Econ 802

Greg Dow — Answers to Final Exam — Dec 2005

1. (a) First, we show that $\pi(p) \geq \sum_{j=1}^{m} \pi_j(p)$.

Let $y_j(p)$ be an optimal production plan for firm j at prices p. Write $y = \sum_{j=1}^{m} y_j(p)$. Since $y_j(p) \in Y_j$ for all j, we have $y \in Y$. Thus one feasible profit level for the aggregate firm is $py = p\sum_{j=1}^{m} y_j(p) = \sum_{j=1}^{m} \pi_j(p)$. This implies that the maximum profit $\pi(p)$ for the aggregate firm must be at least as large.

We now rule out the possibility $\pi(p) > \sum_{j=1}^{m} \pi_j(p)$. Suppose this is true, so there is some $y' \in Y$ such that $py' > \sum_{j=1}^{m} py_j(p)$. Since $y' \in Y$, it can be written in the form $y' = \sum_{j=1}^{m} y_j'$ where $y_j' \in Y_j$ for all j. Thus

$$\sum_{j=1}^{m} py_j' > \sum_{j=1}^{m} py_j(p)$$

which implies that at least one firm has $py_j' > py_j(p)$ where $y_j' \in Y_j$. This contradicts the fact that $y_j(p)$ is an optimal plan.

(b) From WAPM we must have

$$p^1 y_j^1 \geq p^1 y_j^2 \quad \text{for all } j$$

$$\text{and} \quad p^2 y_j^2 \geq p^2 y_j^1 \quad \text{for all } j.$$

We want to know whether $y^1 = \sum_{j=1}^{m} y_j^1$ and $y^2 = \sum_{j=1}^{m} y_j^2$ satisfy similar inequalities.
Let's check whether $p^1 y^1 \geq p^1 y^2$ holds.
This is true if

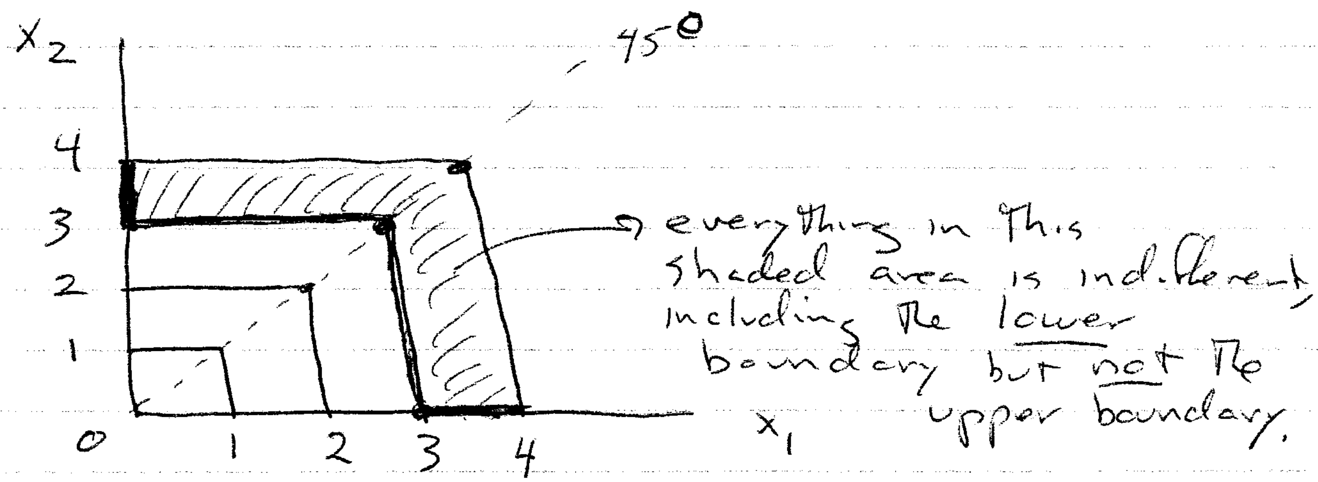
$$p^1 \sum_{j=1}^{m} y_j^1 \geq p^1 \sum_{j=1}^{m} y_j^2$$

or $\sum_{j=1}^{m} p^1 y_j^1 \geq \sum_{j=1}^{m} p^1 y_j^2$.

