Econ 802

Answers to Final Exam

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1. (a) Write $L = -wx + \lambda(ax - y) + \mu x$
(note that this is the Lagrangian for a maximization problem; since we want min wx, we max $-wx$)

FOC: $\frac{\partial L}{\partial x_i} = -w_i + \lambda a_i + \mu_i = 0$; $\mu_i \geq 0$; $x_i \geq 0$; $\mu_i x_i = 0$
(all $i = 1 \ldots n$)

If $x_i^* > 0$ then $\mu_i = 0 \Rightarrow \lambda = \frac{w_i}{a_i}$
If $x_i^* = 0$ and $\mu_i = 0 \Rightarrow$ same thing
If $x_i^* = 0$ and $\mu_i > 0 \Rightarrow -w_i + \lambda a_i < 0 \Rightarrow \lambda < \frac{w_i}{a_i}$

This implies that for any i such that $\frac{w_i}{a_i} > \lambda = \min_{i=1..n} \left\{\frac{w_i}{a_i}\right\}$ we must have $x_i^* = 0$.
Among the i for which $\frac{w_i}{a_i}$ takes on the minimum value, we can choose any input levels that add up to $y = \sum_i a_i x_i$.

[One could reach this conclusion without using the Kuhn-Tucker approach. Suppose you produced y units of output using only input i. This implies that $x_i = \frac{y}{a_i}$ and the cost is $y\left(\frac{w_i}{a_i}\right)$. To minimize total cost, clearly we can only use those inputs for which $\frac{w_i}{a_i}$ is at a minimum.]

The cost function is $c(w, y) = y \cdot \min_{i=1..n} \left\{\frac{w_i}{a_i}\right\}$

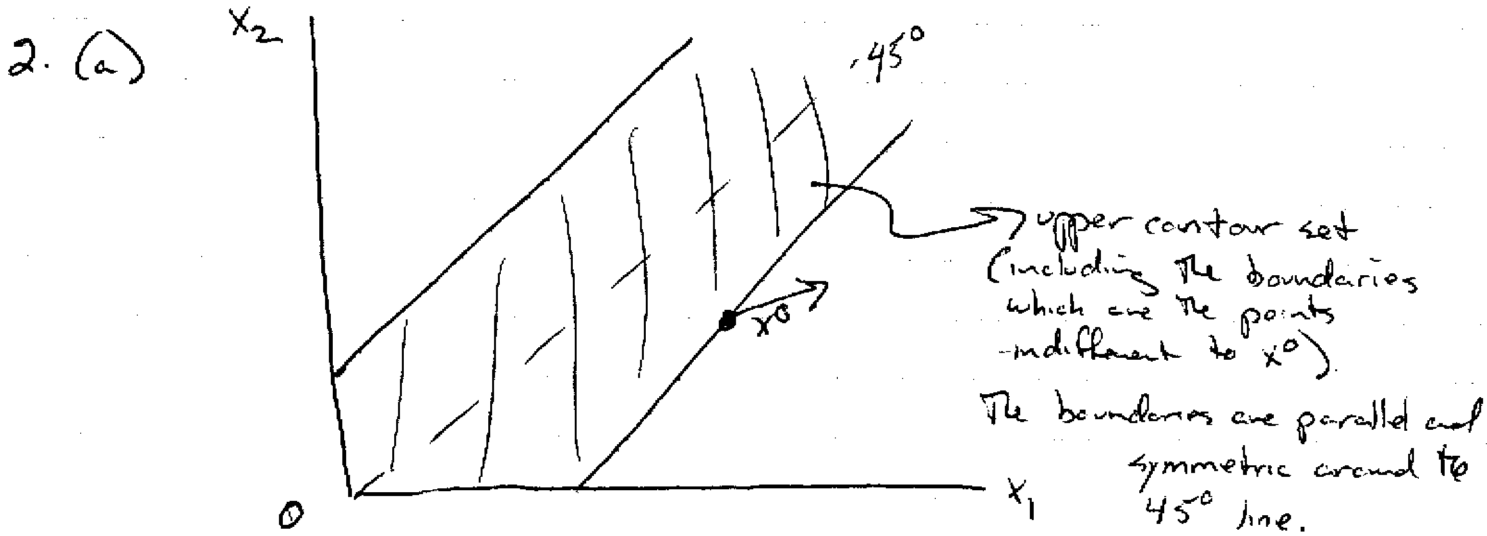
(b). Yes, this is possible. Suppose $\frac{w_n}{a_n} > \frac{w_i}{a_i}$ for some i (and there is no other input j with $\frac{w_j}{a_j} < \frac{w_i}{a_i}$)

