Because most natural rivers are capable of deforming their channels by eroding and depositing sediment, at first it might seem a bit odd that we should concern ourselves with the behaviour of water flowing through channels with rigid boundaries. But there are two good reasons for taking this approach.

First, some channels do have rigid boundaries and they deserve our attention. In many cases the first few orders of channels in a river network may be flowing on bedrock. No matter how large the discharge carried through such bedrock channels, the boundary deforms so slowly by erosion that, for all practical short-term purposes, it can be regarded as rigid or non-alluvial in character.

The second and more important reason is that rivers are such exceedingly complex physical systems that we cannot hope to understand them without first simplifying reality in order to grasp the character of the general forces at work in channels. When we understand the workings of the simple case, we can then consider increasingly more complex ones which more closely match the behaviour of real rivers. The most important complexity in this regard is the ability of alluvial rivers to mould their channels to accommodate the forces in flowing water, a matter we
will take up again in Chapter 4. Meanwhile we need to back up and have a close look at how flowing water responds to the forces acting on it.

Some of the models of flow developed in this chapter do not have much real-world application because they are just too much of a simplification of reality. Nevertheless, they do serve the useful purpose of revealing the quality of important forces at work in the flow even if they do not allow us to quantify them precisely. Other quite simple models turn out to be remarkably good at predicting real river behaviour. Part of our job here is to learn to tell the difference between them.

Much of the discussion to follow assumes that you are familiar with the nature of basic dimensions used in mechanics and with the requirement of dimensional homogeneity in equations describing physical systems. If these notions are not familiar you may find it very useful before proceeding to review these matters in the previous chapter.

**The nature of river water**

The *water* in rivers is actually a complex mixture of water, dissolved matter of both organic and inorganic origin, and suspended particles ranging in size from clay to sands and in some cases even gravel. Although this fluid mixture varies from one river to another, the properties of pure water so dominate its character that, for purposes of our model building, we can regard rivers simply as moving bodies of pure water. We will need to relax this assumption in certain circumstances, but it holds reasonably well in general.

Because water is so abundant and its simple two-element formula so familiar, there is perhaps a tendency to think of it as a very simple compound. It turns out, however, that water is rather complicated stuff. It consists of the molecule H\(_2\)O in which two small hydrogen atoms are covalently bonded to the same hemisphere of the relatively large oxygen atom. Although electrically balanced overall, the asymmetry of this covalent bond means that there is a relative abundance of electrons and an excess negative local charge on one end of the H\(_2\)O structure counterbalanced by an excess local positive charge at the other. In consequence the water molecule behaves rather like a weak dipole magnet. Some but not all of these molecules are in
turn joined to form tetrahedral clusters in which positively charged hydrogen ions are ionically bonded to negatively charged oxygen ions. These clusters of molecules are separated one from the other by unbonded water molecules that move freely and serve to lubricate the bonded substructures, allowing flowage to occur. Furthermore, the whole structure of bonded and unbonded molecules is remarkably dynamic with molecules exchanging rapidly between clusters and flow layers such that a given intermolecular hydrogen bond breaks and reforms some $10^{12}$ times each second!

It is the strength of the bonding of hydrogen to oxygen ions in particular in water - a sort of molecular glue - that is expressed in the physical property of viscosity. Viscosity is a measure of the ease with which a fluid will deform when subjected to some stress. In order for water to 'flow', the electrical force bonding the water molecules one to the other must be overcome. The rate at which water flows (deforms or changes shape) reflects the balance between the stress acting on it (such as pipe pressure or gravity) and the internal molecular forces resisting the deformation. Obviously these molecular or viscous forces must be very great in fluids like oil or molasses which flow only sluggishly, and much smaller in others such as water and alcohol which under the same stress flow quite readily.

Some physical properties of pure water, including viscosity, are listed in Figure 2.1. Clearly, these common properties of water (and of most other liquids) are quite temperature dependent.

<table>
<thead>
<tr>
<th>Temperature (°C)</th>
<th>Specific weight (γ) Nm$^{-3}$</th>
<th>Density (ρ) kgm$^{-3}$</th>
<th>Dynamic viscosity (μ × 10$^{-3}$) Nsm$^{-2}$</th>
<th>Kinematic viscosity (ν × 10$^{-6}$) m$^2$s$^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9 805</td>
<td>999.8</td>
<td>1.781</td>
<td>1.785</td>
</tr>
<tr>
<td>5</td>
<td>9 807</td>
<td>1000.0</td>
<td>1.518</td>
<td>1.519</td>
</tr>
<tr>
<td>10</td>
<td>9 804</td>
<td>999.7</td>
<td>1.307</td>
<td>1.306</td>
</tr>
<tr>
<td>15</td>
<td>9 798</td>
<td>999.1</td>
<td>1.139</td>
<td>1.139</td>
</tr>
<tr>
<td>20</td>
<td>9 789</td>
<td>998.2</td>
<td>1.002</td>
<td>1.003</td>
</tr>
<tr>
<td>25</td>
<td>9 777</td>
<td>997.0</td>
<td>0.890</td>
<td>0.893</td>
</tr>
<tr>
<td>30</td>
<td>9 764</td>
<td>995.7</td>
<td>0.798</td>
<td>0.800</td>
</tr>
<tr>
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<td>988.0</td>
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<tr>
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<tr>
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<td>0.326</td>
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<tr>
<td>100</td>
<td>9 399</td>
<td>958.4</td>
<td>0.282</td>
<td>0.294</td>
</tr>
</tbody>
</table>

2.1: Some physical properties of water (SI units)

2.3
At about 4°C thermal agitation of the water molecules is minimal and the number of hydrogen bonds, and thus molecule cluster size, are at a maximum, corresponding to peak density for the liquid phase. As water temperature increases, more of the hydrogen bonds are broken so that molecular cluster size declines along with density and specific weight. Internal strength (internal resistance to deformation, or viscosity) also declines; for example, Figure 2.1 indicates that, at 20°C, the absolute or dynamic viscosity of water is little more than half that at 0°C, and about twice that at 50°C.

Two different coefficients of viscosity are listed in Figure 2.1. The first is the absolute or dynamic viscosity ($\mu$) and has the dimensions Nsm$^{-2}$ while the second is the kinematic viscosity ($\nu = \mu/\rho$) with the dimensions m$^2$s$^{-1}$. Unless otherwise specified, discussions involving the viscosity of water usually refer to the absolute or dynamic viscosity.

The temperature of water is quite variable among rivers (near freezing in high alpine or arctic streams and tepid in the tropics) and may vary significantly on a seasonal basis in a given river. It follows that the related physical properties of water listed in Figure 2.1 will vary in nature as well.

**The energy approach to fluid mechanics**

The physics of flow can be studied in various ways. Hydrodynamics involves the study of the behaviour of ideal fluids deduced mathematically from various assumptions about the physics involved. As such it is the theoretical branch of the science concerned with fluid flow. Engineers faced with solving real rather than theoretical problems in relation to flow have adapted certain ideas in hydrodynamics but more often have replaced them altogether with alternative engineering solutions because the theory of flow is incomplete and therefore not very practical in application. These practical engineering solutions form the applied field of hydraulics. Most river science is concerned with both theory and practice and useful notions of both sorts are combined in the study of the fluid mechanics of open channel flow. We are careful to specify *open-channel* flow because much of fluid mechanics was developed to
understand the behaviour of flow in *closed conduits* (pipes). Indeed, many of our models of open-channel flow have been directly adapted from earlier thinking about the behaviour of flow in pipes.

**Some Definitions**

It is sometimes useful to visualize the pattern of flow direction in moving water in terms of *streamlines*. Streamlines are imaginary lines across which there is no flow component, so that the velocity vector at any instant is tangential to every point on it (Figure 2.2a). A *streamtube* or *flow filament* may be thought of as an imaginary tube in which the walls are made up of contiguous streamline bundles across which there can be no flow (Figure 2.2b). A common application of this concept is to imagine that the entire flow is a streamtube so that the conduit boundary and the streamtube walls coincide.

Flow through a streamtube may be *steady* or *unsteady*. Unsteady flow varies with time so that an observer standing on the bank of a river, for example, would see temporal fluctuations in the velocity and the depth of flow. Such fluctuations imply that both positive and negative acceleration characterize such flow. On the other hand, if the flow were steady in the river, our observer would note only a constant velocity and constant depth over time. Only in a steady flow does a streamline coincide exactly with the actual *pathline* followed by an individual water particle; in unsteady flow streamlines and pathlines are distinctly different. We should note that

![Diagram of streamline and streamtube](image-url)
it is quite possible for certain flow phenomena to be regarded as either steady or unsteady, depending on the observer's point of view. For example, a wave moving through the water surface will pass our riverbank observer as unsteady flow while another observer moving downstream with the same wave in a canoe would see the wave as part of a steady flow. Indeed, changing our frame of reference can greatly simplify (or complicate!) the solutions to certain flow problems.

Flow through a circular streamtube of constant diameter (Figure 2.2) is said to be spatially constant or uniform. Thus uniform flow may be steady or unsteady but it must remain constant with respect to distance along the streamtube. Uniform flow through a river channel means that velocity and depth do not vary with distance down the channel and that all streamlines therefore are parallel to both the bed of the channel and to the free water-surface (Figure 2.3a). In such uniform flows the pressure everywhere in the flow simply is hydrostatic and thus depends only on the flow depth beneath the free surface. Conversely, in flows where streamlines converge or diverge the fluid is respectively accelerating and decelerating and the flow is nonuniform. In cases of curvilinear flow, additional centrifugal forces act on the flow and pressures are no longer simply hydrostatic. In the vertical longitudinal section (Figure 2.3b) concave streamlines imply a downward acting centrifugal force which augments gravity and gives rise to pressures in excess of hydrostatic while the converse relationship holds for convex streamlines.

Even though flow may not be strictly steady and uniform, practically it can be conveniently considered to be so over short distances if changes in direction and spacing of the streamlines are gradual enough. In such cases we speak of gradually varied flow, in contrast with rapidly varied flow in which the assumption of uniformity is no longer tenable even as an approximation.
2.3: Longitudinal streamline profiles illustrating uniform and nonuniform flow in an open channel.