But since $p^1 y_j^1 \geq p^1 y_j^2$ holds for all j, the aggregate inequality must also hold. Using the same method we can show that $p^2 y^2 \geq p^2 y^1$ also holds.

Therefore as long as the individual firms always obey WAPM, the "aggregate" firm must do the same.

2. (a) George's indifference "curves" look like this:



(i) His preferences are transitive because $u(x) \geq u(y)$ and $u(y) \geq u(z)$ implies $u(x) \geq u(z)$.

(ii) His preferences are not continuous because the set $\{x : x \precsim (3,3)\}$ is not closed (it includes the shaded area but not its upper boundary)

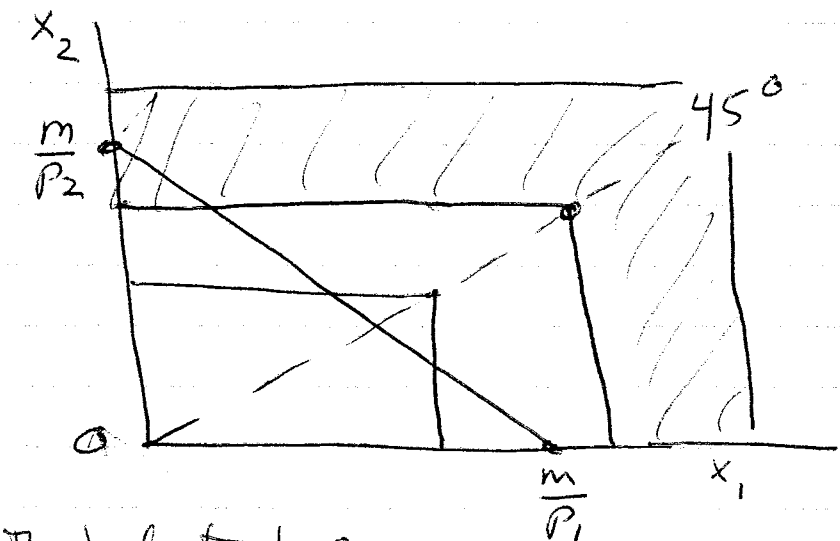
(iii) George's preferences are <u>not</u> locally non-satiated. Consider the point (3.5, 3.5). We can find a small neighborhood of this point in which all bundles are indifferent; there is no bundle in the neighborhood that is strictly preferred.

(iv) His preferences are <u>not</u> strongly monotonic either. Strong monotonicity implies local non-satiation and we already know this is false.

(v) His preferences are clearly <u>not</u> convex. We have $(3,0) \succsim (3,3)$ and $(0,3) \succsim (3,3)$. But intermediate points on the line segment between $(3,0)$ and $(0,3)$ are <u>worse</u> than $(3,3)$.

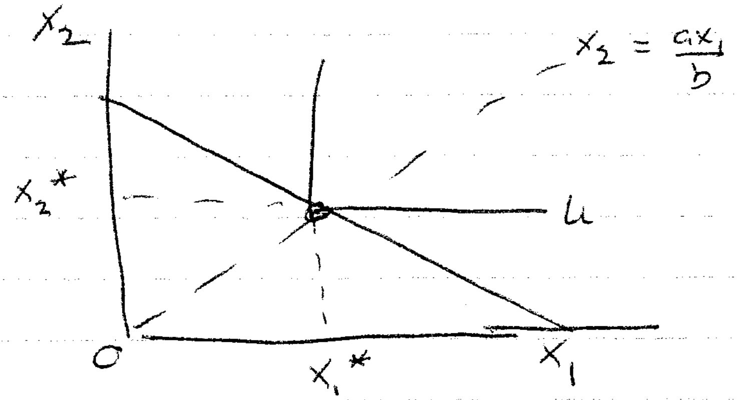
(b) Consider a graph:

If $\frac{m}{p_2} \geq \frac{m}{p_1}$
then $I\left(\frac{m}{p_2}\right) \geq I\left(\frac{m}{p_1}\right)$



and the bundle $\left(0, \frac{m}{p_2}\right)$ is at least as good as $\left(\frac{m}{p_1}, 0\right)$. It is clear from the graph in (a) that no other point on the budget line can be preferred to $\left(0, \frac{m}{p_2}\right)$. Therefore this bundle is optimal (note that no point <u>below</u> the budget line can be better either). Conversely, if $\frac{m}{p_1} \geq \frac{m}{p_2}$ then $\left(\frac{m}{p_1}, 0\right)$ is an optimal bundle. So a solution always exists. However, it is usually not unique. For instance, if $\frac{m}{p_2} = 3.5 > \frac{m}{p_1} = 2.5$, then any bundle of the form $(0, x_2)$ where $3.0 \leq x_2 \leq 3.5$ is feasible and just as good. Thus George may not spend all his income at an optimal bundle.

3. (a) Let $u(x) = \min\{ax_1, bx_2\}$ with $a > 0$, $b > 0$



To minimize the cost of achieving utility u, we set $u = ax_1 = bx_2$
$\Rightarrow x_1^* = \frac{u}{a}$, $x_2^* = \frac{u}{b}$

Thus the expenditure function is $e(p,u) = p_1 \frac{u}{a} + p_2 \frac{u}{b}$
which is a linear function of prices. $= u\left(\frac{p_1}{a} + \frac{p_2}{b}\right)$

The Hicksian demands
$h_1(p,u) = \frac{\partial e(p,u)}{\partial p_1} = \frac{u}{a}$ and $h_2(p,u) = \frac{\partial e(p,u)}{\partial p_2} = \frac{u}{b}$
have the special property that they are constants (they don't depend on prices). This reflects the absence of any substitution effects. The Marshallian demands are obtained by setting
$ax_1 = bx_2$ and $p_1x_1 + p_2x_2 = m \Rightarrow p_1x_1 + p_2\frac{ax_1}{b} = m$

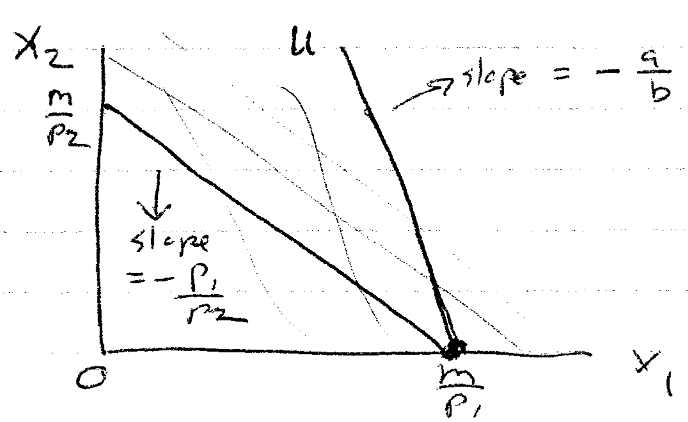
$$\Rightarrow x_1(p,m) = \frac{m}{p_1 + \frac{a}{b}p_2}$$

$$x_2(p,m) = \frac{m}{\frac{b}{a}p_1 + p_2}$$

These involve only income effects when a price changes (the substitution effect in the Slutsky equation is zero)

$\Rightarrow$ perfect complements.

(b) Let $u(x) = ax_1 + bx_2$ with $a > 0$, $b > 0$



If $-\frac{a}{b} \le -\frac{p_1}{p_2}$ or $\frac{a}{b} \ge \frac{p_1}{p_2}$ then $\left(\frac{m}{p_1}, 0\right)$ is optimal.
If $\frac{a}{b} \le \frac{p_1}{p_2}$ then $\left(0, \frac{m}{p_2}\right)$ is optimal in the Marshallian problem.