Then cost min in the short run implies that $a_n x_n^o$ units of output are produced with the fixed input n, and the remaining units $y - a_n x_n^o$ are produced with the cost-minimizing input i. Total cost is $w_n x_n^o + \left[y - a_n x_n^o\right]\frac{w_i}{a_i}$.

In the long run all units of output would be produced using the cheap input i and total cost would be $y\frac{w_i}{a_i}$.

In both cases $MC = \frac{w_i}{a_i}$ but ~~total~~ total (and thus average) cost are higher in the short run since $w_n x_n^o - a_n x_n^o \frac{w_i}{a_i} = x_n^o a_n\left[\frac{w_n}{a_n} - \frac{w_i}{a_i}\right] > 0$.

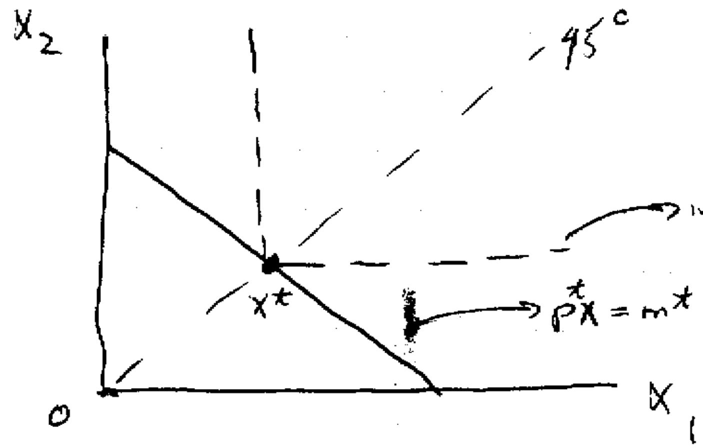
2. (a)



Reason: x^o is indifferent to any point x for which $|x_1 - x_2| = |x_1^o - x_2^o|$. This is true if $x_1 - x_2 = |x_1^o - x_2^o|$ or $x_2 = x_1 - |x_1^o - x_2^o|$ (a straight line with slope $= +1$ below the 45° line) or if $x_2 - x_1 = |x_1^o - x_2^o| \Rightarrow$ $x_2 = x_1 + |x_1^o - x_2^o|$ (a straight line with slope $+1$ above the 45° line)

(i) not weakly or strongly monotonic: one can add more of both goods and get a less-preferred point (see arrow)
(ii) not locally non-satiated because along the 45° line one has the highest possible utility (nothing is better than such a point)

(b) The data could never contradict the friend's hypothesis. When we max $u(x)$ s.t. $p^t x \le m^t$, we must choose a point on the $45°$ line (This is always feasible and gives the highest possible utility). One such point is located on the budget line → by assumption the consumer is spending all her income, so the observed bundle is the one where the budget line and the $45°$ line intersect. But this is exactly the same behavior as we get by maxing $w(x) = \min\{x_1, x_2\}$; this also leads to the point where the budget line and $45°$ line intersect (see graph).



In both cases we observe the point x^t.