A fundamental concept in steady uniform flow mechanics is the continuity principle. It states that the discharge (Q) must remain constant along a streamtube and at all points is defined as

\[ Q = A \bar{V} \] ...........................\( (2.1) \)

where A is cross-sectional area of the flow and \( \bar{V} \) is mean flow velocity. At successive cross sections 1, 2, 3, 4,.....n, the continuity principle requires that (Figure 2.2):

\[ Q = A_1 \bar{V}_1 = A_2 \bar{V}_2 = A_3 \bar{V}_3 = A_4 \bar{V}_4 = \ldots \ldots A_n \bar{V}_n \] ...........................\( (2.2) \)

We will often find it useful to consider the particular case of a rectangular channel, for which the appropriate form of equation (2.1) is

\[ Q = WD \bar{V} \] ...........................\( (2.3) \)

where D is the depth of flow.

At other times we will find it useful to look, not at the three-dimensional flow picture described by equations (2.1) - (2.3), but rather at longitudinal slices through the flow. Such two-dimensional flow has no width and the two-dimensional discharge \( q = Q/W \) in the case of our rectangular channel is obtained by dividing both sides of equation (2.3) by \( W \):

2.7
Thus we can think of \( q \) as the discharge per unit width of a rectangular channel.

Water moves through a channel as a result of net applied forces of one sort or another. As we noted in Chapter 1, if we are to concern ourselves with forces we must turn to the study of dynamics and therefore to Newton's laws of motion. One of the most important in this regard, is Newton's second law which states that:

\[
F = ma \tag{1.7}
\]

or that the force \( F \) necessary to accelerate a mass \( m \) at a certain rate \( a \) is equal to the product \( ma \).

When a force acts over some distance \( s \) we say that work \( (w = Fs) \) has been done. The relationship between work and energy becomes apparent if both sides of equation (1.7) are multiplied by a length \( s \) in the direction of the force:

\[
Fs = w = mas \tag{2.5}
\]

It follows from the principles of uniformly accelerated motion discussed in Chapter 1 that, since \( v_2^2 = v_1^2 + 2as \), by rearrangement,

\[
a = \frac{v_2^2 - v_1^2}{2s} \tag{2.6}
\]

Substituting this expression for \( a \) in equation (2.5) yields

\[
Fs = w = m \left( \frac{v_2^2 - v_1^2}{2s} \right) s = \frac{1}{2} m(v_2^2 - v_1^2) \tag{2.7a}
\]

We have in effect integrated both sides of equation (2.5) with respect to length \( s \) so that in general,

\[
q = d\bar{v} \tag{2.4}
\]
This is the energy equation which states that the work done on a body as it moves from \( s_1 \) to \( s_2 \) is equal to the kinetic energy \( (KE = \frac{1}{2} mv^2) \) acquired by the body. This is a quite fundamental equation in dynamics and has many applications to problems in fluid mechanics as we shall soon see.

The force \( F \) in equation (2.7) and its derivatives is a resultant or net force. That is, it is the difference between the total impressed force and any force such as friction or viscous drag. For the moment, however, you are asked to set aside the reality that water exhibits various properties including internal resistance to motion, and assume that it is an ideal fluid, incompressible and inviscid (without viscosity). Such imaginary Newtonian fluids, as they also are known, move in such a way that their energy is completely conserved (without loss in time or space). Although these assumptions may seem to be outrageous excursions from the real world you may be surprised at just how much we can learn about river flows by exploring them.

Because equations (2.6) and (2.7) were developed, and will be familiar to you, in terms of solid-body motion it is easier to deal with their application to fluids if we conceptualize them as applying to a fluid element such as that shown in Figure 2.4.

This element is \( \Delta n \) high, \( \Delta s \) long, and has an implied unit width (normal to the page); its motion is referred to a streamline along which it moves distance \( s \). There are only two forces acting on such an element: that resulting from the pressure gradient in the direction of motion, \(- (\partial p / \partial s) \Delta s \Delta n\), and that due to its weight resolved in the direction of motion, \((\gamma \Delta s \Delta n) \sin \theta\). Alternatively, this weight component of force may be expressed with respect to the vertical height \( z \) above the datum, or \(- (\gamma \Delta s \Delta n) \partial z / \partial s\).
Thus equation (2.7) in the case of this element can be written:

\[-(\partial p/\partial s)\Delta s\Delta n - (\gamma \Delta s\Delta n) \partial z/\partial s = ma \] \hspace{1cm} (2.8)

The partial derivatives are necessary here because the two component forces may be changing with time as well as with distance s.

If \( \rho \) is fluid density so that \( m = \rho \Delta s\Delta n \), and \( a_s \) is its acceleration in the s direction, equation (2.8) becomes

\[-(\partial p/\partial s)\Delta s\Delta n - (\gamma \Delta s\Delta n) \partial z/\partial s = \rho \Delta s\Delta na_s \] \hspace{1cm} (2.9)

which simplifies to

\[ \partial/\partial s (p+\gamma z) + \rho a_s = 0 \] \hspace{1cm} (2.10)

Equation (2.10) is the Euler equation, named for Leonhard Euler, a famous 18th century Swiss mathematician. Although the Euler equation is not as useful in direct application as its integrated form, the Bernoulli equation, it is introduced here because it will serve to clearly remind us of the physics involved in the flow. Two important properties of this relationship need to be appreciated.

First, the term \( (p + \gamma z) \) in equation (2.10), known as the piezometric pressure, is constant throughout a body of still water \( (a_s = 0) \) so that \( \partial/\partial s (p+\gamma z) = 0 \), regardless of the direction s. In other words, in the case of still water where there is a free water surface a distance \( y \) above the datum for \( z \) (i.e., the water depth), pressure \( p \) at any height \( z \) is hydrostatic so that

\[ p = \gamma (y - z) \] \hspace{1cm} (2.11)

and therefore,

\[ (p + z\gamma) = \text{a constant} = \gamma y \] \hspace{1cm} (2.12)

Should the fluid element begin to move, however, the acceleration term \( \rho a_s \) becomes non-zero so that the piezometric pressure clearly no longer will remain constant throughout the fluid.
Second, the acceleration term \( a_s \) in equation (2.10) is a result of velocity variation in both space and time (non-uniform and unsteady flow). Thus we can say that

\[
\frac{dv}{dt} = a_s = \frac{ds}{dt} \frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v \frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} \tag{2.13}
\]

That is, to an observer moving in the direction of flow (s) at velocity \( \frac{ds}{dt} = v \) with the water element shown in Figure 2.4, the rate of change in the resultant velocity with respect to time (i.e., acceleration, \( a_s \)) is the sum of the velocity change attributable to the change in position (i.e., distance \( s \)), \( \frac{\partial v}{\partial s} \), and that change in \( v \) specifically attributable to the change in time, \( \frac{\partial v}{\partial t} \). The velocity change related to distance is termed the \textit{advective acceleration} while that related to time is termed the \textit{local acceleration}.

In steady flows, by definition there is no local acceleration and for this special case, \( a_s = v \frac{\partial v}{\partial s} \), which upon substitution in equation (2.10) yields:

\[
\frac{\partial}{\partial s}(p + \gamma z) + \rho v \frac{\partial v}{\partial s} = 0 \tag{2.14}
\]

Equation (2.14) can be integrated directly with respect to distance \( s \) to yield the Bernoulli equation (named for Daniel Bernoulli, the Swiss physicist and mathematician who developed the kinetic theory of gases in the 18th century):

\[
p + \gamma z + \frac{1}{2} \rho v^2 = \text{constant} \tag{2.15a}
\]

or

\[
p/\gamma + z + v^2/2g = \text{constant } H \tag{2.15b}
\]

Since equation (2.15) was obtained by integrating a force equation with respect to distance, it is an energy equation and the constant \( H \) is termed the \textit{total energy} or \textit{total head}. The derivation employed here is quite general but it usefully highlights the dependence of the Bernoulli equation on steady flow conditions; otherwise, of course, there would be an additional term from the integration of the expression for the local acceleration, a mathematical operation involving considerable difficulties and one best avoided if possible.
Because equation (2.15) is an energy equation, it could have been derived directly from equation (2.7). Indeed, this derivation is a useful exercise for us to undertake because it helps to clarify the meaning of the terms in the Bernoulli equation and their relation to certain measurable quantities routinely used to describe the flow and the geometry of the conduit. It also will be instructive if we initially consider this direct derivation in terms of the pipe flows for which the ideas were first developed. Once we have accomplished this task we can set about adapting the Bernoulli concepts to the free-surface flows we encounter in rivers.

Figure 2.5 shows a run of circular piping (although any cross-sectional shape could be employed) in which water flows from a large-diameter section to a smaller diameter section. Our analysis will focus on a control volume of the fluid as it moves through the transition. We assume that no energy is lost through this pipe transition (it is an ideal fluid moving through a frictionless pipe transition) and that the sum of potential and kinetic energy in section A (PE_A + KE_A) plus the additional kinetic energy acquired as a result of the work (W) done in moving the water through the pipe transition, must equal the sum of kinetic and potential energy in section B (PE_B + KE_B).

