Econ 802
Second Midterm Exam
Answer Key

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1. a) Define $w'' = tw + (1-t)w'$
and let x'' be the cost-minimizing input vector corresponding to w''.

Then $c(w'', y) = w''x'' = twx'' + (1-t)w'x''$

where $wx'' \geq wx = c(w,y)$ because x'' is not necessarily
and $w'x'' \geq w'x' = c(w',y)$ optimal at w, or at w'.

Thus $c[tw + (1-t)w', y] \geq tc(w,y) + (1-t)c(w',y)$.

Interpretation: The cost function is <u>concave</u> in prices.

b) Use Shephard's Lemma to show that

$$\frac{\partial c(w,y)}{\partial w_i} = x_i(w,y) \quad \text{for all } i = 1 \dots n.$$

(we assume differentiability)

Then take second derivatives to get

$$\frac{\partial^2 c(w,y)}{\partial w^2} = \begin{bmatrix} \frac{\partial x_1}{\partial w_1} & \cdots & \frac{\partial x_1}{\partial w_n} \\ \vdots & & \\ \frac{\partial x_n}{\partial w_1} & \cdots & \frac{\partial x_n}{\partial w_n} \end{bmatrix} = \frac{\partial x(w,y)}{\partial w} = \text{conditional substitution matrix.}$$

Since $c(w,y)$ is concave in w (see part (a)), it has a negative semidefinite Hessian matrix. This implies all diagonal elements are non-positive.

$\frac{\partial x_i}{\partial w_i} \leq 0$ for all i. $\Longleftarrow$

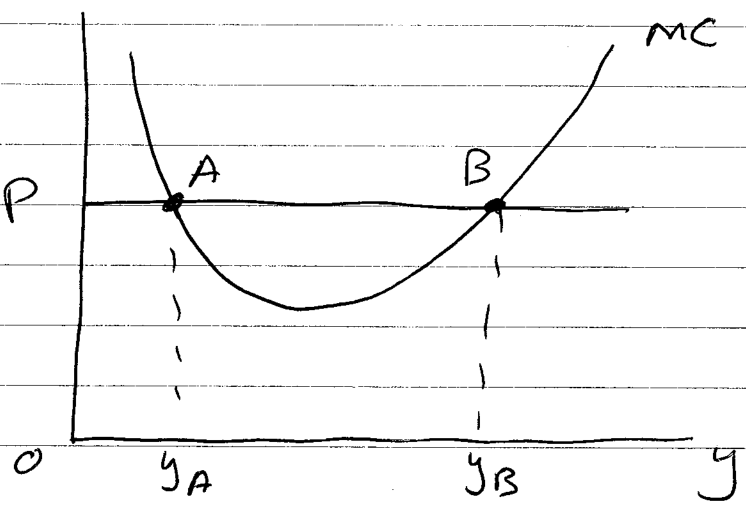
c) The firm maximizes $py - c(w,y)$

Assuming an interior solution the FOC is

$$p - \frac{\partial c(w,y)}{\partial y} = 0 \quad \text{or} \quad P = MC.$$

The SOC (necessary) is $-\frac{\partial^2 c(w,y)}{\partial y^2} \leq 0$ or

$$\frac{\partial^2 c(w,y)}{\partial y^2} \geq 0 \quad \text{(MC cannot be falling)}$$



y_A satisfies FOC but not SOC because MC is falling at A. point B satisfies both FOC and SOC.

2. a) The Lagrangean is $L = g(x_1 \ldots x_n) - \lambda\,[px - m]$

with FOC $\frac{\partial g(x)}{\partial x_j} = \lambda p_j$ for $j = 1 \ldots J$

and $px = m$

Fix p and suppose x^0 is optimal when income is m^0.

Thus $\frac{\partial g(x^0)}{\partial x_j} = \lambda^0 p_j$ for all j and $px^0 = m^0$.

Now change to a new income $m' = tm^0$ for $t > 0$.

Clearly $x' = tx^0$ satisfies the budget constraint since $p(tx^0) = (tm^0)$. Because $g(x)$ is HD1 its derivatives are HD0 and $\frac{\partial g(tx^0)}{\partial x_j} = \frac{\partial g(x^0)}{\partial x_j} = \lambda^0 p_j$ for all j.