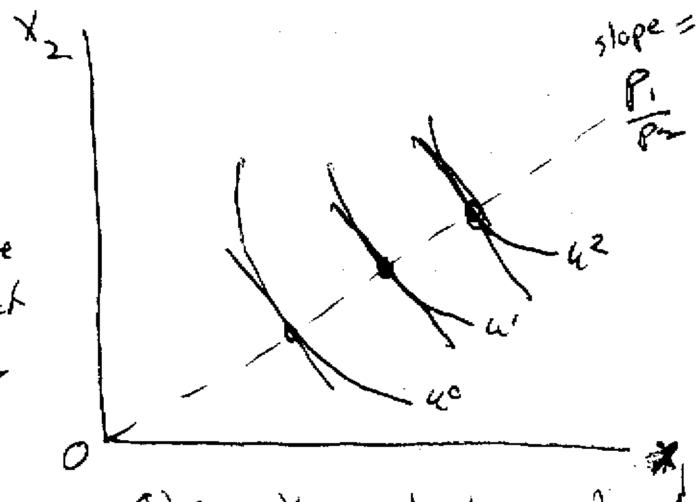
3. (a)

$$h_1(p,u) = \frac{\partial e(p,u)}{\partial p_1} = 2u^{1/2}\left(\tfrac{1}{2}\right)p_1^{-1/2}p_2^{1/2} = u^{1/2}\left(\frac{p_2}{p_1}\right)^{1/2}$$

$$h_2(p,u) = \frac{\partial e(p,u)}{\partial p_2} = u^{1/2}\left(\frac{p_1}{p_2}\right)^{1/2}$$

$$\frac{h_1(p,u)}{h_2(p,u)} = \frac{p_2}{p_1}$$

<u>Interpretation</u>: (h_1, h_2) is the solution to a cost minimization problem where we are trying to get to the lowest isocost line subject to a utility constraint. $\frac{h_1}{h_2}$ depends only on the price ratio but not the level of utility because the expansion path is a ray from the origin (homothetic prefs)



Observable implication: by duality the Marshallian demands have the same property.

(b) Since $e(p,u) = 2u^{1/2} p_1^{1/2} p_2^{1/2}$ we can invert this to get

$$v(p,m) = \frac{m^2}{4p_1p_2}$$

Roy's identity $\Rightarrow$ $$x_1(p,m) = -\frac{\frac{\partial v(p,m)}{\partial p_1}}{\frac{\partial v(p,m)}{\partial m}} = -\frac{\left[-\frac{m^2}{4p_1^2p_2}\right]}{\frac{m}{2p_1p_2}} = \frac{m}{2p_1}$$

$$x_2(p,m) = \frac{m}{2p_2}$$

The cross price effect $\frac{\partial x_1}{\partial p_2} = 0$. We decompose this as follows:

$$\frac{\partial x_1(p,m)}{\partial p_2} = 0 = \frac{\partial h_1[p, v(p,m)]}{\partial p_2} - \frac{\partial x_1(p,m)}{\partial m} \cdot x_2(p,m)$$

$$0 = \underbrace{\left(\tfrac{1}{2}\right) p_2^{-1/2} p_1^{-1/2} [v(p,m)]^{1/2}}_{\frac{1}{2p_1^{1/2}p_2^{1/2}} \cdot \frac{m}{2p_1^{1/2}p_2^{1/2}} = \boxed{\frac{m}{4p_1p_2}}} - \underbrace{\frac{1}{2p_1} \cdot \frac{m}{2p_2}}_{\boxed{-\frac{m}{4p_1p_2}}}$$

Thus an increase in p_2 has a <u>positive</u> substitution effect $\frac{m}{4p_1p_2}$ on the demand for x_1 but there is an exactly offsetting <u>negative</u> income effect from $p_2 \uparrow$ so the net effect is <u>zero</u>.