Thus we can write an equation to describe the circumstances in Figure 2.5 as follows:

\[ PE_A + KE_A + W = PE_B + KE_B \] ...................................... (2.16)

or more specifically,

\[ mgz_1 + \frac{1}{2} mv_1^2 + W = mgz_2 + \frac{1}{2} mv_2^2 \] ...................................... (2.17)

Work (W = Fs) here is the net work done by the pressure forces (the opposing pressure acting over the respective cross-sectional areas) and the distances moved by the control volume are respectively L_1 and L_2. So the net work involved is

2.12
Chapter 2: The energy equation for open-channel flow

\[ W = F_1L_1 - F_2L_2 = p_1A_1L_1 - p_2A_2L_2 \]

Thus we can substitute this expression for \( W \) in equation (2.17) yielding:

\[ mgz_1 + \frac{1}{2} \rho v_1^2 + [p_1A_1L_1 - p_2A_2L_2] = mgz_2 + \frac{1}{2} \rho v_2^2 \]……………….. (2.18)

Dividing equation (2.18) throughout by the fluid volume (\( V_o = A_1L_1 = A_2L_2 \)), and noting that \( m/V_o = r \), results in:

\[ \rho gz_1 + \frac{1}{2} \rho v_1^2 + [p_1 - p_2] = \rho gz_2 + \frac{1}{2} \rho v_2^2 \]…………………….. (2.19)

Noting further that \( \rho g = \gamma \), we can make this substitution and regroup the terms in equation (2.19) to yield:

\[ \frac{p_1}{\gamma} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{v_2^2}{2g} + z_2 \]…………………….. (2.20)

which is simply a particular version of the general form of the Bernoulli equation:

\[ \frac{p}{\gamma} + \frac{v^2}{2g} + z = H \]…………………….. (2.15b)

The equation (2.15b) components each have the dimension of length and these lengths may be related to the observed physical properties of pipe flow as shown in Figure 2.6. If a small open tube is connected to such a pipe the static pressure will force the fluid up the tube until the height of the free water-surface reaches \( p/\gamma \) m above the centre of the pipe. Early experiments by the French engineer H. Pitot in 1732 showed that the sum of the velocity head and the pressure head could be measured by placing a second small open tube (now known as a pitot tube) in the flow with its open end facing into the flow. The difference between the height of the water column in the static tube and that in the pitot tube is the velocity head. The velocity head commonly is measured in this way in laboratory experiments so that the pressure difference becomes the basis for computing flow velocity.
Since energy in this system is conserved, the locus of the free water-surface elevations in a series of pitot tubes along the pipe describes the horizontal (invariant) energy line. The locus of the free water-surface in a corresponding series of static tubes describes the hydraulic grade line which will rise and fall in response to velocity changes along the pipe. In the limit, as velocity goes to zero, the elevation of the free water-surface in the static and pitot tubes converges.

The application of these ideas to open channel flow is direct and straightforward, although not made without some important assumptions; the application of equation (2.20) to the longitudinal section of a rectangular open channel is illustrated in Figure 2.7. The velocity head, $v^2/2g$, has direct correspondence to the pipe flow component provided we assume that the velocity is constant over the entire cross-section, an assumption never strictly met in real channels but closely enough approximated that we do not introduce severe errors by making it. Provided the downstream bed and water-surface slopes are not too great (say, less than $5^\circ$), the pressure head, $p/\gamma$, at any point in the flow can be taken as the hydrostatic head and therefore is simply equal to the depth below the water surface ($y - z$); see equation (2.11). Thus the sum $(p/\gamma + z)$ for any point in the flow must represent the height of the water surface above datum and if $z$ is taken as the bed elevation, $p/\gamma$ must equal the flow depth, $y$.

2.6: Definition diagram for the Bernoulli energy equation terms as they apply to flow through a pipe.
We can now rewrite equation (2.15) in these terms:

$$y + \frac{v^2}{2g} + z = H \quad \text{........................................... (2.21)}$$

The requirement of a low water-surface slope is necessary because the pressure head is strictly the \textit{vertical} distance below the water surface, \(h\), and not \(y\) (the flow depth measured normal to the water surface). The approximation is close for small water-surface inclines (see Figure 2.7), however, and \(y/h = \cos \theta\) (for \(\theta = 5^\circ\), \(y/h = 0.996\) and for \(\theta = 10^\circ\), \(y/h = 0.985\)).

The Euler equation \(\left(\frac{\partial}{\partial s} (p + \gamma z) + \rho a_s = 0\right)\) serves as an important reminder that the assumption of hydrostatic pressure distribution can only be true in the vertical direction \(s\) if the acceleration term, \(a_s\), is zero. In other words, there must be no vertical acceleration associated with strongly curved streamlines such as those depicted in Figure 2.3.

Fortunately, the conditions of relatively low water-surface slope and hydrostatic pressure distribution are very often closely approximated in rivers so that equation (2.21) enjoys quite wide application. Nevertheless, we must remember these limitations and carefully assess each application to assure ourselves that the assumptions underlying equation (2.21) are sensibly met.

\[\text{2.7: A longitudinal profile of a two-dimensional open-channel flow illustrating the application of the terms of the Bernoulli equation.}\]
Another assumption that deserves our careful assessment here is the fundamental notion of energy conservation. You will note that Figure 2.7 now includes a head-loss term, $\Delta H$, but that it is ignored in equation (2.21). Head loss is that component of the energy which is consumed overcoming the frictional resistance to flow. Thus, it represents a 'leakage' of energy from our assumed energy-conserving system and obviously can only be ignored if it is insignificant in relation to the total energy of the flow. It turns out, fortunately, that this is not an unreasonable assumption provided we are dealing with short reaches of channel only. For problems involving long reaches of channel, however, the total energy at the end of the reach must be discounted by the frictional head loss in order for the Bernoulli equation to apply without error. Later we will return to examine much more closely this whole business of flow resistance.

Equation (2.21) can be used to solve a variety of channel transition problems where we want to know how the flow will change in response to certain downstream changes in the boundary conditions. For example, Sample Problem 2.1 shows a solution to a simple problem where both upstream and downstream flow depths are known.

### Sample Problem 2.1

Problem: A tree falls across a 5 m-wide rectangular stream channel and causes the water to back up on the upstream side to a depth of 2 m while water discharges under the tree trunk in a 0.5 m deep flow and continues downstream. For this short channel transition the channel bed can be regarded as horizontal. What is the discharge of the stream?

Solution: Initially, we treat this as a two-dimensional problem. Since the bed is horizontal ($z_1 = z_2$) from equation (2.21) we get

$$2.0 + \frac{v_1^2}{2g} = 0.5 + \frac{v_2^2}{2g}$$

and from two-dimensional continuity

$$2.0v_1 = 0.5v_2 \quad \text{and therefore} \quad v_2 = 4.0v_1$$

It follows from substitution for $v_2$ that

$$2.0 + \frac{v_1^2}{2g} = 0.5 + \frac{(4.0v_1)^2}{2g} = 0.5 + 16.0\frac{v_1^2}{2g}$$
Chapter 2: The energy equation for open-channel flow

Sample Problem 2.1 (cont)

\[
1.5 = 16 \frac{v_1^2}{2g} - \frac{v_1^2}{2g} (16.0 - 1.0) = 15.0 \frac{v_1^2}{2g}
\]

Noting that \(g = 9.806 \text{ ms}^{-2}\), it follows that \(v_1^2 = \frac{1.5}{15.0} (2g) = 1.961\); \(v_1 = \sqrt{1.961} = 1.4 \text{ ms}^{-1}\)

Therefore, \(q = (1.4)(2.0) = 2.8 \text{ m}^2\text{s}^{-1}\) and \(Q = (2.8)(5.0) = 14.0 \text{ m}^3\text{s}^{-1}\)

A more commonly encountered flow transition problem is outlined in Sample Problem 2.2. Here we are interested in knowing how the velocity and depth of flow will change as the river flows.

**Sample Problem 2.2**

\[
\begin{align*}
\text{Problem:} & \quad \text{Water flows through a 5 m-wide rectangular channel and over a 20 cm step up in the bed. If the discharge is } 15 \text{ m}^3\text{s}^{-1}, \text{ and the initial depth upstream of the step is } 1.5 \text{ m}, \text{ what will be the depth and velocity of the flow downstream of the transition?} \\
\text{Solution:} & \quad \text{Initially, we treat this as a two-dimensional problem. Noting that } q = Q/w = 15/5 = 3 \text{ m}^2\text{s}^{-1} \text{ and } V_1 = q/y_1 = 3/1.5 = 2 \text{ ms}^{-1}, \text{ we set up the Bernoulli equation for this transition as follows:} \\
y_1 + \frac{v_1^2}{2g} + z_1 = y_2 + \frac{v_2^2}{2g} + z_2 \\
1.500 + \frac{4.00}{2g} + 0 = y_2 + \frac{v_2^2}{2g} + 0.200
\end{align*}
\]

You will note that bed elevation upstream of the transition is the datum so that \(z_2 = 0.200 \text{ m} \) (implicit in the 3-digit lengths is a measurement accuracy to the nearest millimetre). For \(g = 9.806 \text{ ms}^{-2}\), this Bernoulli equation simplifies to

\[
1.504 = y_2 + \frac{v_2^2}{19.612}
\]

From continuity, \(y_2v_2 = 3\), so that \(v_2 = 3/y_2\). Substituting for \(v_2\) above and rearranging yields

\[
y_2^3 - 1.504y_2^2 + 0.459 = 0
\]

which by iteration (see Appendix 3.1) has three solutions: \(y_2 = 1.167 \text{ m}, 0.818 \text{ m}, \text{ and -0.481 m}\).

The negative depth clearly has no physical meaning and of the positive solutions, \(y_2 = 1.167 \text{ m}\) is taken as correct because it is the closest to the pre-transition condition (where \(y_1 = 1.5 \text{ m}\)). The solution \(y_2 = 1.167 \text{ m}\) implies that the water surface must drop by 0.133 m over the step \([(1.167 + 0.20) -1.50] = -0.133 \text{ m}]\). From continuity it also follows that \(v_2 = 3/1.167 = 2.571 \text{ ms}^{-1}\).
over an upward step in the bed. Once again we set the Bernoulli terms for each side of the transition equal to each other and use continuity to close the set of equations. In this case, however, we are left with a cubic equation for which there are three solutions. Setting aside the mathematical difficulty of solving a cubic equation for the moment, deciding which of our solutions to Sample Problem 2.2 is correct and physically possible presents a problem. Clearly the negative solution is not a physically real solution but it is not readily apparent, however, which of the two positive solutions is correct. It turns out that $y_2$, closest to the initial $y_1$, is the appropriate solution (i.e., $y_2 = 1.167$ m) but in order to understand why this is so we must approach this particular type of problem from the perspective of the specific energy, an extremely useful concept introduced to fluid mechanics by B.A. Bakhmeteff in 1912.

Specific energy and alternative flow regimes

Specific energy, $E$, is defined as the energy of flow in relation to the bed (rather than to an external datum), and thus is described by the expression:

$$E = y + \frac{v^2}{2g} \hspace{1cm} (2.22)$$

In a sense equation (2.22) recognizes that, for short reaches of channel, changes in specific energy related to the downstream decline in bed elevation or water-surface slope (both measured with respect to an external horizontal datum) are negligible compared to that related to local changes in depth and velocity. Thus we can adopt the bed itself as a datum, greatly simplifying the energy equation and allowing us to explore the relation between the velocity and depth heads.

