeneous of degree one.

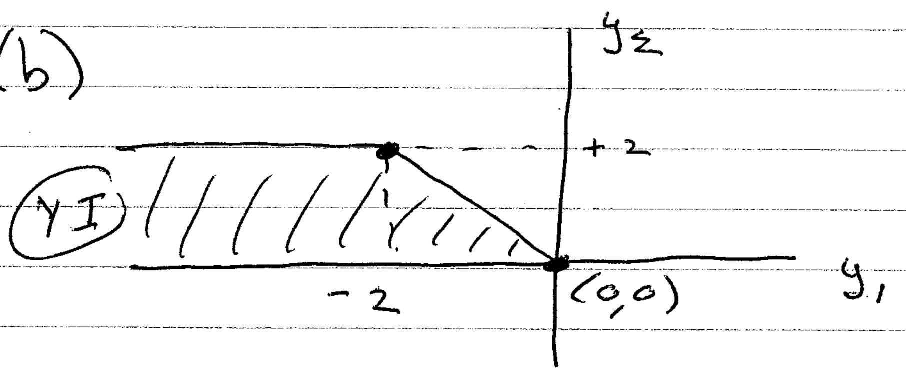
4. (a) In periods when $p_1^t < p_2^t$ the firm's profit was $2p_2^t - 2p_1^t = 2(p_2^t - p_1^t) > 0$. If the firm had instead chosen $(0,0)$, its profit would have been zero.

In periods when $p_1^t \geq p_2^t$ the firm's profit was zero. If the firm had instead chosen $(-2, +2)$ its profit would have been

$$2(p_2^t - p_1^t) \leq 0.$$

Therefore WAPM is satisfied.

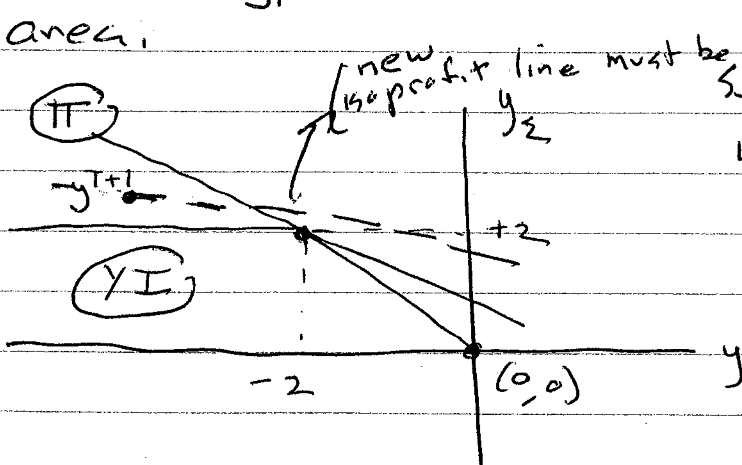
(b)



$(0,0)$ and $(-2,+2)$ are both in YI, and so is any point on the line segment between them due to convexity of YI.

Monotonicity implies that any point along the horizontal line to the left of $(-2,+2)$ is in YI and so are all points below the upper boundary such that $y_1 \leq 0$ and $y_2 \geq 0$. So YI is the shaded area.

(c)



Suppose the flattest isoprofit line that has occurred so far is π. If we see a price vector p^{T+1} and a production plan y^{T+1} outside YI, this would prove that Y is larger than YI but it would not violate WAPM (it would not have been optimal to choose y^{T+1} at any of the previous price vectors p^t).

5.(a) $$e(x) = \frac{\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot x_i}{f(x)}$$

where $$\frac{\partial f}{\partial x_1} = \frac{1}{n} x_1^{\frac{1}{n}-1} (x_2 x_3 \ldots x_n)^{1/n}$$

$$\frac{\partial f}{\partial x_2} = \frac{1}{n} x_2^{\frac{1}{n}-1} (x_1 x_3 \ldots x_n)^{1/n}$$

and so on. Therefore

$$\frac{\partial f}{\partial x_i} \cdot x_i = \frac{1}{n} (x_1 x_2 \ldots x_n)^{1/n} \quad \text{for } \underline{\text{all}} \; i = 1 \ldots n$$

$$\Rightarrow \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot x_i = (x_1 x_2 \ldots x_n)^{1/n}$$

$$\Rightarrow e(x) = \frac{(x_1 x_2 \ldots x_n)^{1/n}}{(x_1 x_2 \ldots x_n)^{1/n}} = 1$$

So the elasticity of output with respect to scale is a constant equal to one everywhere. Interpretation: The production function is homogeneous of degree one (it is just a Cobb-Douglas function with n inputs)

(b) $$\sigma = -\frac{\partial \left(\frac{x_1}{x_2}\right)}{\partial \left(\frac{w_1}{w_2}\right)} \cdot \frac{\left(\frac{w_1}{w_2}\right)}{\left(\frac{x_1}{x_2}\right)}$$

Solve the cost min problem by using a Lagrangean:

$$L = w_1 x_1 + w_2 x_2 - \lambda \left[x_1^{1/2} x_2^{1/2} - y \right]$$

$$\left.\begin{aligned} \frac{\partial L}{\partial x_1} &= w_1 - \frac{\lambda}{2} x_1^{-1/2} x_2^{1/2} = 0 \\ \frac{\partial L}{\partial x_2} &= w_2 - \frac{\lambda}{2} x_1^{1/2} x_2^{-1/2} = 0 \end{aligned}\right\} \quad \text{Divide to get } \frac{w_1}{w_2} = \frac{x_2}{x_1}$$

So $$\frac{x_1}{x_2} = \left(\frac{w_1}{w_2}\right)^{-1}$$

$$\frac{\partial\left(\frac{x_1}{x_2}\right)}{\partial\left(\frac{w_1}{w_2}\right)} = -\left(\frac{w_1}{w_2}\right)^{-2}$$

So $$\sigma = \left(\frac{w_1}{w_2}\right)^{-2} \cdot \frac{\left(\frac{w_1}{w_2}\right)}{\left(\frac{w_1}{w_2}\right)^{-1}} = 1$$

Interpretation: The elasticity of substitution is constant and equal to 1 for all prices and ~~input~~ output levels. This is not surprising: $y = x_1^{1/2} x_2^{1/2}$ is a Cobb-Douglas function which we know has a constant elasticity of substitution equal to one.

(c) It would stay the same. You can prove this using the fact that $\sigma = 1$ or you can use the FOC from the cost min problem:

$$\frac{w_1}{w_2} = \frac{x_2}{x_1} \Rightarrow w_1 x_1 = w_2 x_2$$

Expenditure on input 1 as a fraction of total cost is

$$\frac{w_1 x_1}{w_1 x_1 + w_2 x_2} = \frac{1}{2}$$

Clearly this is a constant that does not depend on input prices (the firm always spends ½ of total cost on each input.)