Econ 802

Answers to Second Midterm

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1. (a) Suppose x^* is cost minimizing for y^*. Then x^* must satisfy the necessary FOC for a minimum:

$$\left.\begin{aligned} w_1 &= \lambda \frac{\partial f(x^*)}{\partial x_1} \\ w_2 &= \lambda \frac{\partial f(x^*)}{\partial x_2} \end{aligned}\right\} \Rightarrow \frac{w_1}{w_2} = \frac{\frac{\partial f(x^*)}{\partial x_1}}{\frac{\partial f(x^*)}{\partial x_2}}$$

Since $f(x)$ is homothetic, $f(x) = h[g(x)]$ where $g(x)$ is homogeneous of degree one. Thus

$$\frac{\partial f(x^*)}{\partial x_1} = h'[g(x^*)]\frac{\partial g(x^*)}{\partial x_1}$$

$$\frac{\partial f(x^*)}{\partial x_2} = h'[g(x^*)]\frac{\partial g(x^*)}{\partial x_2}$$

So $$\frac{w_1}{w_2} = \frac{\frac{\partial g(x^*)}{\partial x_1}}{\frac{\partial g(x^*)}{\partial x_2}}$$

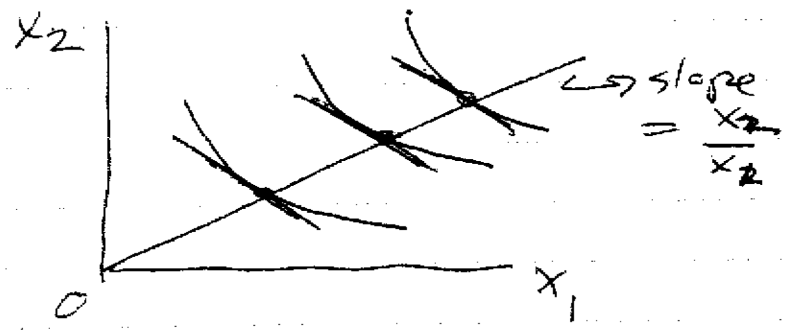
Since g is homogeneous of degree 1, its derivatives are homogeneous of degree zero. Therefore

$$\frac{\partial g(x_1^*, x_2^*)}{\partial x_1} = \frac{\partial g(\frac{x_1^*}{x_2^*}, 1)}{\partial x_1}$$

and $$\frac{\partial g(x_1^*, x_2^*)}{\partial x_2} = \frac{\partial g(\frac{x_1^*}{x_2^*}, 1)}{\partial x_2}$$

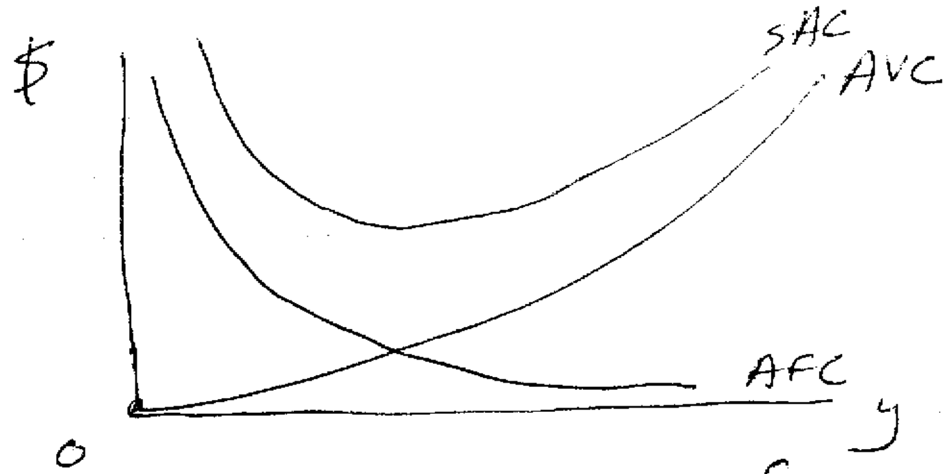
Therefore the equation $\dfrac{w_1}{w_2} = \dfrac{\dfrac{\partial g(\frac{x_1}{x_2}, 1)}{\partial x_1}}{\dfrac{\partial g(\frac{x_1}{x_2}, 1)}{\partial x_2}}$

uniquely determines
the ratio x_1/x_2. Since y does not appear in this equation, the level of output does not affect the cost-minimizing input ratio.
Graphically, the expansion path is a ray from the origin:



(b) Yes. Suppose $y = x_1^{\alpha} x_2^{1-\alpha}$ which is homogeneous of degree 1. Let $x_2 = 1$ be fixed in the short run. The short run production function is then $y = x_1^{\alpha}$ and the short run demand for input 1 is $x_1 = y^{1/\alpha}$.