Since the simplest case allows the clearest development of this concept of specific energy, for now we will consider the two dimensional version of flow in a rectangular channel of fixed width in which $q = Q/w$, the discharge per unit width of channel. Thus equation (2.22) can be rewritten in the form:
Chapter 2: The energy equation for open-channel flow

\[ E = y + \frac{q^2}{2gy^2} \] ................................. (2.23)

or for the case where discharge is constant along the channel,

\[ (E - y)y^2 = \frac{q^2}{2g} = \text{a constant} \] ........................................ (2.24)

The graph of the relationship between E and y described by equation (2.22) as it applies to flow over an upward step in the bed (see Sample Problem 2.2) is shown in Figure 2.8.

In the physically meaningful (positive y) domain the graph of the cubic equation (2.24) is bounded by the 45° angle formed by the asymptotes \((E-y) = 0\) and \(y = 0\) in the first quadrant. Prior to the step, the water possesses specific energy \(E_1\) and flows at a depth \(y_1\), conditions corresponding to point A on the specific energy curve. It is clear from this curve, however, that a second alternative combination of depth and velocity also is possible at \(E_1\), corresponding to point A'. Although the specific head is the same at this alternative condition, depths are much lower and velocity must therefore be higher than at point A (since \(q\) is constant). As the flow moves over the step the head declines by the length of the step height so that \(E_1 - E_2 = \Delta z\). As the flow accelerates over the step, energy is transferred from the depth head to the velocity...
head and we move from points A to B on the specific energy curve in Figure 2.8. Because the total energy is conserved, these changes must be accompanied by a drop in the elevation of the water surface over the step, a perhaps surprising result because most people find it to be rather counter-intuitive. Once again, we find that an alternative depth and velocity occurs, this time at point B' (the second positive and smaller depth solution noted in Sample Problem 2.2).

These circumstances beg an obvious question: why does the flow adjust to condition B rather than condition B'? Both alternative points B and B' represent physically real equilibrium flow conditions. The answer to our question lies in the accessibility of the alternative depths and velocities to the precursor flow. First we should note that the cubic equation graphed as a heavy line in Figure 2.8 represents our specified condition of a constant discharge. Other curves can be drawn for higher (or lower) discharges, as shown by the faint-line curves, but all specific energy changes in our example must follow the heavy-line curve. For example, it is not possible to jump from point A to point B' (nor to A') simply because such a trajectory would require an increase in the two-dimensional discharge. Such a jump could be achieved, however, if the flow also encountered a local channel contraction at the step. A constriction of just the right degree would increase local $q$ and provide direct access to the alternative flow conditions at B'. Nevertheless, if discharge is constant and the channel has a fixed width, all specific energy changes must follow a given specific energy curve.

The second possibility that might at first appear plausible is that point B' is reached by flow adjustments which simply follow the curve around the apex to settle on these alternative conditions. But such an adjustment is not physically possible because it requires that the specific energy drop below $E_2$ and return to it again. Such an effect could be achieved only by a local rise in the upstream bed just high enough above the step to achieve a depth $y_c$ (at point C) but it is not a possibility for a simple upward-step transition of the kind shown in Figure 2.8.

So we may conclude that, if width remains unchanged through a simple upward-step transition, the point B' in Figure 2.8 simply is not accessible from an upstream flow represented by point A. Similar physical reasoning leads us to conclude that B is not accessible from A'. From all of this follows a useful rule: of the two solutions to the energy equation applied to step transitions, the
appropriate depth/velocity condition in the transition will be that nearest the initial conditions upstream.

We might also expect that a 'negative step' or downward step in the bed will yield results consistent with the processes described above. Indeed, Sample Problem 2.3 shows that where approaching flow is represented conceptually by a point on the upper limb of the energy

---

**Sample Problem 2.3**

Water flows through a 10 m-wide rectangular channel and over a 20 cm step down in the bed. If the discharge is 25 m$^3$s$^{-1}$, and the initial depth upstream of the step is 1.0 m, what will be the depth and velocity of the flow downstream of the transition?

**Solution:** Once again, we initially treat this as a two-dimensional problem. Noting that $q = Q/w = 25/10 = 2.5$ m$^2$s$^{-1}$ and $v_1 = q/y_1 = 2.5/1.0 = 2.5$ ms$^{-1}$, we set up the Bernoulli equation for this transition as follows:

$$y_1 + \frac{v_1^2}{2g} + z_1 = y_2 + \frac{v_2^2}{2g} + z_2$$

$$1.000 + \frac{6.25}{2g} + 0 = y_2 + \frac{v_2^2}{2g} - 0.200$$

You will note that bed elevation upstream of the transition is taken as the datum so that $z_2 = -20$ cm = -0.200 m (implicit in the 3-digit lengths is a measurement accuracy to the nearest millimetre). Of course we could just as easily set the lower bed as datum and avoided a negative $z_2$. For $g = 9.806$ ms$^{-2}$, this Bernoulli equation simplifies to

$$1.519 = y_2 + \frac{v_2^2}{19.612}$$

From continuity, $y_2v_2 = 2.5$, so that $v_2 = 2.5/y_2$. Substituting for $v_2$ above and rearranging yields

$$y^3 - 1.519y^2 + 0.319 = 0$$

which by iteration (see Appendix 1) yields the positive solutions: $y = 1.342$ m and 0.584 m

Only the solution $y_2 = 1.342$ m is accessible from $y = 1.0$ implying that the water surface must rise by 0.142 m over the downward step [1.342 - (1.0 + 0.2) = +0.142 m]. From continuity it also follows that $v_2 = 2.5/1.342$ m = 1.863 ms$^{-1}$.

---
equation, a negative step results in a reduction in the velocity head, an increase in the depth head, and a rise in the water surface over the transition. Thus we can see that, for these conditions described by the upper limb of the energy equation, the water surface is out of phase with the bed, rising over pools and falling over shoals.

But what about the case where the upstream flow approaching a transition has a depth and velocity combination corresponding to the lower-limb condition A' in Figure 2.8? Sample Problems 2.4 and 2.5 illustrate the flow response in just such a case. Here the nearest

**Sample Problem 2.4**

![Diagram](image)

**Problem:** Water flows through a 5 m-wide rectangular channel and over a 20 cm step up in the bed. If the discharge is 25 m³s⁻¹, and the initial depth upstream of the step is 1.0 m, what will be the depth and velocity of the flow downstream of the transition?

**Solution:** Noting that \( q = Q/w = 25/5 = 5 \text{ m}^3\text{s}^{-1} \) and \( V_1 = q/y_1 = 5/1 = 5 \text{ m} \text{s}^{-1} \), we set up the Bernoulli equation for this transition as follows:

\[
1.000 + \frac{25.00}{2g} + 0 = y_2 + \frac{v_2^2}{2g} + 0.200
\]

Simplifying,

\[
2.075 = y_2 + \frac{v_2^2}{19.612}
\]

From continuity, \( y_2v_2 = 5 \), so that \( v_2 = 5/y_2 \). Substituting for \( v_2 \) above and rearranging yields

\[
y_2^3 - 2.075y_2^2 + 1.275 = 0
\]

which by iteration has two real solutions: \( y_2 = 1.531 \text{ m} \) and \( 1.224 \text{ m} \).

The solution \( y_2 = 1.224 \text{ m} \) is taken as correct because it is the closest to the pre-transition condition (where \( y_1 = 1.0 \text{ m} \)). In this case the solution \( y_2 = 1.224 \text{ m} \) implies that the water surface must rise by 0.424 m over the step \([(1.224 + 0.20) - 1.0 = - 0.424 \text{ m}] \). From continuity it also follows that \( v_2 = 5/1.224 = 4.085 \text{ m} \text{s}^{-1} \).
solution to the initial conditions also is on the lower limb of the specific energy curve. In this case the energy equation indicates that the water surface rises over the upward-step transition and falls over a negative step, the reverse of the case on the upper limb. Clearly, quite different responses result, depending on whether the upper or lower limb of the energy curve is accessed by the flow. In this latter case we note that the water surface and bed geometry are in-phase with the water surface, rising over shoals and falling over pools.

We should also note here that the particular problem treated in Sample Problem 2.5 involves a

Sample Problem 2.5

![Diagram of channel flow](image)

**Problem:** Water flows through a 100 m-wide rectangular channel and over a 20 cm step down in the bed. If the discharge is $338 \text{ m}^3\text{s}^{-1}$, and the initial depth upstream of the step is 0.75 m, what will be the depth and velocity of the flow downstream of the transition?

**Solution:** Noting that $q = \frac{Q}{w} = 338/100 = 3.38 \text{ m}^2\text{s}^{-1}$ and $v_1 = \frac{q}{y_1} = 3.38/0.75 = 4.507 \text{ ms}^{-1}$, we set up the Bernoulli equation for this transition as follows:

$$y_1 + \frac{v_1^2}{2g} + z_1 = y_2 + \frac{v_2^2}{2g} + z_2$$

$$0.750 + \frac{20.313}{2g} + 0.20 = y_2 + \frac{v_2^2}{2g} + 0$$

which simplifies to

$$1.986 = y_2 + \frac{v_2^2}{19.612}$$

From continuity, $y_2v_2 = 3.38$, so that $v_2 = 3.38/y_2$. Substituting for $v_2$ above and rearranging yields

$$y_2^3 - 1.986y_2^2 + 0.583 = 0$$

which by iteration yields $y = 1.807 \text{ m}$ and $0.664 \text{ m}$

In this case only the solution $y_2 = 0.664 \text{ m}$ is accessible from $y_1 = 0.75 \text{ m}$ implying that the water surface must fall by $0.286 \text{ m}$ over the downward step $[0.664 - (0.75 + 0.2) = -0.286 \text{ m}]$. From continuity it also follows that $v_2 = 3.38/0.664 \text{ m} = 5.090 \text{ ms}^{-1}$.  