Thus tx^0 satisfies all FOC at income m^0 and due to strict quasi-concavity, tx^0 must be optimal.

b) Start with $m^0 = 1$. Then $v(p,1) = \max g(x)$ subject to $px = m^0$.

Let the solution to this problem be x^0 so $v(p,1) = g(x^0)$.

Any other income level m can be written as $m = tm^0 = t$, and from the result in part (a), tx^0 is optimal when income is tm^0.

Thus $v(p,m) = \underbrace{g(tx^0) = tg(x^0)}_{\text{using homogeneity}} = tv(p,1)$

Thus $v(p,m) = v(p,1)m$ for all m.

c) No, it does not imply that aggregate Marshallian demands depend only on aggregate income $M = \sum_i m_i$. For this to be true each individual indirect utility would have to be in the Gorman form: $v_i(p,m) = a_i(p) + b(p)m_i$.

From part (b) we can write

$v_i(p,m_i) = v_i(p,1)m_i$ [note that $a_i(p) \equiv 0$ for everyone]

The problem is that we need $v_i(p,1) = b(p)$ for all i where $b(p)$ is identical for everyone. Since preferences are not identical, $v_i(p,1)$ is not identical, and the distribution of total income M among the consumers will affect the market demands for particular goods.

[To see an example consider 2 consumers with utility functions $x_1^{1/2}x_2^{1/2}$ and $x_1^{1/3}x_2^{2/3}$. Both are linearly homogeneous but the distribution of income affects aggregate demand for each good]

3 a) This is a Leontief function. Utility max requires $ax_1 = bx_2$ and $p_1x_1 + p_2x_2 = m$

$\Rightarrow x_2 = \frac{ax_1}{b} \Rightarrow p_1x_1 + p_2\left(\frac{ax_1}{b}\right) = m$

(i) $\Rightarrow x_1 = \dfrac{m}{p_1 + p_2\left(\frac{a}{b}\right)}$ This is Marshallian demand for good 1.

(ii) $\dfrac{\partial x_1}{\partial p_1} = \boxed{-\dfrac{m}{D^2}}$ where $D \equiv p_1 + p_2\left(\frac{a}{b}\right)$

(iii) Slutsky says $\dfrac{\partial x_1}{\partial p_1} = \dfrac{\partial h_1}{\partial p_1} - \dfrac{\partial x_1}{\partial m} \cdot x_1$

The income effect is $-\dfrac{\partial x_1}{\partial m} \cdot x_1 = -\dfrac{1}{D} \cdot \dfrac{m}{D} = \boxed{-\dfrac{m}{D^2}}$

This implies the substitution effect is $\dfrac{\partial h_1}{\partial p_1} = 0$

b) This is a log transformation of a Cobb Douglas

FOC $\Rightarrow \dfrac{a}{x_1} = \lambda p_1$ and $\dfrac{b}{x_2} = \lambda p_2$

$\Rightarrow a = \lambda p_1 x_1$ and $b = \lambda p_2 x_2 \Rightarrow a + b = \lambda m$

(i) $\Rightarrow \lambda = \dfrac{a+b}{m} \Rightarrow \boxed{x_1 = \dfrac{a}{\lambda p_1} = \dfrac{ma}{(a+b)p_1}}$

Marshallian demand

(ii) $\dfrac{\partial x_1}{\partial p_1} = -\dfrac{ma}{(a+b)p_1^2}$

(iii) The income effect is $-\dfrac{\partial x_1}{\partial m} \cdot x_1$

$= -\dfrac{a}{(a+b)p_1} \cdot \dfrac{ma}{(a+b)p_1} = -\dfrac{ma^2}{(a+b)^2 p_1^2}$ < 0

substitution effect $\dfrac{\partial h_1}{\partial p_1} = \dfrac{\partial x_1}{\partial p_1} + \dfrac{ma^2}{(a+b)^2 p_1^2}$

$\dfrac{-ma}{(a+b)p_1^2} + \dfrac{ma^2}{(a+b)^2 p_1^2} = \dfrac{ma}{(a+b)p_1^2}\left[-1 + \dfrac{a}{a+b}\right]$