The short run cost function is
$$c_s(w, y) = w_1 y^{1/\alpha} + w_2$$
and short run average cost is
$$\underbrace{\frac{c_s(w, y)}{y}}_{SAC} = \underbrace{w_1 y^{\frac{1}{\alpha} - 1}}_{AVC} + \underbrace{\frac{w_2}{y}}_{AFC}$$
Suppose $\alpha < 1/2$ so $\frac{1}{\alpha} - 1 > 1$. Then graphically we have



so SAC is U-shaped. [Note: This is just one example you could use some other function.]

(c) No. A function that is homogeneous of degree k has $f(tx) = t^k f(x)$ for all $t > 0$.
The elasticity with respect to scale is

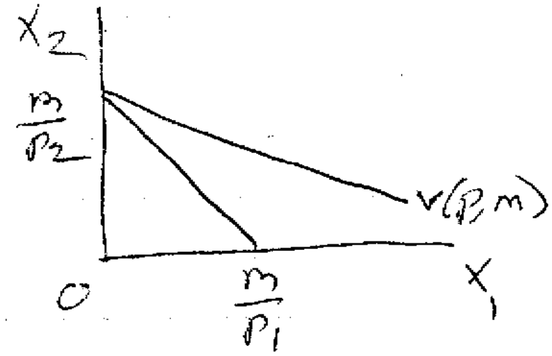
$$e(x) = \frac{d f(tx)}{dt} \cdot \frac{t}{f(tx)}\bigg|_{t=1} = \frac{k t^{k-1} f(x)\, t}{f(tx)}\bigg|_{t=1}$$

$\Rightarrow e(x) = k$. (a constant)

But we know that $e(x^*) > 1 \Rightarrow$ LAC is falling
$e(x^*) = 1 \Rightarrow$ LAC is horizontal
$e(x^*) < 1 \Rightarrow$ LAC is rising

Because k is a constant, one of these three things must be true globally and this rules out a U-shaped LAC curve.

2. (a) The indifference curves are linear with slope $= -\frac{a}{b}$. The budget line has the slope $= -\frac{p_1}{p_2}$. If $\frac{p_1}{p_2} > \frac{a}{b}$ then the budget line is steeper and we get a graph like this:

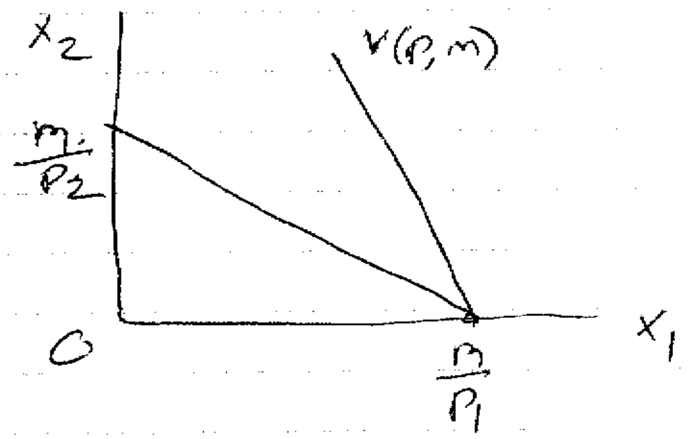


Solution: $x_1 = 0,\ x_2 = \frac{m}{p_2}$

$\Rightarrow v(p,m) = \frac{bm}{p_2}$

$\Rightarrow e(p,u) = \frac{up_2}{b}$

But if $\frac{p_1}{p_2} < \frac{a}{b}$ then the budget line is flatter than the indifference curves and we get



