Mathematical Preliminaries

Ling 324

Reading: *Basic Concepts of Set Theory*

Outline

- Set theory
- Ordered pairs and cartesian products
- Relations
- Functions

What is a Set?

- A SET is a collection of objects. It can be finite or infinite.

  $A = \{a, b, c\}$
  $N = \{1, 2, 3, ...\}$

- An object is an ELEMENT of a set A if that object is a member of the collection A.

  Notation: "$\in$" reads as "is an element of", "is a member of", or "belongs to".

  $a \in A$
  $2 \in N$

  A set can have another set as a member.

  Let $B = \{a, b, c, \{d, e\}\}$, then $\{d, e\} \in B$

- A set with only one member is called a SINGLETON.

- A set with no members is called the EMPTY SET or NULL SET.

  Notation: $\emptyset$ or $\{\ \}$

Specification of Sets

- List notation

  A set consists of the objects named, not of the names themselves.

  $B = \{$The Amazon River, George Washington, 3$\}$
  $C = \{$The Amazon River, ‘George Washington’, 3$\}$

  A set is unordered.

  Writing the name of a member more than once does not change its membership status. For a given object, either it is a member of a given set or it is not.

  $\{a, b, c, e, e, e\}$
  $\{a, b, c, e\}$

Specification of Sets (cont.)

- Predicate notation

  A better way to describe an infinite set is to indicate a property the members of the set share.

  $\{x \mid x$ is an even number greater than 3$\}$ is read as
  "the set of all x such that x is an even number greater than 3."

  'x' is a variable.

- Recursive rules

  A rule for generating elements 'recursively' from a finite basis.

  a) 4 $\in E$
  b) if $x \in E$, then $x + 2 \in E$
  c) Nothing else belongs to E.

  QUESTION: Give a list notation for the above recursive rules.

Set-theoretic Identity and Cardinality

- Two sets are IDENTICAL if and only if they have exactly the same members.

  $\{1, 2, 3, 4\} = \{x \mid x$ is a positive integer less than 5$\}$

  $\{x \mid x$ is a member of the Japanese male olympic gymnastics team in 2004$\}$
  = $\{x \mid x$ is a member of the team that won the gold medal in the 2004 olympics for the male all-around gymnastics competition$\}$

- The number of members in a set A is called the CARDINALITY of A.

  Notation: $|A|$

  Let $A = \{1, 3, 5, a, b\}$.

  $|A|$ =

Subsets

- A set A is a SUBSET of a set B if all the elements of A are also in B.

  Notation: $A \subseteq B$

- PROPER SUBSET

  Notation: $A \subset B$

- $A \not\subseteq B$ means that A is not a subset of B.

- A set A is a subset of itself. $A \subseteq A$.

- If $A \subseteq B$, and $B \subseteq C$, then $A \subseteq C$.

Subsets (cont.)

QUESTION: Fill in the blank with either $\subseteq$ or $\not\subseteq$.

a) $\{a,b,c\}$ $\{s,b,a,e,g,i,c\}$
b) $\{a,b,j\}$ $\{s,b,a,e,g,i,c\}$
c) $\emptyset$ $\{a\}$
d) $\{a,\{a\}\}$ $\{a,b,\{a\}\}$
e) $\{\{a\}\}$ $\{a\}$
f) $\{a\}$ $\{\{a\}\}$
g) $\{\emptyset\}$ $\{a\}$
h) A A

QUESTION: Are the following statements true or false?

Let $A = \{b,\{c\}\}$.

a) $c \in A$
b) $\{c\} \in A$
c) $\{b\} \subseteq A$
d) $\{c\} \subseteq A$
e) $\{\{c\}\} \subseteq A$
f) $\{b\} \notin A$
g) $\{b,\{c\}\} \subset A$
h) $\{\{b,\{c\}\}\} \subseteq A$

Power Sets

- The POWER SET of A, $\wp(A)$, is the set whose members are all the subsets of A.

  The set A itself and the nulll set are always members of $\wp(A)$.

  Let $A = \{a, b\}$.
  $\wp(A) = \{\{a\}, \{b\}, \{a, b\}, \emptyset\}$

- $|\wp(A)| = 2^n$

- From the definition of power set, it follows that $A \subseteq B$ iff $A \in \wp(B)$.

QUESTION: Let $B = \{a, b, c\}$. What is $\wp(B)$?

Set-theoretic Operations

- $A \cap B$: The INTERSECTION of two sets A and B is the set containing all and only the objects that are elements of both A and B.

  $A \cap B \cap C = \bigcap\{A, B, C\}$

- $A \cup B$: The UNION of two sets A and B is the set containing all and only the objects that are elements of A, of B, or of both A and B.