< 0

(This is a quasi-linear utility function)

c) $\max\ u = x_1^a + x_2$ s.t. $p_1 x_1 + p_2 x_2 = m$

$$\Rightarrow x_2 = \frac{m - p_1 x_1}{p_2}$$

$$\max\ x_1^a + \frac{m}{p_2} - \frac{p_1 x_1}{p_2}$$

FOC: $a x_1^{a-1} = \frac{p_1}{p_2} \Rightarrow x_1 = \left(\frac{p_1}{a p_2}\right)^{\frac{1}{a-1}}$

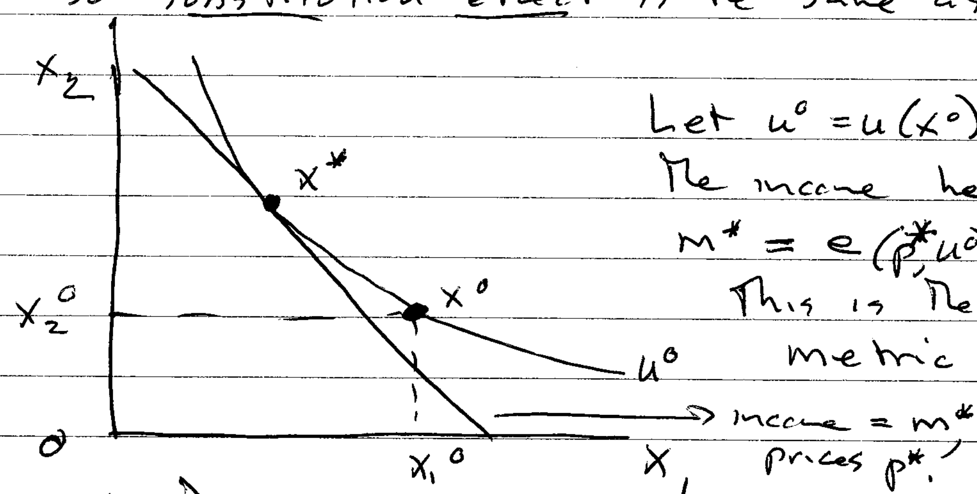
(i) Marshallian demand $\Rightarrow x_1 = \left(\frac{a p_2}{p_1}\right)^{\frac{1}{1-a}}$

(ii) $\frac{\partial x_1}{\partial p_1} = \left(\frac{1}{1-a}\right)\left(\frac{a p_2}{p_1}\right)^{\frac{1}{1-a}-1}\left(-\frac{a p_2}{p_1^2}\right) < 0$

(iii) Income effect $-\frac{\partial x_1}{\partial m} \cdot x_1 = 0$ because $\frac{\partial x_1}{\partial m} = 0$ (income does not affect x_1)

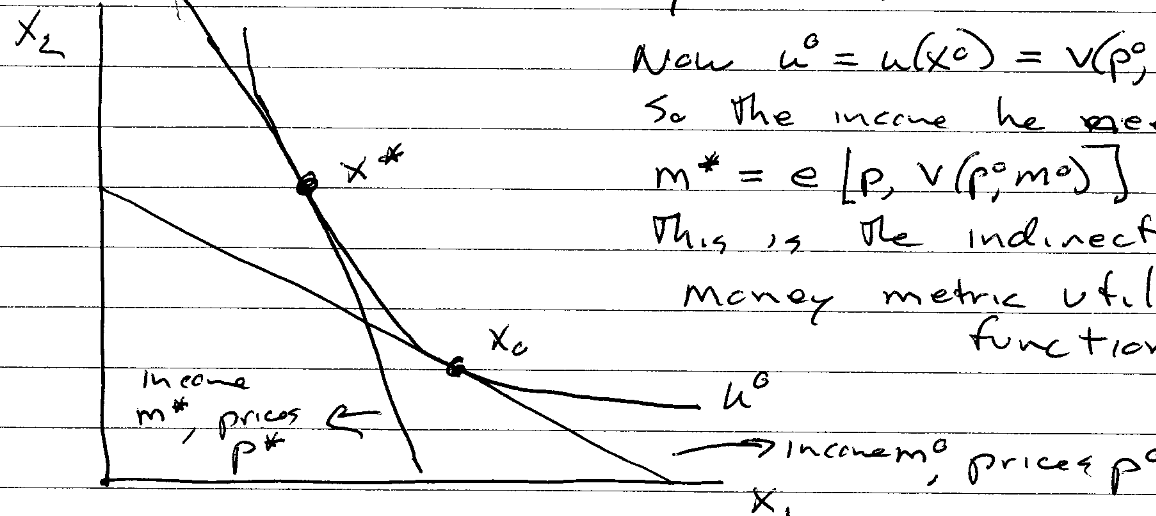
So substitution effect is the same as $\frac{\partial x_1}{\partial p_1}$