Similarly, in the expenditure min (Hicksian) problem,

$\frac{a}{b} \geq \frac{p_1}{p_2} \Rightarrow$ set $x_2 = 0$ and $x_1 = \frac{u}{a}$

and $\frac{a}{b} \leq \frac{p_1}{p_2} \Rightarrow$ set $x_1 = 0$ and $x_2 = \frac{u}{b}$

Thus $e(p,u) = \frac{u p_1}{a}$ if $\frac{a}{b} \geq \frac{p_1}{p_2}$ or $\frac{p_2}{b} \geq \frac{p_1}{a}$

$e(p,u) = \frac{u p_2}{b}$ if $\frac{a}{b} \leq \frac{p_1}{p_2}$ or $\frac{p_2}{b} \leq \frac{p_1}{a}$

So $e(p,u) = u \min\left\{\frac{p_1}{a}, \frac{p_2}{b}\right\}$ which has the Leontief form in prices.

In this case the goods are perfect substitutes and we generally get corner solutions in both the Marshallian and Hicksian cases. As long as we are not at the discontinuity point where $\frac{p_1}{a} = \frac{p_2}{b}$, we can differentiate $e(p,u)$ and show that small changes in prices have no effect on the Hicksian demands.

4. (a) In the short run, method 1 gives $y = \min\{aL, b\}$ so $y = aL$ if $aL \leq b$ and $y = b$ if $aL \geq b$. We can ignore the case $aL > b$ so we have $L = \frac{y}{a}$ $\Rightarrow$ variable cost is $wL = \frac{wy}{a}$, and marginal cost is $\frac{w}{a}$ = AVC. The firm is willing to produce any output $0 \leq y \leq b$ if $p = \frac{w}{a}$, ~~and~~ it produces 0 for $p < \frac{w}{a}$, and it produces $y = b$ if $p > \frac{w}{a}$.

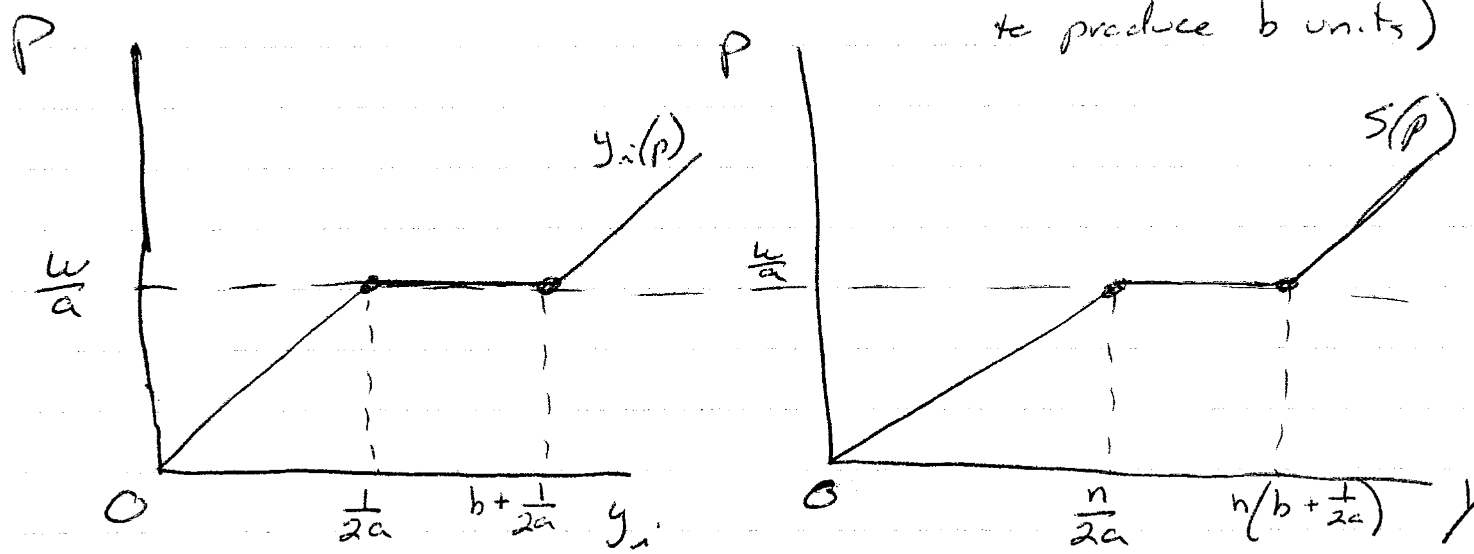
Method 2 gives $L = y^2 \Rightarrow$ variable cost is $wy^2 \Rightarrow$ marginal cost $= 2wy > wy =$ AVC. So the firm always chooses y for method 2 to satisfy $p = 2wy \Rightarrow y = \frac{p}{2w}$