2.23
transition over which the streamlines almost certainly are strongly curvilinear and associated with some vertical acceleration. Since we can no longer assume that the acceleration term \( a_s \) in the Euler equation is zero, it would be prudent to treat the solutions obtained here with some caution. Experience suggests that, although the indicated direction of change in the variables is correct, the precise degree of predicted change is less accurate. Thus, Sample Problem 2.5 is nudging the application limit of the Bernoulli equation and as we shall see later, such problems sometimes are more appropriately solved in terms of the momentum exchanges involved rather than by approximating the energy balance.

Setting aside these potential errors for the moment, these examples illustrate that flow behaviour on the upper limb of the specific energy curve in Figure 2.8 is quite fundamentally different from that associated with the lower limb. For this reason the entire domain of relatively low velocities and large depths \((y>y_c)\) is known as lower regime flow (or subcritical flow) and the corresponding domain of alternative high velocities and small depths \((y<y_c)\) is known as upper regime flow (or supercritical flow).

Clearly we need to know more about the critical condition C in Figure 2.8, corresponding to \(y_c\), and which separates these two domains, one from the other.

**Critical Flow**

In order to derive the equations for critical flow, first we need to note from Figure 2.8 that critical flow occurs at point C where the specific energy, \(E\), is a minimum. Thus, the defining equation for critical flow can be obtained by differentiating equation (2.23) with respect to depth and setting the differential expression equal to zero (the subscript \(c\) indicates critical flow conditions):

Equation (2.23) states that
\[
E = y + \frac{q^2}{2gy^3} = y + \frac{q^2}{2g}y^{-2} \tag{2.23}
\]

Differentiating,
\[
\frac{dE}{dy} = 1 - \frac{q^2}{g}y^{-3} = 1 - \frac{q^2}{gy^3} = 0
\]

from which it follows that,
\[ q^2 = g y_c^3 \] \hspace{1cm} (2.25)

Dividing both sides by \( y_c^2 \),
\[ v_c^2 = g y_c \] \hspace{1cm} (2.26)

Another useful form of equation (2.25) is obtained by rearrangement:
\[ y_c = \frac{q^2}{\sqrt{g}} \] \hspace{1cm} (3.27)

The relationship between \( y_c \) and \( E \) can be derived from equations (2.23) and (2.25).

For critical flow,
\[ E_c = y_c + \frac{q^2}{2 g y_c^2} \]

or alternatively
\[ E_c = y_c + \frac{v_c^2 y_c^2}{2 g y_c^2} = y_c + \frac{v_c^2}{2g} \]

From equation (2.26)
\[ \frac{v_c^2}{g} = y_c \]

and therefore
\[ \frac{v_c^2}{2g} = \frac{y_c}{2} \]

Making the appropriate substitution in equation (2.28) gives
\[ E_c = y_c + \frac{y_c}{2} \quad \text{or} \quad E_c = \frac{3}{2} y_c \]

and by rearrangement,
\[ y_c = \frac{2}{3} E_c \] \hspace{1cm} (2.29)

These equations relating \( E \) and \( y \) in critical flow have been developed for the case of a fixed discharge, \( q \). It is also of considerable practical interest to consider how \( q \) varies with \( y \) for a given specific energy. In terms of the specific energy curve in Figure 2.8, if we fix \( E = E_1 \), for example, we can focus on the changes in \( q \) that will occur as we move vertically from the upper subcritical asymptote at \( E = y \) to the lower limiting supercritical asymptote at \( y = 0 \). At the two

2.9: The discharge-depth curve for a fixed specific energy, \( E_0 \).
asymptotes, \( q = 0 \), at the lower limit because \( y = 0 \), and at the upper limit because the entire specific head is in the form of the depth head so that once again \( y = 0 \). Between these two limits, however, a vertical line through \( A \) and \( A' \) in Figure 2.8 passes through isolines of increasing discharge to reach a maximum discharge \( q_{\text{max}} \), beyond which \( q \) declines to zero once again. The general form of the \( q/y \) relationship for some fixed specific energy \( E_0 \), therefore, must appear as depicted in Figure 2.9.

The maximum discharge is shown in Figure 2.9 as occurring at the critical depth, \( y_c \). The proof of this correspondence is shown readily by differentiating an appropriate form of equation (2.24) and solving \( \frac{dq}{dy} = 0 \), thus:

Rearranging equation (2.24),

\[
q^2 = 2gE_0y^2 - 2gy^3
\]

and differentiating,

\[
2q \frac{dq}{dy} = 4gE_0y - 6gy^2
\]

When \( 4gE_0y - 6gy^2 = 0 \)

\[
4E_0 = 6y
\]

and \( y = \frac{2}{3} E_0 \) ...................................................... (2.30)

Equation (2.30) essentially is equivalent to equation (2.29) describing the conditions in critical flow. Thus we can conclude that, not only does critical flow occur at the minimum specific energy for a given discharge, but it also corresponds with the maximum discharge for a given specific energy. In Figure 2.9 the locus of critical depth across a range of specific energies and discharges (generalizing to all points \( C \)) plots as the straight line \( y = \frac{2}{3} E \).
Critical flow, wave celerity, and the Froude number

As we have seen, the condition of critical flow is associated with a critical depth and also with a corresponding critical velocity, \(v_c\). Where specific energy \(E_0\) is fixed and discharge can vary we might also think in terms of a critical discharge (\(q_c = q_{\text{max}}\)).

The relationship between the critical velocity and the physical process of depth adjustment in open channels is fundamental to understanding the nature of the flow in rivers. In most rivers the water surface is smooth in the sense that changes in the water-surface elevation are spatially gradual. Most of us simply take this circumstance as normal even though changes in the boundary (such as upward and downward steps, contractions and expansions) may be quite abrupt. But why is this so?

The answer to this question lies in the fact that, although boundary irregularities generate disturbances at a point in the flow, the resulting water-surface elevation changes are quickly and continually dispersed throughout the water surface. Small elevation changes are propagated radially outwards from the point of disturbance in the form of small gravity waves. In a sense we might think of these waves as the physical mechanism by which information about the boundary is transmitted to the rest of the flow.

We will not explore wave theory in this account and simply take as our starting point the general expression for the velocity of an oscillatory wave in the free water-surface that you will find developed in most texts on the subject:

\[
c^2 = \frac{gL}{2\pi} \tanh \frac{2\pi y}{L} \tag{2.31}
\]

Here \(c\) and \(L\) are respectively wave celerity (velocity in standing water) and wavelength in water of depth \(y\). Typically the waves propagated by disturbances in open channels are long waves of low amplitude (or 'shallow-water waves') in which \(2\pi y/L\) is small so that \(\tanh \frac{2\pi y}{L} = \frac{2\pi y}{L}\). In these circumstances equation (2.31) reduces to
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\[ c^2 = \frac{gL \ 2\pi y}{2\pi \ L} = gy \] .................................. (2.32a)

and therefore
\[ c = \sqrt{gy} \] ................................................................. (2.32b)

Clearly, equations (2.26) and (2.32) are identical, indicating yet another important property of critical flow. We conclude that critical velocity equals exactly the velocity with which disturbances in the flow are propagated through the free-water surface. The relationship between \( v_c \) and \( c \) is expressed in an important dimensionless ratio called the Froude number, \( F \), as
\[ F = \frac{v}{\sqrt{gy}} \]...........................................................................(2.33)

Thus, the definitional property that we should now recognize is that, in critical flow, \( F = 1.0 \), \( v = v_c \), and \( y = y_c \). It also follows that, in subcritical flow, \( F<1.0 \), \( v<v_c \), and \( y>y_c \), while in supercritical flow, \( F>1.0 \), \( v>v_c \), and \( y<y_c \). Thus, to return to specific energy diagram in Figure 2.8, the lower regime alternative depths at A and B occur in the subcritical flow domain where Froude numbers are less than unity and the upper regime alternative depths A' and B' occur in the supercritical flow domain where Froude numbers are greater than unity.

The importance of the Froude number in specifying the state of flow is implicit in the specific energy equation and we might note here that, from equation (2.23) we get
\[ E = y + \frac{v^2}{2g} = y + \frac{q^2}{2gy^2} = y + \frac{v^2}{2} \ F^2 \] ........................................... (3.34)

and
\[ \frac{dE}{dy} = 1 - F^2 \] ...........................................................................(2.35)

We can recognize critical flow in rivers by the presence of standing waves. We have all seen how disturbances to the surface of standing water spread out radially. In subcritical flow this radial pattern of wave propagation persists but is distorted because of the influence of the mean downstream velocity of the river. To an observer standing on the river bank, waves moving in
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2.29

the downstream direction will travel relatively fast with an enhanced resultant downstream velocity \( \sqrt{\frac{g}{y}} + \vec{V} \). Waves moving in the upstream direction, however, will appear to our observer to be moving much more slowly at the reduced upstream resultant velocity \( \sqrt{\frac{g}{y}} - \vec{V} \). In critical flow, on the other hand, water-surface disturbances will move downstream at velocity \( 2\sqrt{\frac{g}{y}c} \) (since \( c = \bar{v}_e = \sqrt{\frac{g}{y}c} \)) but in the upstream direction they will stand still with respect to an observer on the bank because their upstream resultant velocity is zero \( (\sqrt{\frac{g}{y}c} - \sqrt{\frac{g}{y}c} \).

It is quite common, for example, to see in steep gravel-bed rivers, trains of standing (or stationary) waves spread out for some distance downstream from a site where a large boulder on the bed produces local acceleration to critical flow. Of course, in fully developed supercritical flow in which \( F > 1.0 \), all surface disturbances will be swept downstream because \( \bar{v} > \sqrt{\frac{g}{y}c} \).

Subcritical and supercritical flow: transitions and controls

The introduction of the Froude number allows a general confirmation of the water-surface/bed phases we already have noted in the sample problems. Recalling the Bernoulli equation (2.21) applied to a rectangular channel we can write:

\[
H = y + z + \frac{v^2}{2g} = E + z = \text{a constant, } H.......................... (2.36)
\]

which can be differentiated with respect to distance \( x \) along the channel, giving

\[
\frac{dE}{dx} + \frac{dz}{dx} = 0
\]

which might also be written

\[
\frac{dy}{dx} \frac{dE}{dy} = -\frac{dz}{dx}
\]

Expressing \( \frac{dE}{dy} \) in terms of the Froude number (from equation (2.35)) yields

\[
\frac{dy}{dx} (1 - F^2) = \frac{dz}{dx} ........................................ (2.37)
\]

Once again we see that, if there is a downward step in the bed (i.e., \( dz/dx \) is negative), then the left-hand side of equation (2.37) must be positive. It follows that, when the flow is subcritical (\( F < 1.0 \)), \( dy/dx \) must be positive, indicating a rise in the water surface over the step. Similarly,
for supercritical flow (F>1.0), dy/dx must be negative, implying a drop in the water-surface over the step. Similar reasoning for the case of an upward step (positive dz/dx) confirms that subcritical flow is out of phase with the bed and drops over the step while supercritical flow is in phase and rises over the step.