Solution: $x_1 = \frac{m}{p_1},\ x_2 = 0$

$\Rightarrow v(p,m) = \frac{am}{p_1}$

$\Rightarrow e(p,u) = \frac{up_1}{a}$

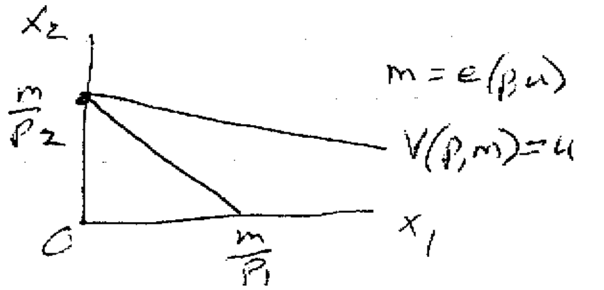
(b) When $p_1/p_2 > \frac{a}{b}$, $v(p,m) = \frac{bm}{p_2}$, $e(p,u) = \frac{up_2}{b}$

Roy $\Rightarrow x_1(p,m) = 0$

$$x_2(p,m) = -\frac{\frac{\partial v(p,m)}{\partial p_2}}{\frac{\partial v(p,m)}{\partial m}} = \frac{\frac{bm}{p_2^2}}{\frac{b}{p_2}} = \frac{m}{p_2}$$

Shephard $\Rightarrow h_1(p,u) = 0$

$$h_2(p,u) = \frac{\partial e(p,u)}{\partial p_2} = \frac{u}{b}$$



The point $x_1 = 0,\ x_2 = \frac{m}{p_2}$ gives $u = \frac{bm}{p_2}$; plugging this into the Hicksian

demand for good 2 gives $h_2(p,u) = \frac{m}{p_2}$
which is the same as
Marshallian demand at income m. You can do the same thing in reverse by substituting the expenditure function into the Marshallian demand. This works because of duality: $x_1^* = 0$ and $x_2^* = \frac{m}{p_2}$ simultaneously maxes utility subject to (p, m) and minimizes expenditure subject to (p, u).

(c) At (p^0, m^0) we have $x_1 = 0$ and $x_2 = \frac{m^0}{p_2^0}$
so $u^0 = \frac{b m^0}{p_2^0}$.

At (p', m') we have $x_1 = \frac{m'}{p_1'}$ and $x_2 = 0$

so $u' = \frac{a m'}{p_1'}$

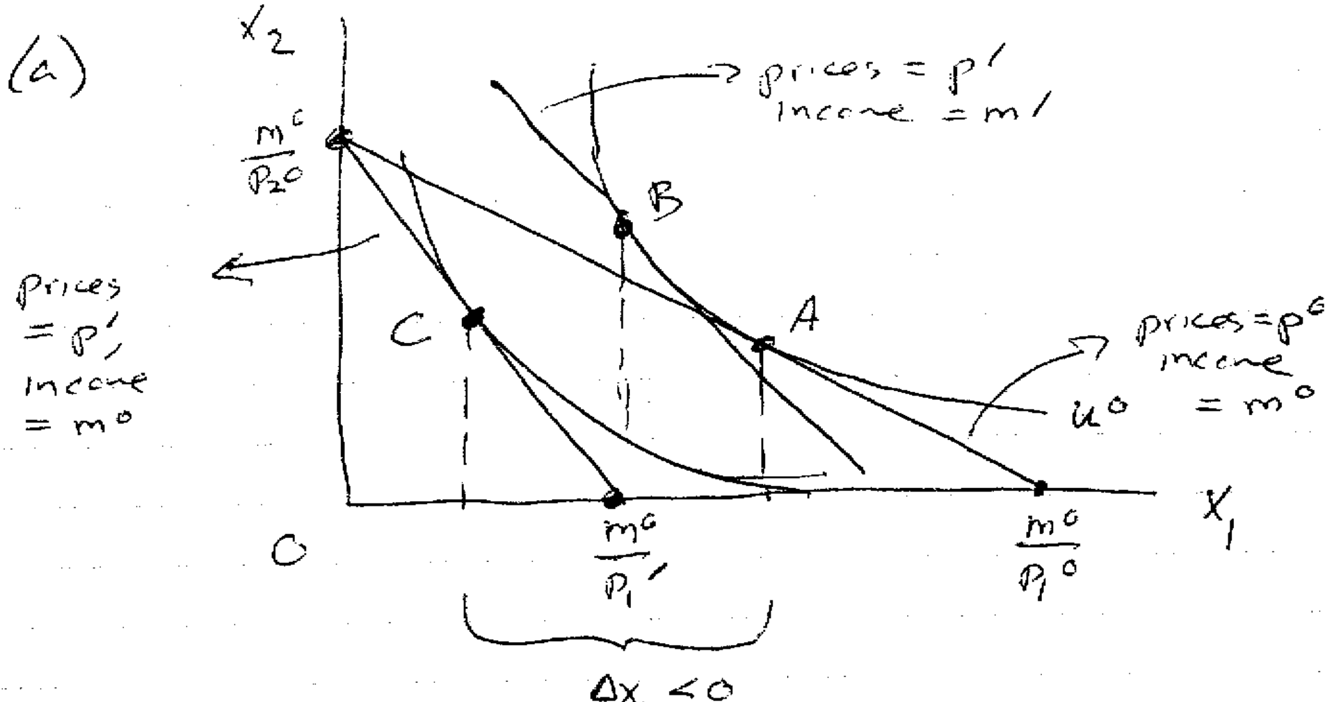
But we want $u^0 = u'$ so $\frac{b m^0}{p_2^0} = \frac{a m'}{p_1'}$