4 (a) Each i maxes $-\frac{1}{x_i} + y_i$ subject to $p x_i + y_i = m_i$

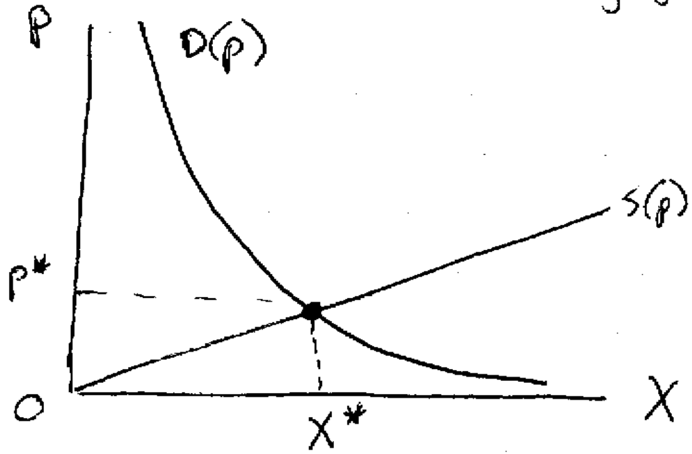
By substitution we max $-\frac{1}{x_i} + m_i - p x_i$

FOC: $\frac{1}{x_i^2} = p \Rightarrow x_i(p) = \frac{1}{\sqrt{p}} \qquad i = 1 \cdots n$

So total demand is $D(p) = \sum_i x_i(p) = \frac{n}{\sqrt{p}}$

Each j maxes $p z_j - \frac{z_j^2}{2}$; FOC: $p = z_j \Rightarrow z_j(p) = p,\ j = 1 \cdots m$

So total supply is $S(p) = \sum_j z_j(p) = mp$



Set $D(p^*) = S(p^*)$

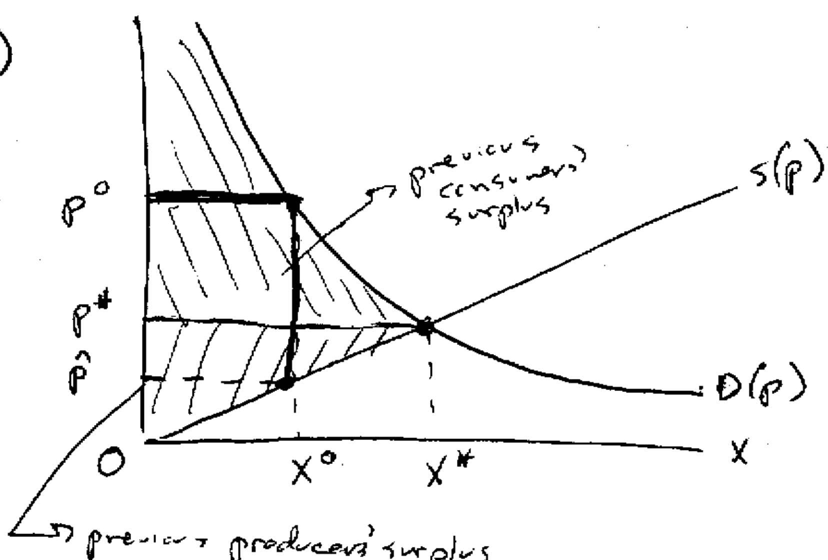
$\Rightarrow \frac{n}{\sqrt{p}} = mp$

$\Rightarrow \frac{n}{m} = p^{3/2}$

$\Rightarrow \boxed{p^* = \left(\frac{n}{m}\right)^{2/3}}$

$\boxed{X^* = mp^* = n^{2/3} m^{1/3}}$

(b)



① Consumer surplus previously was the area below D and above p^*; it falls and is now the area below D and above p^0

② producer surplus was previously the area above S and below p^*; There is both a gain and a loss; PS ↑ because the rectangle between p^0 and p^* is shifted from CS to PS; but PS ↓ because we lose the area above S and below p^* to the right of X^0.

③ total surplus falls because the shaded area below D and above S between X^0 and X^* disappears (it is deadweight loss)

To see whether the net effect on PS is positive or negative, write

$$PS(p_0) = p^0 X^0 - \frac{1}{2}\hat{p}\, X^0 \quad \text{(see graph)}$$

from part (a) $X^0 = \frac{n}{\sqrt{p^0}}$ and $\hat{p} = \frac{X^0}{m}$

So

$$PS(p^0) = n(p^0)^{1/2} - \frac{1}{2}\frac{(X^0)^2}{m} = n(p^0)^{1/2} - \frac{1}{2}\frac{n^2}{mp^0}$$

$$\frac{d\, PS(p^0)}{dp^0} = \frac{n}{2}(p^0)^{-1/2} + \frac{n^2}{2m}(p^0)^{-2} > 0$$ So producer surplus $\uparrow$ as $p^0 \uparrow$.