This relationship between the propagation rate for disturbances and the mean flow-velocity highlights a fundamental difference between subcritical and supercritical flow. Subcritical flow can be influenced by downstream conditions because the related effects are transmitted upstream at a faster rate than the river flows downstream. Consequently, an obstruction or waterfall might produce a respective upstream backwater or drawdown effect in subcritical flow in a way that simply is not possible in supercritical flow. Subcritical flow is able to adjust to a channel transition somewhat before it actually arrives at the source of disturbance so that the complete process of adjustment often is made through gradually varied flow. In summary, we say that subcritical flow is subject to downstream control.

Although this and the following explanation may seem a little anthropomorphic, it captures the physics of the phenomenon to say that, because supercritical flow is moving faster downstream than the upstream propagation of disturbances ahead of it, it arrives unexpectedly at sources of disturbance downstream so that necessary flow adjustments must be made abruptly at relatively severe transitions. Because supercritical flow must be induced by some upstream condition which raises the Froude number above unity, we say that supercritical flow is subject to upstream control. These are not intuitively comfortable notions to most people and you might find it useful to remember an old engineering saying that goes, “unlike subcritical flow, supercritical flow doesn't know what it’s doing 'til it gets there”; or a variant that says, “supercritical flow gets to where it's going before it knows that it's there!”

In this regard it may be helpful to consider the control exerted on the flow by a simple sluice gate. Such gates commonly are used to control flows in laboratory flumes, irrigation canals and reservoir outfalls; Figure 2.10 illustrates the case where such a sluice gate is slowly lowered into an open-channel flow. With the gate at position \(a\) above the water surface, the depth and velocity accord with the specific energy level \(E_a\) at point A in the subcritical flow domain (F<1.0). When the gate is lowered to position \(b\), the depth and velocity here are thus forced to
conform to those consistent with the lower specific energy level at point B. Once the flow passes under the gate, however, depth increases as the flow adjusts back along the specific energy line from B to A. Only the subcritical alternative depth is accessible from A because access to the supercritical domain would require further flow acceleration so that specific energy could first fall to critical at $y_c$ and there is no physical mechanism to produce such acceleration of the flow.

![Diagram of sluice gate flow characteristics](image)

2.10: Control characteristics of a sluice gate set at various heights above the bed of a river.

If the sluice gate is lowered further to position $c$, forcing on the flow the critical depth $y_c$, as the depressed water surface moves downstream of the gate it will either again rise back to A through B to reestablish the flow at the original depth $y_a$, or else it will adjust to the alternative depth $y'_a$ corresponding to $A'$ on the specific energy curve. Both alternative depths are accessible from C and which one is accessed depends on the downstream conditions. Generally, if there is no downstream control (such as another sluice gate), the flow will access the supercritical alternative depth. Once the gate has been closed to the critical depth it is not possible to close it further without affecting the flow conditions upstream. Remember, at critical depth $y_c$ the discharge is at a maximum for the given specific energy and the specific energy is at the minimum necessary to maintain that discharge. Consequently, by lowering the sluice gate to position $d$ we make an impossible demand on the flow. Although we now have reduced the depth head, there is not a sufficient corresponding increase in the velocity head necessary to maintain
the discharge required by continuity. In effect, we have slipped off the specific energy curve in Figure 2.10 and onto one corresponding to a lower discharge. The physical consequences of this circumstance is that the flow backs up behind the sluice gate until the upstream water-level rises to the depth $y_c$ where the new specific energy $E_d$ and the required discharge are once again equilibrated.

In summary, several general observations can be made about the behaviour of the flow under a sluice gate. First, although advective acceleration of the flow under the sluice gate lowers the specific energy there, as long as conditions remain in the subcritical flow domain, the flow immediately downstream of the gate simply will return to the initial flow conditions. Second, if flow depth downstream of a sluice gate remains at or less than the sluice-gate opening (i.e., it does not return to the upstream subcritical flow conditions), then flow under and beyond the gate must be supercritical. Third, the lowest possible setting of a sluice gate that does not interfere with the upstream flow occurs at critical depth. Finally, we can see from this example, that a sluice gate can exert a downstream control on subcritical flow but that it exerts an upstream control in the case of supercritical flow.

![Diagram of discharge-depth relations in a lake or reservoir outflow across a broad-crested weir controlled by a sluice gate.]

2.11: Discharge-depth relations in a lake or reservoir outflow across a broad-crested weir controlled by a sluice gate.

It also is quite instructive to consider the case of a sluice gate which dams the outflow of such a large body of water that the lake level remains sensibly unchanged when the gate is opened for only a short period of time. As before, we assume that no energy is lost through the transition and that the outflow has a rectangular cross-section. In this case, shown in Figure 2.11, the lake
outflow is zero when the gate is closed and flow occurs across a broad-crested weir as the gate is opened. We specify a broad-crested weir so that the pressure distribution throughout can be regarded as hydrostatic. Flow over a short or sharp-crested weir would be influenced by the drawdown of the free-falling outflow and would violate the assumptions of our simple model.

When the sluice gate shown in Figure 2.11 is closed at $a$, $q_a = 0$ and the total head is the depth of water $H$ above the weir. When the gate is opened to position $b$ water begins to flow out under the gate and the outflow equilibrates when the depth of water over the weir drops from $y_a$ to $y_b$; the total head $H$ remains constant although specific energy over the weir must be declining. As the sluice gate is gradually opened further, the depth of flow over the weir will continue to decrease as discharge increases to the maximum flow for the given head. Clearly, the discharge/depth curve applicable here is simply the subcritical flow domain of the curve for a fixed specific energy, $E_o$, shown in Figure 2.9. Thus we know that, since position $c$ on the depth/discharge plot in Figure 2.12 corresponds to $q = q_{\text{max}}$, the depth of flow over the weir must be critical at $y = y_c$. Furthermore, if we raise the sluice gate further, the depth of flow over the weir will remain unchanged, and the flow will continue to discharge at the maximum rate fixed by the available head.

A general conclusion we can derive from this example is that all lake or reservoir outflows across uncontrolled broad-crested weirs will discharge at critical depth (and velocity). Indeed, if we know the magnitude of the head $H$ in Figure 2.12, we can use these relationships to solve a variety of flow problems such as those considered in Sample Problems 2.6 and 2.7. We must not forget, however, that we have assumed that there is no resistance to flow in our model. In considering a real outflow we might have to modify our predictions made here in order to account for the fact that $H$ may decline through the outflow as a result of friction. In general such modifications will not alter the general conclusion (that flow will be critical) but it might mean that the critical flow is restricted to a smaller portion of the weir than implied by our uniform flow model.

Implicit in the sluice gate examples considered above is the notion that it is not always possible to simultaneously satisfy both the energy equation and continuity. Just as some settings of a
sluice gate will obstruct an orderly transition of the flow, causing a backwater, so it is that channel width contractions or bed steps above a certain magnitude will cause the same problem. When the flow is thus obstructed by too severe a transition, it is said to be *choked*. The

### Sample Problem 2.6

**Problem:** If the total head \( H = 1.0 \text{ m} \) for the outflow shown in Figure 2.11, what is the two-dimensional discharge at the outflow when the sluice gate is fully raised?

**Solution:** Since the outflow depth must be critical, from equation (2.30) we know that \( y_c = \frac{2}{3}E_0 = \frac{2}{3} \times 1.0 = 0.667 \text{ m} \). We also know from equation (2.33) that, in critical flow \( v_c = \sqrt{gy_c} \) so that in this case \( v_c = \sqrt{9.806 \times 0.667} = 2.557 \text{ ms}^{-1} \). Thus the two-dimensional discharge is \( q = v_c y_c = 1.706 \text{ m}^2\text{s}^{-1} \).

We might note here that, for all problems of this type, the two-dimensional discharge is specified by the general relationship:

\[
q = \frac{2}{3}H \sqrt{\frac{2}{3}gH} \tag{3.38}
\]

### Sample Problem 2.7

**Problem:** If the water-surface of the outflow shown in Figure 2.11 is 45 cm below the level of the lake when the sluice gate is fully raised, what is the discharge at the outflow?

**Solution:** From the energy equation we know that \( H = y + \frac{v^2}{2g} \) and therefore that \( H - y = 0.45 = \frac{v^2}{2g} \). Thus, \( v = \sqrt{0.45 \times 2g} = 2.971 \text{ ms}^{-1} \). But since the flow across the weir must be critical, we also know that \( v = v_c = 2.971 \text{ ms}^{-1} \) and that \( 2.971 = \sqrt{gy_c} \). It follows that \( 8.825 = gy_c \) and that \( y_c = 0.900 \text{ m} \). Hence the two-dimensional discharge \( q = 2.971 \times 0.900 = 2.674 \text{ m}^2\text{s}^{-1} \).

### Sample Problem 2.8

**Problem:** What is the maximum step height possible before a choke forms in the flow transition described in Sample Problem 2.2 (a 5m-wide rectangular channel carrying 15 m$^3$s$^{-1}$ discharge, 1.5 m deep upstream of the step)?