$$\Rightarrow m' = \left(\frac{p_1'}{p_2^0} \frac{b}{a}\right) m^0$$

We must have $m' \leq m^0$ because in the new situation $\frac{p_1}{p_2} \leq \frac{a}{b} \Rightarrow \frac{p_1}{p_2} \cdot \frac{b}{a} \leq 1$

since otherwise he would not be spending all his income on good 1 at the new prices. (Notice that $m' \leq m^0$ makes sense because a price fell, so he doesn't need as much income to get u^0).

3.(a)



A to B $\Rightarrow$ substitution effect
B to C $\Rightarrow$ income effect

(b)
$$\begin{aligned} \Delta x_1 &= x_1(p', m^0) - x_1(p^0, m^0) \\ &= \underbrace{x_1(p', m^0) - x_1(p', m')}_{=\,\Delta x_1^m} \\ &\quad + \underbrace{x_1(p', m') - x_1(p^0, m^0)}_{=\,\Delta x_1^s} \end{aligned}$$

first line is <u>income</u> effect
second line is the <u>substitution</u> effect.

Using Hicksian demands for Δx_1^s we get

$$\begin{aligned} \Delta x_1 &= x_1(p', m^0) - x_1(p', m') \\ &\quad + \underbrace{h_1(p', u^0) - h_1(p^0, u^0)}_{\Delta x_1^s} \end{aligned}$$

(c)
$$\frac{\Delta x_1}{\Delta p_1} = \frac{x_1(p', m^0) - x_1(p', m')}{\Delta p_1} + \frac{h_1(p', u^0) - h_1(p^0, u^0)}{\Delta p_1}$$

As $\Delta p_1 \to 0$, the 2nd term approaches

$$\lim_{\Delta p_1 \to 0} \frac{h_1(p_1^0 + \Delta p_1, p_2^0, u^0) - h_1(p^0, u^0)}{\Delta p_1} = \frac{\partial h_1(p^0, u^0)}{\partial p_1}$$

For the 1st term, note that $m' = e(p', u^0)$
and $m^0 = e(p^0, u^0)$

So $\Delta m = m' - m^0 = e(p', u^0) - e(p^0, u^0)$

$$\Rightarrow \frac{\Delta m}{\Delta p_1} = e(p', u^0) - e(p^0, u^0)$$

In the case where Δp_1 is small, we can approximate:

$$\Delta m \approx \frac{\partial e(p^0, u^0)}{\partial p_1} \Delta p_1$$

But Shephard's Lemma says $\frac{\partial e(p^0, u^0)}{\partial p_1} = h_1(p^0, u^0)$
Substituting this and the
approximation for Δm, we get

$$\frac{\Delta x_1}{\Delta p_1} \approx \left[\underbrace{\frac{x_1(p', m^0) - x_1(p', m')}{\Delta m}}_{\text{in the limit this}} \right] h_1(p^0, u^0) + \frac{\partial h_1(p^0, u^0)}{\partial p_1}$$

$\Rightarrow -\frac{\partial x_1(p^0, m^0)}{\partial m}$ (note that $p' \to p^0$)

and $h_1(p^0, u^0) = x_1(p^0, m^0)$, $u^0 = V(p^0, m^0)$;

$$\text{So } \frac{\partial x_1(p^0, m^0)}{\partial p_1} = \frac{\partial h_1[p^0, V(p^0, m^0)]}{\partial p_1} - \frac{\partial x_1(p^0, m^0)}{\partial m} \cdot x_1(p^0, m^0)$$

which is the Slutsky equation!