  $A \cup B \cup C = \bigcup\{A, B, C\}$

- $A - B$: The DIFFERENCE between two sets A and B subtracts from A all objects which are in B.

- A': The COMPLEMENT of a set A is the set of all the individuals in the universe of discourse except for the elements of A (i.e., $U - A$).

Set-theoretic Operations (cont.)

QUESTION: Let $K = \{a, b\}$, $L = \{c, d\}$, and $M = \{b, d\}$.

a) $K \cup L = \{a, b, c, d\}$
b) $K \cup M =$
c) $(K \cup L) \cup M =$
d) $L \cup \emptyset =$
e) $K \cap L = \emptyset$
f) $L \cap M =$
g) $K \cap K =$
h) $K \cap \emptyset =$
i) $K \cap (L \cap M) =$
j) $K \cap (L \cup M) =$

k) $K - M = \{a\}$
m) $L - M =$
n) $M - L =$
o) $K - \emptyset =$
p) $\emptyset - K =$

Set-theoretic Equalities

- Some fundamental set-theoretic equalities

1. Commutative Laws

   $X \cup Y = Y \cup X$ $\qquad$ $X \cap Y = Y \cap X$

2. Associative Laws

   $(X \cup Y) \cup Z = X \cup (Y \cup Z)$ $\qquad$ $(X \cap Y) \cap Z = X \cap (Y \cap Z)$

3. Distributive Laws

   $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$
   $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

4. Identity Laws

   $X \cup \emptyset = X$ $\qquad$ $X \cup U = U$ $\qquad$ $X \cap \emptyset = \emptyset$ $\qquad$ $X \cap U = X$

5. Complement Laws

   $X \cup X' = U$ $\qquad$ $(X')' = X$ $\qquad$ $X \cap X' = \emptyset$ $\qquad$ $X - Y = X \cap Y'$

6. DeMorgan's Laws

   $(X \cup Y)' = X' \cap Y'$ $\qquad$ $(X \cap Y)' = X' \cup Y'$

Set-theoretic Equalities (cont.)

- Set-theoretic equalities can be used to simplify a complex set-theoretic expression, or to prove the truth of other statements about sets.

  Simplify the expression $(A \cup B) \cup (B \cap C)'$.

1.	$(A \cup B) \cup (B \cap C)'$	
2.	$(A \cup B) \cup (B' \cup C')$	DeMorgan
3.	$A \cup (B \cup (B' \cup C'))$	Associative
4.	$A \cup ((B \cup B') \cup C')$	Associative
5.	$A \cup (U \cup C')$	Complement
6.	$A \cup (C' \cup U)$	Commutative
7.	$A \cup U$	Identity
8.	U	Identity

QUESTION: Show that $(A \cap B) \cap (A \cap C)' = A \cap (B - C)$.

Ordered Pairs and Cartesian Products

- A SEQUENCE of objects is a list of these objects in some order. (cf., Recall that a set is unordered.)

  $< a, b, c >$, $< 7, 21, 57 >$, $< 1, 2, 3, ... >$

- Finite sequences are called TUPLES. A sequence with k elements is a K-TUPLE.

  A 2-tuple is also called an (ordered) PAIR.

  $< a, b >$

- If A and B are two sets, the CARTESIAN PRODUCT of A and B, written as $A \times B$, is the set containing all pairs wherein the first element is a member of A and the second element is a member of B.

- Although each member of a Cartesian product is an ordered pair, the Cartesian product itself is an unordered set of them.

Ordered Pairs and Cartesian Products (cont.)

QUESTION: Let $K = \{a, b, c\}$ and $L = \{1, 2\}$.

$K \times L = \{< a, 1 >, < a, 2 >, < b, 1 >, < b, 2 >, < c, 1 >, < c, 2 >\}$

$L \times K$ =

$L \times L$ =

QUESTION: Let $A \times B = \{< a, 1 >, < a, 2 >, < a, 3 >\}$. What is $\wp(A \times B)$?

Relations

- A RELATION is a set of ordered n-tuples.

  Binary relation: e.g., mother of, kiss, subset

  Ternary relation: e.g., give

  Unary relation: a set of individuals. e.g., being a ling324 student, being a Canadian

- A relation from A to B is a subset of the Cartesian product $A \times B$.

  Rab, or aRb: The relation R holds between objects a and b.

  $R \subseteq A \times B$: A relation between objects from two sets A and B. A relation *from A to B*.

  $R \subseteq A \times A$: A relation holding of objects from a single set A is called a relation *in* A.

Relations (cont.)