Since firm i can use both methods simultaneously it maximizes its overall profit by maximizing profit from each method. Its total output $y_i(p)$ is the sum of the outputs from the two methods. Thus

$y_i(p) = \frac{p}{2w}$ if $p < \frac{w}{a}$ (Method 1 not used)

$y_i(p) =$ any output between $\frac{p}{2w}$ and $\frac{p}{2w} + b$ if $p = \frac{w}{a}$ (indifferent to Method 1)

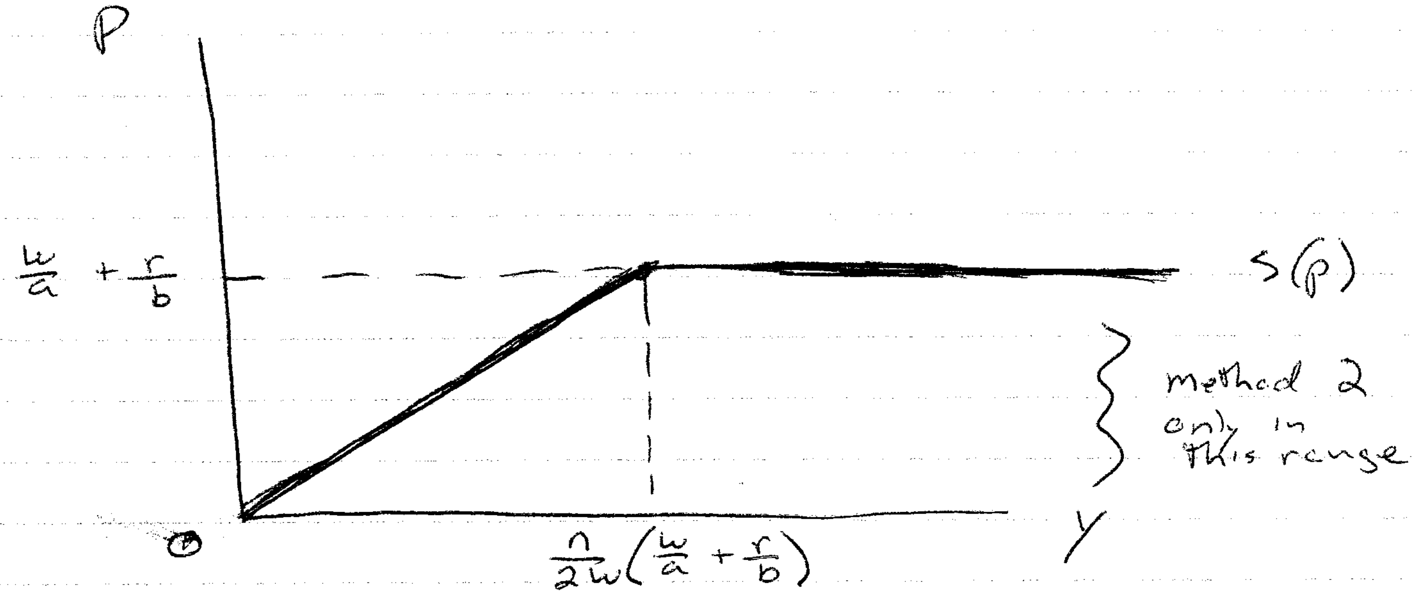
$y_i(p) = \frac{p}{2w} + b$ if $p > \frac{w}{a}$ (Method 1 used to produce b units)