4 a)



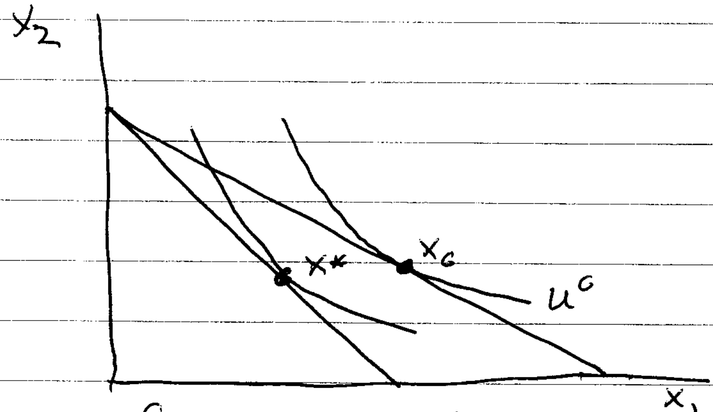
Let $u^0 = u(x^0)$
The income he needs is
$m^* = e(p^*, u^0) = e[p^*, u(x^0)]$
This is the direct money metric utility function.

b)

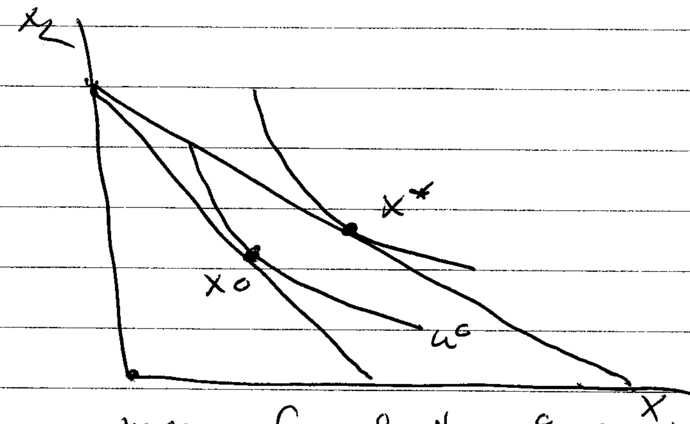


Now $u^0 = u(x^0) = v(p^0, m^0)$
So the income he ~~meeds~~ needs is
$m^* = e[p, v(p^0, m^0)]$
This is the indirect money metric utility function.

(c) Any of the three cases could occur. It depends on whether George is better off, worse off or indifferent at his old income m^0 after the prices change from p^0 to p^*.



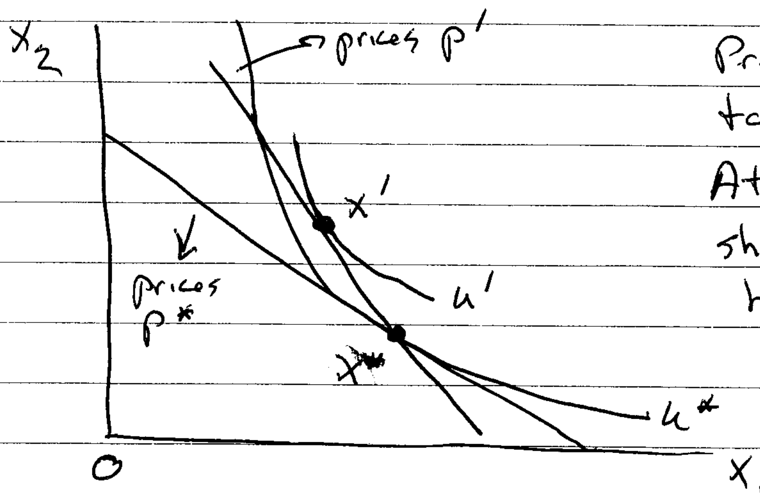
Income fixed at m^0 ⇒ worse off from price change, so needs ~~compensation~~ income $m^* > m^0$ to get back to old indiff curve