- Domain(R) = $\{a \mid$ there is some b such that $< a, b > \in R\}$
  Range(R) = $\{b \mid$ there is some a such that $< a, b > \in R\}$

  Let $A = \{a, b\}$ and $B = \{c, d, e\}$. $R = \{< a, d >, < a, e >, < b, e >\}$.
  Domain(R) = $\{a, b\}$; Range(R) = $\{d, e\}$

  Note: A relation may relate one object in its domain to more than one object in its range.

- The COMPLEMENT of a relation $R \subseteq A \times B$, written R', contains all ordered pairs of the Cartesian product which are not members of the relation R.

  The INVERSE of a relation, written as R^{-1}, has as its members all the ordered pairs in R, with their first and second elements reversed.

  $(R')' = R$; $(R^{-1})^{-1} = R$

  If $R \subseteq A \times B$, then $R^{-1} \subseteq B \times A$, but $R' \subseteq A \times B$.

QUESTION: Let $A = \{1, 2, 3\}$ and $R \subseteq A \times A$ be
$\{< 3, 2 >, < 3, 1 >, < 2, 1 >\}$, which is ‘greater than’ relation in A.
What is R'? What is R^{-1}?

Types of Relations

- Reflexive: for all a in the domain, $< a, a > \in R$. e.g., being the same age as
  Nonreflexive: e.g., like
  Irreflexive: for all a in the domain, $< a, a > \notin R$. e.g., proper subset

- Symmetric: whenever $< a, b > \in R$, $< b, a > \in R$. e.g., being five miles from
  Nonsymmetric: e.g., being the sister of
  Asymmetric: it is never the case that $< a, b > \in R$ and $< b, a > \in R$. e.g., being the mother of

- Transitive: whenever $< a, b > \in R$ and $< b, c > \in R$, $< a, c > \in R$. e.g., being older than
  Nontransitive: e.g., like
  Intransitive: whenever $< a, b > \in R$ and $< b, c > \in R$, it is not the case that $< a, c > \in R$. e.g., being the mother of

- Equivalence: reflexive, transitive and symmetric. e.g., being the same age as
  An equivalence relation PARTITIONS a set A into EQUIVALENCE CLASSES, which are DISJOINT and whose union is identical with A.

Functions

- A relation R from A to B is a FUNCTION from A to B if and only if:

  a) Each element in the domain is paired with just one element in the range.
  b) The domain of R is equal to A (except for PARTIAL FUNCTIONS).

  QUESTION: Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. Which of the following relations are functions?

  a) $P = \{< a, 1 >, < b, 2 >, < c, 3 >\}$

  b) $Q = \{< a, 1 >, < b, 2 >\}$

  c) $R = \{< a, 2 >, < a, 3 >, < b, 4 >\}$

  d) $S = \{< a, 3 >, < b, 2 >, < c, 2 >\}$

- A function that is a subset of $A \times B$ is a function *from* A to B.
  A function in $A \times A$ is a function *in* A.

- $F : A \rightarrow B$ is read as "F is function from A to B".

- $F(a) = b$ is read as "F maps a to b".

- Given $F(a) = b$, a is an ARGUMENT and b is the VALUE.

Functions (cont.)

- Functions from A to B are generally said to be INTO B.
  Functions from A to B such that the range of the function equals B are called ONTO B.

- A function $F : A \rightarrow B$ is called a ONE-TO-ONE function just in case no member of B is assigned to more than one member of A. Otherwise, we will call them MANY-TO-ONE function.

- A function which is both one-to-one and onto is called a ONE-TO-ONE CORRESPONDENCE.
  If a function F is a one-to-one correspondence, F^{-1} is also a function.

- A function with k arguments is called a K-ARY FUNCTION, and k is called the ARITY of the function.

  Unary function takes one argument. $F(a)$.
  Binary function takes two arguments. $F(a, b)$.

- Infix notation: e.g., $a + b$.
  Prefix notation: e.g., $+(a, b)$.

Functions (cont.)

- A PREDICATE or PROPERTY is a function whose range is $\{$True, False$\}$.

  even_number(2) = True, even_number(3) = False.
  take_ling324, male, freshman.

- The specification of a function from a domain D to $\{$True, False$\}$ defines a unique subset of domain D, and the specification of a set defines a unique function.

  For instance, let the domain D be a set of numbers. If we collect all the elements in D that are mapped to True by function $even\_number$, we end up with a set of even numbers, which is a subset of D.

  Also, if a number is an element of the set of even numbers, then it is mapped to True by function $even\_number$, and if it is not an element of the set of even numbers, then it is mapped to False by function $even\_number$.

- We call the unique function that is associated with set A, the CHARACTERISTIC FUNCTION of A.