(b) Method 1 involves constant returns in the long run. To minimize cost each firm sets $y = aL = bK \Rightarrow$ $L = \frac{y}{a}$, $K = \frac{y}{b} \Rightarrow c(w,r,y) = y\left(\frac{w}{a} + \frac{r}{b}\right) \Rightarrow AC = \left(\frac{w}{a} + \frac{r}{b}\right)$.

If $p > \frac{w}{a} + \frac{r}{b}$, all firms produce unboundedly large outputs using Method 1. If $p = \frac{w}{a} + \frac{r}{b}$ then any output is profit-maximizing, and if $p < \frac{w}{a} + \frac{r}{b}$ then Method 1 is not used, but method 2 is still used as before.

The long run market supply curve is



The maximum possible price is $\frac{w}{a} + \frac{r}{b}$ (if the market demand intersects $S(p)$ on the horizontal part). The price could be lower if demand intersects $S(p)$ on its rising part where only Method 2 is used. If entry is permitted then it will occur, because firms can make positive profit from Method 2 whenever $p > 0$ (regardless of whether they use Method 1). Therefore entry will make $p \to 0$.

5. (a) We want to solve $\max \sum_{i=1}^{n} a_i u_i(x_i)$
subj to $\sum_{i=1}^{n} x_{ij} = w_j$ for all $j = 1 .. k$.

The Lagrangian is

$$L = \sum_{i=1}^{n} a_i \sum_{j=1}^{k} b_{ij} \ln x_{ij} - \sum_{j=1}^{k} q_j \left(\sum_{i=1}^{n} x_{ij} - w_j \right)$$

FOC: $\frac{\partial L}{\partial x_{ij}} = \frac{a_i b_{ij}}{x_{ij}} - q_j = 0$ all i, j

Write this as $a_i b_{ij} = q_j x_{ij}$, then sum over i

$$\Rightarrow \sum_{i=1}^{n} a_i b_{ij} = q_j \sum_{i=1}^{n} x_{ij} = q_j w_j \quad \text{(using the constraint)}$$

$$\Rightarrow q_j = \frac{\sum_{i=1}^{n} a_i b_{ij}}{w_j} \Rightarrow x_{ij}^* = \frac{a_i b_{ij} w_j}{\sum_{i=1}^{n} a_i b_{ij}} \quad (\text{all } i, j)$$

Thus person i gets a share of the aggregate supply of good j equal to the fraction $\dfrac{a_i b_{ij}}{\sum_i a_i b_{ij}}$.

(b) Construct prices $p_j^* = q_j = \dfrac{\sum_{i=1}^{n} a_i b_{ij}}{w_j}$

and set the individual endowments equal to the optimal consumption bundles: $w_{ij} = x_{ij}^*$ for all i, j. Clearly supply = demand for each good because $\sum_{i=1}^{n} x_{ij}^* = w_j$ for all j.

To show that we have WE, it is thus sufficient to check that each consumer is maximizing utility by choosing to buy x_i^*. Set up the Lagrangian for person i:

$$L = \sum_{j=1}^{k} b_{ij} \ln x_{ij} - \lambda_i \left[\sum_{j=1}^{k} p_j^* x_{ij} - \sum_{j=1}^{k} p_j^* w_{ij} \right]$$

$$\frac{\partial L}{\partial x_{ij}} = \frac{b_{ij}}{x_{ij}} - \lambda_i p_j^* = 0 \quad \text{for all } j = 1 \ldots k.$$

But from part (a) we have $\dfrac{a_i b_{ij}}{x_{ij}} = q_j$

and $p_j^* = q_j$ for all j

Thus the FOC holds with $\lambda_i = \frac{1}{a_i}$. Since $u_i(x_i)$ is strictly concave this is sufficient to show that x_i^* is optimal.