Income fixed at m^0 ⇒ better off from price change, have to take some income away ($m^* < m^0$) to get back to old indiff curve.

You can also have a combination of price changes that leave him on u^0 ⇒ no compensation needed ⇒ $m^* = m^0$.

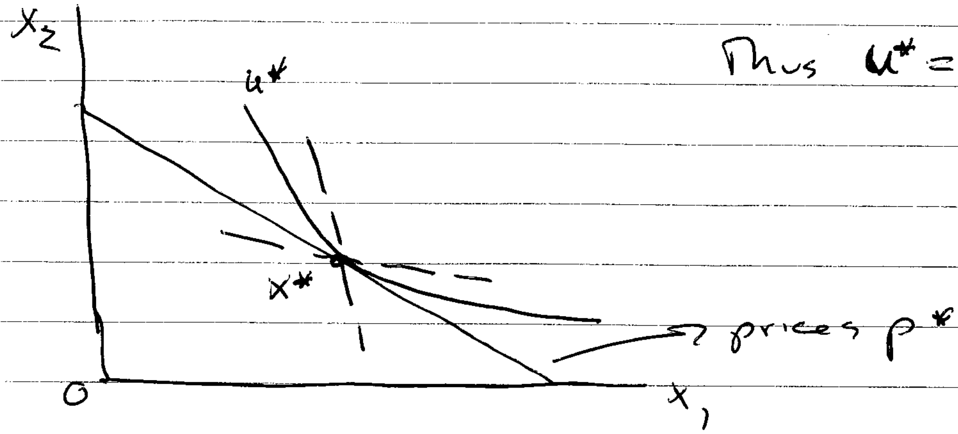
5 a)



Previously she was at a tangency to u^* at x^*. At the new prices p' she can still afford x^* but the budget line is steeper so she goes to x', which is preferred.

There is no violation of GARP because x' was not feasible under the conditions (p^*, m^*)

b) Since x^* is optimal at p^* we have a graph like the one in part (a)



Thus $u^* = v(p^*, 1)$.

The constraint $px^* = 1$ means that we are only considering price vectors such that x^* is affordable. All such budget lines pass through x^* with a steeper or flatter slope (see dashed lines). In every such case, x^* remains feasible so the maximum possible utility must be at least as high as $u^* = u(x^*) = v(p^*, 1)$. Therefore $v(p, 1) \geq v(p^*, 1)$ for all p such that $px^* = 1$.

c) From the FOC, $\dfrac{\partial u(x^*)}{\partial x_i} = \lambda p_i^*$ all $i = 1 \ldots n$

$$\Rightarrow \frac{\partial u(x^*)}{\partial x_i} \cdot x_i^* = \lambda p_i^* x_i^* \qquad \text{Sum over } i \text{ to get}$$

$$\sum_{i=1}^{n} \frac{\partial u(x^*)}{\partial x_i} \cdot x_i^* = \lambda \sum_{i=1}^{n} p_i^* x_i^* = \lambda m^* = \lambda$$

(using $m^* = 1$)

Substituting back into the FOC,

$$p_i^* = \frac{\dfrac{\partial u(x^*)}{\partial x_i}}{\lambda} \quad \text{or} \quad \boxed{p_i^* = \frac{\dfrac{\partial u(x^*)}{\partial x_i}}{\displaystyle\sum_{j=1}^{n} \frac{\partial u(x^*)}{\partial x_j} x_j^*}}$$

This method yields the inverse demand functions $p_i(x)$ $i = 1 \ldots n$ for any given x.