**Solution:** The maximum step height \( D_y \) is the difference between the upstream specific energy, \( E_1 \) and the minimum possible specific energy at critical flow, \( E_c \) (i.e., \( D_y = E_1 - E_c \)). Noting that \( q = Q/w = 15/5 = 3 \text{ m}^2\text{s}^{-1} \) and \( V_1 = q/y_1 = 3/1.5 = 2 \text{ ms}^{-1} \), upstream of the step, specific energy \( E_1 \) is given by

\[
E_1 = y_1 + \frac{v_1^2}{2g} = 1.500 + \frac{4.00}{2 \times 9.806} = 1.704 \text{ m}
\]

The minimum specific energy at critical flow is \( E_c = y_c + \frac{v_c^2}{2g} \) where \( y_c \) can be determined from equation (2.27) as

\[
y_c = \sqrt{\frac{q^2}{g}} = \sqrt{\frac{3 \times 3.086^2}{9.806}} = 0.972 \text{ m}; \text{ and from continuity, } v_c = 3/0.972 = 3.086 \text{ ms}^{-1}.
\]

Thus \( E_c = 0.972 + \frac{3.086^2}{2 \times 9.806} = 1.458 \text{ m} \).

Therefore the maximum permissible step height \( D_y = E_1 - E_c = 1.704 - 1.458 = 0.246 \text{ m} \)
transition limits beyond which choking will occur are readily determined.

Recalling the discussion in relation to Figure 2.8, in the case of a positive step the maximum step height is simply the difference between the upstream specific energy and the minimum possible specific energy (when the flow is critical); an example of such a determination is worked in Sample Problem 2.7.

Sample Problem 2.9 poses a problem which is insoluble. We know from Sample Problem 2.8 that, under the specified flow conditions the maximum possible height of the step is 0.246 m, so clearly a step of 0.355 m cannot be negotiated by the flow because the specific energy over the step must fall below that necessary to maintain the constant discharge. Of course, this physically impossible situation does not prevent us from deriving an equation and herein lies a warning: we must remain alert to the fact that generating a Bernoulli-based cubic expression does not necessarily mean that we have solved the flow transition problem at hand nor does a failure to reach convergence in our mathematical iteration imply that our computations are in error!

---

**Sample Problem 2.9**

**Problem:** Water flows through a 5 m-wide rectangular channel and over a 35 cm step up in the bed. If the discharge is 15 m$^3$s$^{-1}$, and the initial depth upstream of the step is 1.5 m, what will be the depth and velocity of the flow downstream of the transition?

**Solution:** Noting that $q = Q/w = 15/5 = 3$ m$^2$s$^{-1}$ and $V_1 = q/y_1 = 3/1.5 = 2$ ms$^{-1}$, we set up the Bernoulli equation for this transition as follows:

$$y_1 + \frac{v_1^2}{2g} + z_1 = y_2 + \frac{v_2^2}{2g} + z_2$$

$$1.500 + \frac{4.00}{2g} + 0 = y_2 + \frac{v_2^2}{2g} + 0.350$$

which simplifies to

$$1.354 = y_2 + \frac{v_2^2}{19.612}$$

From continuity, $y_2v_2 = 3$, so that $v_2 = 3/y_2$. Substituting for $v_2$ above and rearranging yields

$$y_2^3 - 1.354y_2^2 + 0.459 = 0$$

It is left to the reader to verify that this equation has no meaningful solution (there is a physically meaningless negative solution at $y_2 = -0.498$ m).
Flow transitions in three-dimensions

In the discussion to this point we have kept our analysis simple by assuming a rectangular channel of constant width because such a regular cross-section readily lends itself to a two-dimensional approach to the problems. The flow principles established for this simple case, however, are readily extended to three-dimensional flow and to other forms of channel cross-section (trapezoidal, semi-circular, parabolic, for example). We will not fully develop these alternative models here although they are readily available elsewhere (an excellent source is Henderson, 1966). It will be useful for our purposes, however, to briefly consider the adaption of the energy equation to transitions in rectangular channels of variable width.

The key to dealing with three-dimensional flow transition problems in rectangular channels is to recognize that equations (2.21) and (2.22), respectively for the total and specific energy, can also be expressed in terms of discharge, Q, as follows:

\[
\frac{Q}{wv} + \frac{(Q/wy)^2}{2g} + z = H \quad \text{............................................ (2.39)}
\]

\[
\frac{Q}{wv} + \frac{(Q/wy)^2}{2g} = E \quad \text{.......................................................... (2.40)}
\]

It turns out that, in terms of the specific energy/depth relationships, a channel expansion has exactly the same effect on the flow as an increase in depth (a negative step) and a channel contraction is analogous to a reduction in depth (a positive step). Furthermore, the previously discussed contrasted directions of change associated with subcritical flow on the one hand, and with supercritical flow on the other, also apply in the case of responses to changes in channel width. Thus, an accompanying width change can enhance a response to a bed-elevation change or it can counter it.

Similarly, a flow can be induced to go critical by reducing the channel width just as it can be choked by making it encounter too severe a contraction. These relationships are best explored through some examples such as those developed in Sample Problems 2.10 to 2.14 to follow.
Sample Problem 2.10

Problem: Water flows through an 85 m-wide rectangular channel at a rate of 100 m$^3$s$^{-1}$. If the flow is contracted to 65 m width, what is the depth and velocity in the contracted section of the channel if the approach depth is 2.0 m? Determine the extent and direction of change in the water-surface elevation through the channel transition.

Solution: From continuity we get $v_1 = \frac{Q}{w_1 y_1} = \frac{100.0}{85.00 \times 2.000} = 0.5882$ ms$^{-1}$ and from equation (2.39) we can state that

$$y_1 + \frac{0.5882^2}{2g} + z_1 = y_2 + \frac{(Q/w_2 y_2)^2}{2g} + z_2$$

so that

$$2.000 + \frac{0.5882^2}{2g} + 0 = y_2 + \frac{[100.0/(65.00 \times y_2)]^2}{2g} + 0$$

and

$$2.0176 = y_2 + \frac{0.1207}{y_2^2}$$

or

$$y_2^3 - 2.0176y_2^2 + 0.1207 = 0$$

Solving in the usual way, the two positive solutions are $y_2 = 0.263$ m and 1.987 m. Since the flow upstream of the contraction is in the subcritical flow domain ($F_1 = 0.588/\sqrt{9.806 \times 2.00} = 0.13$), the correct alternative depth is $y_2 = 1.987$ m. From continuity it follows that $v_2 = 100/(65.00 \times 1.987) = 0.774$ ms$^{-1}$.

Sample Problem 2.11

Problem: To what width would the channel in Sample Problem 2.10 have to be contracted to achieve critical flow? How far would the water surface drop under these conditions? What degree of contraction will choke the flow?

Solution: We know from equation (2.30) that, in this case where $E_1 = 2.0176$ m,

$$y_c = 2/3 E_1 = 2/3(2.0176) = 1.345$$

From equation (2.26) it follows that $v_c = \sqrt{gy_c} = \sqrt{9.806 \times 1.345} = 3.632$ ms$^{-1}$.

From continuity,

$$w = \frac{100}{1.345 \times 3.632} = 20.471$$

So in order to achieve critical flow the channel would have to be contracted from 85 m down to about 20 m. The drop in the water surface, $\Delta h$, is given by

$$\Delta h = y_1 - y_2 = 2.000 - 1.345 = 0.655$$

Since a channel width of 20.471 m represents the maximum discharge for the available specific energy, it is also the limiting condition for choking; i.e., any further contraction will here choke the flow.

Sample Problem 2.12
Chapter 2: The Energy Equation for Open-Channel Flow

Problem: Water flows through a rectangular and symmetrical, but otherwise unspecified, channel expansion as shown above. Given \( Q = 180.0 \text{ m}^3/\text{s} \) and a 2.25 m-deep flow in the 60 m-wide approach channel, determine the degree of expansion if the water surface rises by 5.0 cm. Calculate the mean depth and velocity in the expansion.

Solution: At the outset we might note from physical reasoning that, since the water surface drops in a contraction (see Sample Problem 2.10), the rise in the water surface here implies that the channel width must expand to some degree in this transition.

We know that, since the bed is sensibly horizontal, the increase in the water surface, \( \Delta h \), must be equal to the difference in flow depth through the transition, or

\[
\Delta h = y_2 - y_1 \quad \text{and therefore} \quad y_2 = y_1 + \Delta h = 2.25 + 0.05 = 2.30 \text{ m}
\]

From continuity we get

\[
v_1 = \frac{Q}{w_1 y_1} = \frac{180.0}{60.00 \times 2.250} = 1.333 \text{ m/s}
\]

and, noting that \( z_1 = z_2 \), from equation (2.39) we can state that

\[
2.250 + \frac{1.333^2}{2g} = 2.300 + \frac{(Q/w_2 y_2)^2}{2g} = 2.300 + \frac{\left(\frac{180.0}{2.300w_2}\right)^2}{19.612}
\]

which simplifies to

\[
2.341 = 2.300 + \frac{312.297}{w_2^2} \quad \text{and further to} \quad 0.041w_2^2 = 312.297; \quad w_2 = 87.275 \text{ m}
\]

Thus the 5 cm rise in the water surface implies a channel expansion from 60.000 m to 87.275 m.

Continuity dictates that

\[
v_2 = \frac{180.00}{87.275 \times 2.300} = 0.897 \text{ m/s}.
\]

Sample Problem 2.13

Problem: Determine the height of the upward step at the expansion in Sample Problem 2.12 necessary to counteract the 5 cm rise in the water surface.

Solution: In order for the water surface to neither rise nor fall through the expansion it is necessary that

\[
y_2 = 2.25 - \Delta z. \quad \text{Furthermore, we know from continuity that} \quad y_2 v_2 = \frac{Q}{w_2} = \frac{180.0}{87.275} = 2.062 \text{ m/s}^2. \quad \text{Thus,}
\]

we can state that

\[
E_1 = 2.250 + \frac{1.333^2}{2g} = 2.342 = y_2 + \frac{(2.062)^2}{19.612} + \Delta z = y_2 + \frac{0.2168}{y_2^2} + Dz
\]

Since \( \Delta z = 2.25 - y_2 \),

\[
2.342 = y_2 + \frac{0.2168}{y_2^2} + 2.25 - y_2 = \frac{0.2168}{y_2^2} + 2.25
\]

So, \( y_2^2 = \frac{0.2168}{0.092} = 2.36 \) and \( y_2 = 1.535 \text{ m} \). Therefore, the step height to counteract the water surface rise associated with the flow expansion must be \( \Delta z = 2.25 - y_2 = 2.250 - 1.535 = 0.715 \text{ m} \).
Sample Problem 2.14

Problem: Shown on the right are the dimensions of a rectangular channel transition that involves both a negative step and an expansion. If the channel is discharging 500 m$^3$s$^{-1}$ of water and the approaching flow has a mean depth of 3.00 m, calculate the mean Froude number in the channel expansion. Although the expansion is shown to occur abruptly, we will assume, as in previous examples, that frictional head loss is negligible so that energy in the transition is conserved.