4 (a) $v(p, m^x) = \max_{x} \min\{ax_1, bx_2\}$

subject to $p_1x_1 + p_2x_2 = m^x$

This is a standard utility max problem with Leontief preferences. We must have

$ax_1 = bx_2 \Rightarrow x_2 = \frac{ax_1}{b}$

$\Rightarrow px_1 + p_2\left(\frac{ax_1}{b}\right) = m^x \Rightarrow x_1 = \frac{m^x}{p_1 + p_2\left(\frac{a}{b}\right)}$

$x_2 = \frac{m^x}{\left(\frac{b}{a}\right)p_1 + p_2}$

So $v(p, m^x) = \frac{am^x}{p_1 + p_2\left(\frac{a}{b}\right)}$

or $v(p, m^x) = \frac{abm^x}{bp_1 + ap_2}$

Define the composite good $X = v(p, m^x) = \frac{abm^x}{bp_1 + ap_2}$ since expenditure on X is $m^x = X\frac{(bp_1 + ap_2)}{ab}$

The "price" of X is $\frac{bp_1 + ap_2}{ab} = \frac{p_1}{a} + \frac{p_2}{b}$

So now the consumer has to solve

$\max_{X, z} u(X, z)$ subject to $rX + qz = m$

where $r = \frac{p_1}{a} + \frac{p_2}{b}$

is a price index for X.

~~(b) First, it is clear that we must have $ax_1^* = bx_2^*$ for the reasons given in (a); otherwise, there are no prices that make this an optimal bundle.~~

(b) This is a standard question about inverse demand – The fact that x is a composite commodity is irrelevant. We want to find prices (r^*, q^*) such that (x^*, z^*) will satisfy the FOC:

$$\left.\begin{aligned} \frac{\partial u(x^*, z^*)}{\partial x} &= \lambda r \\ \frac{\partial u(x^*, z^*)}{\partial z} &= \lambda q \end{aligned}\right\} \text{ and } r x^* + q z^* = 1.$$

Multiply each FOC by the corresponding quantity:

$$x^* \frac{\partial u(x^*, z^*)}{\partial x} = \lambda r x^*$$

$$z^* \frac{\partial u(x^*, z^*)}{\partial z} = \lambda q z^*$$

Then sum these and use $r x^* + q z^* = 1$ to get

$$\lambda = \underbrace{x^* \frac{\partial u(x^*, z^*)}{\partial x} + z^* \frac{\partial u(x^*, z^*)}{\partial z}}_{\text{call this function } \phi(x^*, z^*)}$$

Then $r^* = \dfrac{1}{\lambda} \dfrac{\partial u(x^*, z^*)}{\partial x} = \dfrac{\dfrac{\partial u(x^*, z^*)}{\partial x}}{\phi(x^*, z^*)}$

$$q^* = \frac{1}{\lambda} \frac{\partial u(x^*, z^*)}{\partial z} = \frac{\dfrac{\partial u(x^*, z^*)}{\partial z}}{\phi(x^*, z^*)}$$

(c) Again we can ignore the fact that X is a composite commodity. We want the individual Marshallian demands to have the form

$$X_i(r,q,m_i) = X_i(r,q,1)\,m_i$$

and

$$z_i(r,q,m_i) = z_i(r,q,1)\,m_i$$

also note that we can drop the i subscript on the functions (though not on m_i) because everyone has identical preferences. Then we can add things up over consumers to get the market demands:

$$\left.\begin{aligned} X(r,q,M) &= X(r,q,1)\,M \\ z(r,q,M) &= z(r,q,1)\,M \end{aligned}\right\} \text{ where } M = \sum_{i=1}^{n} m_i$$

There are various ways to choose the direct utility function so that the Marshallian demands have this form (any utility function that is homothetic will work). A simple example is $u(X,z) = \min\{X, z\}$. In part (a) we already solved for the demand functions for preferences of this kind (just set $a = b = 1$ and replace (p_1, p_2) by (r, q)). The result is

$$X = \frac{m}{r+q} = z \quad \Rightarrow \quad X_i(r,q,m_i) = \frac{m_i}{r+q}$$

$$z_i(r,q,m_i) = \frac{m_i}{r+q}.$$

We can also check that the indirect utility function has the Gorman form. From part (a), the indirect utility is $v(r,q,m_i) = \dfrac{m_i}{r+q}$

This is in the Gorman form with $a_i(p) = 0$ for all p and $b(p)$ replaced by $\frac{1}{r+q}$ (it depends only on prices and is the same for all i.)