6. (a) We want to solve $\max L^{\beta} C^{1-\beta}$
subject to $C = H^{\alpha}$ and $H + L = 1$

$\Rightarrow \max L^{\beta} H^{\alpha(1-\beta)}$ subject to $H + L = 1$

$\Rightarrow \max L^{\beta} (1-L)^{\alpha(1-\beta)}$ with no constraints
(technically there are still non-negativity constraints $L \geq 0$ and $H \geq 0$ but let's ignore those).

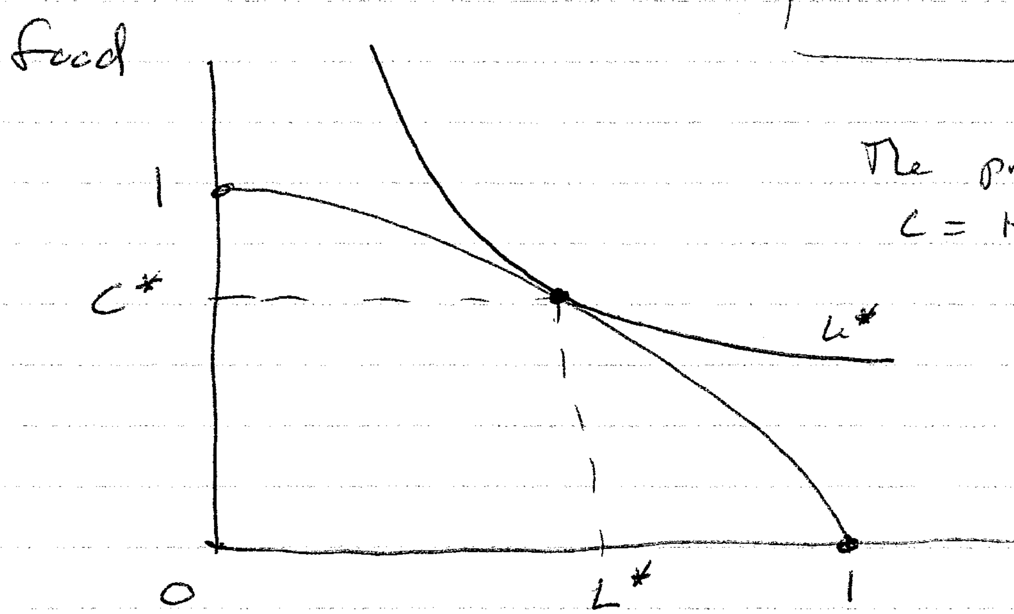
FOC: $\beta L^{\beta-1}(1-L)^{\alpha(1-\beta)} + L^{\beta}\alpha(1-\beta)(1-L)^{\alpha(1-\beta)-1}(-1) = 0$

$$\Rightarrow \frac{\beta}{L} = \frac{\alpha(1-\beta)}{(1-L)} \Rightarrow \beta - \beta L = L\alpha(1-\beta)$$

$$\Rightarrow \beta = L[\beta + \alpha(1-\beta)]$$

$$\boxed{C^* = \left[\frac{\alpha(1-\beta)}{\beta+\alpha(1-\beta)}\right]^{\alpha}} \quad \Leftarrow \quad \boxed{L^* = \frac{\beta}{\beta+\alpha(1-\beta)}}$$

$$\boxed{H^* = \frac{\alpha(1-\beta)}{\beta+\alpha(1-\beta)}}$$



The production function $C = H^{\alpha}$ has DRS since $0 < \alpha < 1$. The utility function is Cobb-Douglas and thus strictly quasi-concave.