5. (a) A's Marshallian demands:

$$\max\ \theta \ln x_{A1} + (1-\theta)\ln x_{A2} \text{ subject to } p_1 x_{A1} + p_2 x_{A2} = m_A$$

FOC:

$$\left.\begin{aligned} \frac{\theta}{x_{A1}} &= \lambda p_1 \\ \frac{(1-\theta)}{x_{A2}} &= \lambda p_2 \end{aligned}\right\} \left(\frac{\theta}{1-\theta}\right)\frac{x_{A2}}{x_{A1}} = \frac{p_1}{p_2} \Rightarrow x_{A2} = \frac{p_1 x_{A1}}{p_2}\left(\frac{1-\theta}{\theta}\right)$$

$$\Rightarrow p_1 x_{A1} + p_1 x_{A1}\left(\frac{1-\theta}{\theta}\right) = m_A \Rightarrow \boxed{x_{A1} = \frac{m_A}{p_1\left[1+\left(\frac{1-\theta}{\theta}\right)\right]} = \frac{\theta m_A}{p_1} \qquad x_{A2} = \frac{(1-\theta)m_A}{p_2}}$$

$$m_A = \frac{p_1}{2} + \frac{p_2}{2}$$

$$\Rightarrow x_{A1} = \frac{\theta}{2}\left(1+\frac{p_2}{p_1}\right) \qquad x_{A2} = \frac{(1-\theta)}{2}\left(1+\frac{p_1}{p_2}\right)$$

B's indifference curves are linear with slope -1 so if B is going to hold both goods the price ratio must be $\frac{p_1}{p_2} = +1$. Another possible approach is to set $MRS_A = MRS_B \Rightarrow \dfrac{\frac{\theta}{x_{A1}}}{\frac{1-\theta}{x_{A2}}} = 1 \Rightarrow \left(\frac{\theta}{1-\theta}\right) = \frac{x_{A1}}{x_{A2}}$ so the solution must lie along this ray.

Either way we have

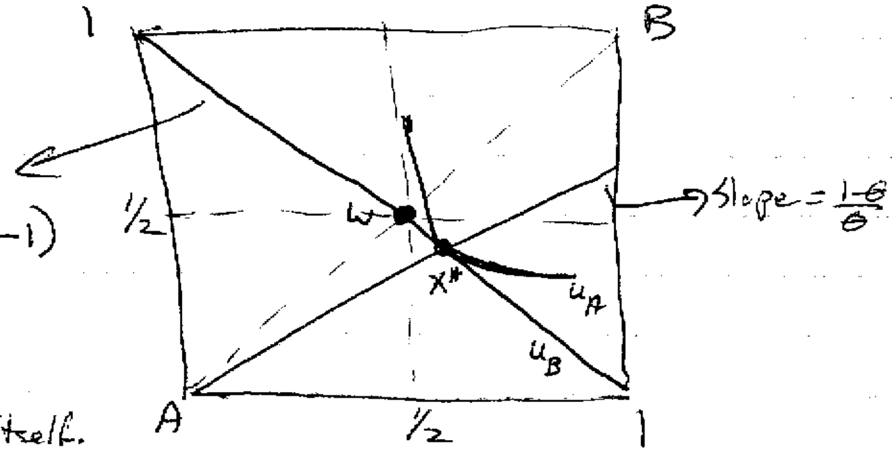
$$x_{A1}^* = \theta \text{ and } x_{A2}^* = 1-\theta \quad \text{where } x_{A1}^* > x_{A2}^* \text{ because } \theta > 1/2$$

To clear both markets we must have

$$x_{B1}^* = 1-\theta \text{ and } x_{B2}^* = \theta.$$

Edgeworth box:

The endowment is w and equilibrium is x^*, u_A is tangent to the budget line x^*; B's indiff curve through x^* is the budget line itself.