Solution: The Bernoulli equation for this transition is

$$y_1 + \frac{v_1^2}{2g} + z_1 = y_2 + \frac{v_2^2}{2g} + z_2$$

$$3.000 + \frac{\left(\frac{500.0}{100.0 \times 3.0}\right)^2}{19.612} + 0 = y_2 + \frac{\left(\frac{500.0}{120.0 \times y_2}\right)^2}{19.612} - 0.5$$

which simplifies to

$$3.142 = y_2 + \frac{0.885}{y_2^2} - 0.5$$

or

$$y_2^3 - 3.642y_2^2 + 0.885 = 0$$

Solving by iteration yields two positive solutions at $y_2 = 0.534$ m and 3.573 m. Since the approach flow is subcritical ($F_1 = \frac{1.67}{\sqrt{9.806 \times 3.000}} = 0.307$), the correct solution is 3.573 m. From continuity we get $v_2 = \frac{500.0}{120.0 \times 3.573} = 1.166$ ms$^{-1}$ & in the channel expansion, $F_2 = \frac{1.166}{\sqrt{9.806 \times 3.573}} = 0.197$.

On the other hand, if the discharge through the channel is 2000 m$^3$s$^{-1}$, the Bernoulli equation becomes

$$3.000 + \frac{\left(\frac{2000.0}{100.0 \times 3.0}\right)^2}{19.612} + 0 = y_2 + \frac{\left(\frac{2000.0}{120.0 \times y_2}\right)^2}{19.612} - 0.5$$

which simplifies to

$$5.766 = y_2 + \frac{14.164}{y_2^2}$$

or

$$y_2^3 - 5.766y_2^2 + 14.164 = 0$$

Once again, solving by iteration yields two positive solutions at $y_2 = 1.919$ m and 5.253 m. But here the approach flow is supercritical ($F_1 = \frac{6.667}{\sqrt{9.806 \times 3.000}} = 1.229$) and $y_2 = 1.919$ m is the correct alternative depth.

From continuity, $v_2 = \frac{2000}{120.0 \times 1.919} = 8.685$ ms$^{-1}$ and $F_2 = \frac{8.685}{\sqrt{9.806 \times 1.919}} = 2.002$.

For $Q = 500$ m$^3$s$^{-1}$ the subcritical flow through the transition involves an increase in the water-surface elevation of 0.073 m [3.573 - (3.000 + 0.500)] but in the supercritical flow at $Q = 2000$ m$^3$s$^{-1}$ the water surface drops by an impressive 1.581 m [1.919 - (3.000 + 0.500)] as the flow enters the expansion. Of course, a discharge of 2000 m$^3$s$^{-1}$ would not be contained by this channel if the flow were not supercritical because the equivalent subcritical flow depth in the expansion (5.253 m) would significantly exceed the bank height.
Accounting for energy losses

In all of these flow-transition problems employing the energy equation, the solution follows from the assumption of energy conservation in which \( E_1 = E_2 \). In real rivers this can never be true because energy is always “lost” in overcoming frictional drag at the boundary and is drained further to drive turbulence and secondary flow (lateral rather than downstream flow). In channel transitions where the boundary is smooth and the boundary transition is such that it causes just gradually varied flow, the loss in energy or head may be very small or even negligible. Flow over rough boundaries through abrupt channel transitions, however, may involve considerable head loss and the energy equation will only describe flow accurately if there is an accounting for this loss.

In problems where there is significant energy loss (\( \Delta E \)) we modify the energy balance so that

\[
E_1 = E_2 + \Delta E
\]

thereby recognizing that the initial energy \( E_1 \) is partitioned into that available for allocation to the depth and velocity heads in the flow after the transition (\( E_2 \)) and a component used in overcoming frictional drag and turbulence.

In most flow-transition problems we know the amount of initial energy (\( E_1 \)) available and we need to compute the flow parameters for \( E_2 \). Rearranging equation (2.42) yields

\[
E_1 - \Delta E = E_2
\]

reminding us that, whenever energy is lost in a channel transition, \( E_1 < E_2 \) by the amount of the energy loss. For example, one of our first two-dimensional step problems (Sample Problem 2.2) is reworked below (Sample Problem 2.15) to allow for a two per cent head loss as the flow adjusts to an upward step in the bed (ie, \( E_1 \) is discounted to 98%).

The two per cent head-loss in Sample Problem 2.15 only amounts to about 3.4 cm but the effect on the energy-based solutions for depth and velocity over the step are significant (see Figure 2.12).

Although the directions in which changes occur remain unchanged from the fully energy-conserved case, the reduction in total head forces the flow to adopt a higher velocity and reduced flow depth in order to maintain continuity of flow through the transition. In other words, the flow state has moved much closer to critical flow. Indeed, if we were to impose a
### Table: Comparison of flow parameter changes in Sample Problems 2.2 and 2.15 illustrating the effects of a 2% head loss through the transition.

<table>
<thead>
<tr>
<th>Flow parameter</th>
<th>Sample Problem 2.2 (no head loss)</th>
<th>Sample Problem 2.15 (2% head loss)</th>
<th>Relative Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Available head</td>
<td>1.704 m</td>
<td>1.670 m</td>
<td>-2.0%</td>
</tr>
<tr>
<td>Flow depth, $y_2$</td>
<td>1.167 m</td>
<td>1.066 m</td>
<td>-8.7%</td>
</tr>
<tr>
<td>Flow velocity, $v_2$</td>
<td>2.571 m/s</td>
<td>2.814 m/s</td>
<td>+9.5%</td>
</tr>
<tr>
<td>Water-surface elev.</td>
<td>-0.133 m</td>
<td>-0.234 m</td>
<td></td>
</tr>
<tr>
<td>Froude number, $F_2$</td>
<td>0.76</td>
<td>0.87</td>
<td>+14.5%</td>
</tr>
</tbody>
</table>

2.12: Consider the three-dimensional flow transition problem treated in Sample Problem 2.14. If we now account for the previously ignored energy loss resulting from the rapid expansion of the
flow, the problem can be reworked as in Sample Problem 2.16. Here we allow for a rather extreme ten per cent loss of initial energy to friction and turbulence (i.e., $E_1$ is discounted to 90%). The comparative flow parameters for each case are summarized in Figure 2.13.

Sample Problem: 2.16

**Problem:** Shown on the right are the dimensions of a rectangular channel transition which involves both a negative step and an expansion. If the channel is discharging 500 m$^3$·s$^{-1}$ of water and the approaching flow has a mean depth of 3.000 m, calculate the mean Froude number in the channel expansion. Because the channel expansion occurs abruptly, we will assume that there is a 10% frictional head loss incurred as the flow moves through the transition.

**Solution:** The Bernoulli equation for this transition is

$$y_1 + \frac{v_1^2}{2g} + z_1 = y_2 + \frac{v_2^2}{2g} + z_2 + \frac{500.0}{100.0 \times 3.0} + 0 = y_2 + \frac{500.0}{120.0 \times y_2} - 0.5$$

so that

$$3.142 = y_2 + \frac{0.885}{y_2^2} - 0.5$$

Here $E_1 = 3.142$ m so discounting for the energy loss (0.90 x 3.142 = 2.828 m) yields

$$2.828 = y_2 + \frac{0.885}{y_2^2} - 0.5$$

which simplifies to

$$y_2^3 - 3.328y_2^2 + 0.885 = 0$$

Solving by iteration gives a (subcritical) solution for $y_2 = 3.244$ m. From continuity we get

$$v_2 = \frac{500.0}{1200 \times 3.244} = 1.284 \text{ m/s}$$

in the channel expansion, $F_2 = \frac{1.284}{\sqrt{y_2}} = 0.228$.

<table>
<thead>
<tr>
<th>Flow parameter</th>
<th>Sample Problem 2.14 (no head loss)</th>
<th>Sample Problem 2.16 (10% head loss)</th>
<th>Relative Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Available head</td>
<td>3.142 m</td>
<td>2.828 m</td>
<td>-10.0%</td>
</tr>
<tr>
<td>Flow depth, $y_2$</td>
<td>3.573 m</td>
<td>3.224 m</td>
<td>-9.8%</td>
</tr>
<tr>
<td>Flow velocity, $v_2$</td>
<td>1.166 m/s</td>
<td>1.284 m/s</td>
<td>+10.1%</td>
</tr>
<tr>
<td>Water-surface elev.</td>
<td>+0.073 m</td>
<td>-0.276 m</td>
<td></td>
</tr>
<tr>
<td>Froude number, $F_2$</td>
<td>0.20</td>
<td>0.23</td>
<td>+15.0%</td>
</tr>
</tbody>
</table>

2.13: Comparison of flow parameter changes in Sample Problems 2.14 and 2.16 illustrating the effects of a 10% head loss through the transition.
Here again, the reduced available head has resulted in a lower depth of flow and an increased velocity in order to maintain flow continuity through the transition. In this low-Froude-number flow the relative changes in depth and velocity are about the same as the reduction in the available head. Note that, because less energy is available to allocate to the depth head, the water-surface elevation now actually drops through the expansion, contrary to our expectations in the energy-conserved case.

As we noted at the outset, the degree of energy loss through short gradual transitions in river channels is negligible and usually can be ignored. If it is suspected that energy loss is significant then an allowance should be made for it. Losses of a few per cent of the initial head are common and a ten per cent loss is rather extreme. As in many aspects of river science, experience is the ultimate guide to the appropriate discounting factor.

**Some concluding remarks**

There are many circumstances where energy loss in a river will always be significant and difficult to evaluate. Because there is a sensitive relationship between changes in the head and the responses in the flow depth and velocity, the energy approach is not always the most robust basis for describing the flow. It is particularly suspect at high Froude numbers where energy-exchange responses are at their most sensitive. But we have some other tools available for such circumstances and one of these, the momentum approach, is the subject of the next chapter.

**References**