Crusoe can set $L = 1 \Rightarrow H = 0 \Rightarrow C = 0$, which gives the horizontal intercept, or he can set $L = 0 \Rightarrow H = 1 \Rightarrow C = 1$ which gives the vertical intercept.

(b) Crusoe's budget constraint will be

$$pC = w(1-L) + \underbrace{\pi(p,w)}_{\text{profit of firm}}$$

$$\Rightarrow \quad pC + wL = \underbrace{w + \pi(p,w)}_{\text{treated as parametric}}$$

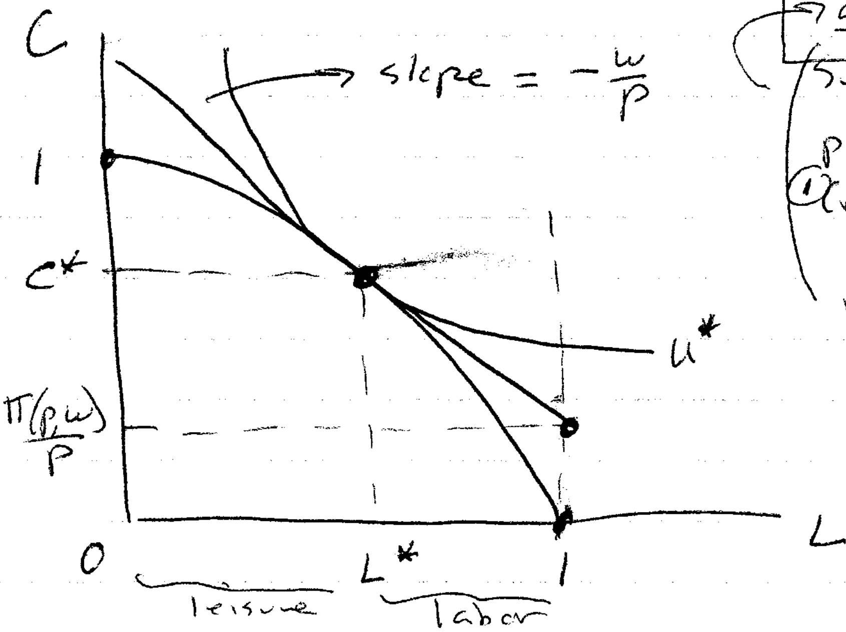
So the slope of his budget line in (L, C) space is $-\frac{w}{p}$. We want this to equal the slope of the indifference curve at (L^*, C^*) → or equivalently the slope of the production frontier at this point.

The slope of the indifference curve is $-\dfrac{\frac{\partial u(L,C)}{\partial L}}{\frac{\partial u(L,C)}{\partial C}}$

$$= -\frac{\beta L^{\beta-1} C^{1-\beta}}{(1-\beta) L^{\beta} C^{-\beta}} = -\left(\frac{\beta}{1-\beta}\right)\frac{C}{L}$$

So we set $\dfrac{w}{p} = \left(\dfrac{\beta}{1-\beta}\right)\dfrac{C^*}{L^*} = \left(\dfrac{\beta}{1-\beta}\right) \dfrac{\left[\dfrac{\alpha(1-\beta)}{\beta+\alpha(1-\beta)}\right]^{\alpha}}{\left[\dfrac{\beta}{\beta+\alpha(1-\beta)}\right]}$ ✱

This gives



✱ (2) and firm is maxing profit (3) and both markets clear.

Subject to the budget constraint $pC + wL = w + \pi(p,w)$, (1) Crusoe is maxing utility at (L^*, C^*). The firm must have positive profit because if $L = 1$, the budget constraint gives $pC = \pi(p,w)$ or $C = \frac{\pi(p,w)}{p} > 0$ as shown.

This occurs because $C = H^{\alpha}$ has DRS ⇒ the firm can get positive profit (you can check this directly by solving the profit max problem).