(b) $$L = n_A\left[\theta \ln x_{A1} + (1-\theta)\ln x_{A2}\right] + n_B\left(x_{B1} + x_{B2}\right)$$
$$- \lambda_1\left[n_A x_{A1} + n_B x_{B1} - \frac{(n_A+n_B)}{2}\right] - \lambda_2\left[n_A x_{A2} + n_B x_{B2} - \frac{(n_A+n_B)}{2}\right]$$

$$\left.\begin{aligned}\frac{\partial L}{\partial x_{A1}} &= \frac{n_A\theta}{x_{A1}} - \lambda_1 n_A = 0\\ \frac{\partial L}{\partial x_{A2}} &= \frac{n_A(1-\theta)}{x_{A2}} - \lambda_2 n_A = 0\end{aligned}\right\} \Rightarrow \boxed{\begin{aligned}x_{A1}^* &= \theta\\ x_{A2}^* &= 1-\theta\end{aligned}}$$

$$\left.\begin{aligned}\frac{\partial L}{\partial x_{B1}} &= n_B - \lambda_1 n_B = 0\\ \frac{\partial L}{\partial x_{B2}} &= n_B - \lambda_2 n_B = 0\end{aligned}\right\} \Rightarrow \boxed{\lambda_1 = \lambda_2 = 1}$$

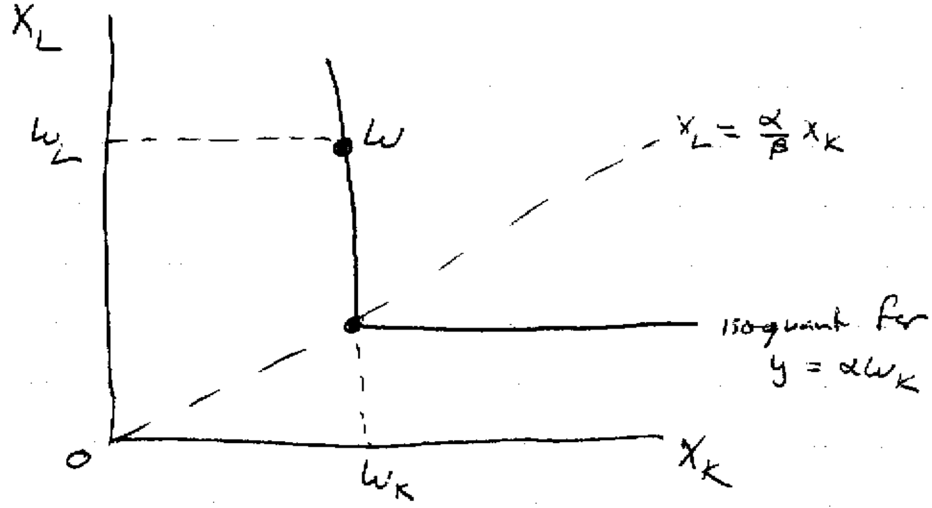
So the optimal allocation is to give each person of type A this bundle and give each person of type B the fraction $\frac{1}{n_B}$ of what is left over; ie

$$\left.\begin{aligned}x_{B1}^* &= \frac{1}{n_B}\left[\frac{(n_A+n_B)}{2} - \theta n_A\right]\\ x_{B2}^* &= \frac{1}{n_B}\left[\frac{(n_A+n_B)}{2} - (1-\theta)n_A\right]\end{aligned}\right\}$$

note: These are non-negative because $n_B \geq n_A$ and $1 > \theta > 1-\theta$

To achieve this allocation as a WE, choose the price ratio $\frac{p_1}{p_2} = 1$ as in part (a) and assign each person their ~~equilibrium~~ optimal consumption bundle as an endowment. Clearly everyone can afford their optimal bundle, and it can be shown that no one can increase their utility by choosing some other feasible (affordable) bundle, because $MRS_A = MRS_B = 1 = \frac{p_1}{p_2}$.

6. (a)



The endowment point w lies above the ray $x_L = \frac{\alpha}{\beta} x_K$ (so above the corner of the isoquant) because $\beta w_L > \alpha w_K \Rightarrow w_L > \frac{\alpha}{\beta} w_K$.
The output at the endowment point is $y = \min\{\alpha w_K, \beta w_L\} \Rightarrow y = \alpha w_K$

Since Crusoe wants to maximize food output using his endowment, the best he can do is to produce $y = \alpha w_K$ using his entire endowment of seeds as an input, and any x_L such that $\alpha w_K \leq \beta x_L \leq \beta w_L$ (must use <u>at least</u> $x_L = \frac{\alpha}{\beta} w_K$ hours of labor but using more is OK too).

(b) First obtain the cost function for food. If $p_K > 0$ and $p_L > 0$ the cost minimizing solution is $y = \alpha x_K = \beta x_L$ for any given y
$\Rightarrow x_K = \frac{y}{\alpha}$ and $x_L = \frac{y}{\beta} \Rightarrow c(w,y) = y\left[\frac{p_K}{\alpha} + \frac{p_L}{\beta}\right]$

To have equilibrium in the food market we must have $p_F = \frac{p_K}{\alpha} + \frac{p_L}{\beta} = 1$
1. LAC $> 1 \Rightarrow$ firm's profit maximizing response is $y = 0$
$\Rightarrow$ zero demand for both inputs $\Rightarrow$ both inputs have zero prices (they are in excess supply) which contradicts LAC > 1.
2. LAC $< 1 \Rightarrow$ firm wants $y = +\infty$ (no solution to profit max problem) both inputs are in excess demand $\Rightarrow$ cannot clear input markets.

So $\boxed{\frac{p_K}{\alpha} + \frac{p_L}{\beta} = 1.}$

The firm has zero profit and *any* output level is optimal.
Crusoe spends all his income on food and his income is $p_K w_K + p_L w_L$
(firm has zero profit so we can ignore income from this source)
Since $p_F = 1$, Crusoe demands $p_K w_K + p_L w_L$ units of food,
and market clearing $\Rightarrow$ this must be the supply from the firm.

How to nail down p_K^* and p_L^*? If both are *positive*, the firm demands $x_K = \frac{y}{\alpha}$ and $x_L = \frac{y}{\beta}$ where $y = p_K w_K + p_L w_L$ as above

To clear the K market we need $x_K^* = \frac{p_K w_K + p_L w_L}{\alpha} = w_K$

and to clear the L market we need $x_L^* = \frac{p_K w_K + p_L w_L}{\beta} = w_L$

$\Rightarrow p_K w_K + p_L w_L = \alpha w_K$ <u>and</u> $p_K w_K + p_L w_L = \beta w_L$

<u>BUT</u> we know $\alpha w_K < \beta w_L$ so this doesn't work. Why? There is an <u>excess supply</u> of L. To deal with this, set $p_L^* = 0$ so the firm is indifferent about how much x_L it uses. Now $p_F = 1 = LAC \Rightarrow$
$1 = \frac{p_K}{\alpha} \Rightarrow \boxed{p_K^* = \alpha}$ $\boxed{p_L^* = 0}$

We clear the K market because $x_K^* = \frac{p_K^* w_K + p_L^* w_L}{\alpha} = w_K$ $\left(\begin{matrix}\text{demand}\\ = \text{supply}\end{matrix}\right)$
and the L market can be cleared because
the firm is indifferent toward any x_L such that $\alpha w_K \leq \beta x_L$.
In particular, we can set $x_L^* = w_L > \frac{\alpha}{\beta} w_K$ $\left(\begin{matrix}\text{again, demand}\\ = \text{supply}\end{matrix}\right)$

or: could have the firm demand $x_L^* = \frac{\alpha}{\beta} w_K$ and let there be excess supply of labor (this is OK because $p_L^* = 0$).

The preceding argument shows that (i) Crusoe is maxing utility (he spends all his income on food); (ii) the firm is maxing profit (it is minimizing cost and is indifferent about the scale of output); and (iii) all markets clear.