Tutorial 2. *Game Theory: Basic Concepts. Simultaneous-Move Games.*

Solutions.

Warm-up questions: which of the following situations involve strategic thinking and can be modeled as a game?

1. OPEC members choosing their annual output.

   Definitely a game: players' optimal decisions are interdependent and payoffs are affected by the decisions of other players. Each country's market share is large enough to affect the market price. At the same time a firm's profit maximizing output depends on price: when deciding how much to produce each country needs to know how much other countries are going to export.

2. Two manufacturers: one of nuts and one of bolts decide whether to use metric or American standards.

   There is no conflict of interest between the players in this game, but still, the actions of one affect the optimal decision of the other. The two firms prefer to coordinate on the standards: if one firm chooses metric, the best decision for the other firm is to also choose metric.

3. An electric company deciding whether to order a new power plant given its estimate of demand for electricity in ten years.

   The decision might be a complex one, however, the company does not interact with any players - all consumers are price-takers and don't act strategically.

Problem 1. Consider a "Boxed Pigs" Game[1]: There is a Big Pig (player 1) and a Small Pig (player 2). For experimental purposes they are put in a box. In the box there is a panel in one end of the box and a food dispenser at the other end of the box. When a pig presses the panel, food is dispensed at the other end of the box. It costs 2 utility units to press the panel and total utility worth of the food dispensed is 10 units. Big Pig is strong and eats really fast - if he gets to the dispenser first - he eats 9 units of food and only 1 unit of food is left for the Small Pig. If Small Pig gets to the dispenser first - he can eat 4 units. If they arrive to the food dispenser at the same time Small Pig gets 3 units of food. The actions available to the pigs are to Press and Wait (by the food dispenser). If both choose W nothing happens. If one presses and the other waits, the one who presses will get to the dispenser second.

(a) Represent this game in normal form.

[1] Baldwin and Meese, 1979, you do not have to read the article

At first let's compute the payoffs for all possible outcomes:

- *If both Wait, each gets zero: no disutility from pressing the panel and no food.*
- *Big Pig chooses Press and Small Pig chooses Wait: Small Pig: 4 - 0 = 4; Big Pig: 6 - 2 = 4.*
- *Big Pig chooses Wait and Small Pig decides to Press. Small Pig: 1 - 2 = -1; Big Pig: 9 - 0 = 9.*
- *Both Press. Small Pig: 3 - 2 = 1; Big Pig: 7 - 2 = 5.*

Small Pig

		Press	*Wait*
Big Pig	Press	5; *1*	4; *4*
	Wait	9; *-1*	0; *0*

(b) For each player determine whether there is a dominant strategy.

Small Pig has a dominant strategy: regardless of what Big Pig does, strategy 'Wait' gives a higher payoff: $4 > 1$ *(if Big Pig plays 'Press') and* $0 > -1$ *(if Big Pig plays 'Wait'). There is no dominant strategy for Big Pig: 'the best thing to do' depends on the choice that Small Pig makes. Big Pig's best response to Small Pig playing 'Wait' is to 'Press':* $4 > 0$. *Best response to Small Pig playing 'Press' is 'Wait':* $9 > 5$.

(c) Find pure strategy NE in this game.

To find all NE check all combinations of strategies: in each outcome can any player do better by changing his strategy given the other players continue to play their strategies? Once you've checked all cells your conclusion should be that the unique pure strategy NE of this game is (Press, Wait): given the strategy of the other pig, no pig has an incentive to play a different strategy. For all other combinations of strategies one of the pigs wants to change his strategy, so they are not a NE.
Recall that in the lecture we talked about Iterated Elimination of Dominated Strategies as a method for finding NE. In this game Small Pig has a dominated strategy: to 'Press' gives lower payoff no matter what the Big Pig does. Therefore, a rational Small Pig will never play 'Press'. After we 'crossed out' 'Press' for the Small Pig, it does not make sense for the Big Pig to 'Wait' - we can eliminate this strategy. What left is Big Pig playing 'Press' and Small Pig playing 'Wait'.

Problem 2. When I was teaching ECON 383 in 2007 one of the graded components on the course outline was 'surprise quizzes', however, the number of quizzes was not specified (obviously so weren't the dates). Let's model this situation as a game. The resulting simultaneous game we played each week was as follows. Students have two pure strategies: prepare (P) and relax (R). Iryna also has two pure strategies: quiz (Q) and no quiz (N). Assume students do not care about learning and just want to maximize "payoff = grade - disutility from studying". It is costly for Students to study, so they want to study only if there is a quiz; if there is no quiz they prefer to relax. It is costly for Iryna to make and grade a quiz, therefore, if Students are prepared Iryna prefers not to give quiz, but if students are not prepared, Iryna prefers quiz. The payoffs that satisfy these preferences over the outcomes can be as follows:

Iryna

		Q	N
Students	P	15; -10;	-20, 0
	R	-20; 15	15, -15

(a) If Iryna chooses Q, what is Students' best response (BR)? If Iryna chooses N, what is Students' BR?

If Iryna chooses Q Stdents' BR is to come prepared to that tutorial. If there is no quiz Students are better off not studying.

(b) For each player determine whether there is a dominant strategy.

From part (a) it follows that there no DS for Students. There is no DS for Iryna.

(c) Find pure strategy NE in this game. - *Examine each cell*:

(P, Q): Students are OK because they choose their BR, however, GIVEN Students' choice is P, Iryna has incentive to change her strategy and play N.

(P, N): Iryna chose her BR - does not have incentive to change her strategy GIVEN that Students' choice is P, but students are not happy with their choice - GIVEN Iryna plays N, they prefer to relax.

(R, N): Is not a NE because Iryna now prefers to give quiz.

(R, Q): Is not a NE because if there is a quiz Students' BR is to study.

In conclusion, there is no combination of pure strategies that satisfies the definition of NE. Or, there is no pure strategy NE in this game.

(d) Find the NE in mixed strategies. Denote p =probability that students study; q =probability with which Iryna gives the quiz.

To find the mixed strategy NE use the indifference condition we discussed in the lecture: each player will only play a mixed strategy if he is indifferent between the pure strategies (everything is a BR.)

For Students: *given probability of quiz is q, if Students are prepared, their expected payoff is* $EU^P = 15q + (1-q)(-20) = 35q - 20$. *Notice that it is positively related to q, which makes sense: the higher is the probability of quiz, the more rewarding it is to come prepared. If students decide not to study their expected payoff is* $EU^R = (-20)q + (1-q)15 = -35q + 15$, *again notice the relationship between q and expected payoff of partying: the higher is the probability of quiz, the more costly it is to come unprepared.*
set $EU^P = EU^R$: $35q - 20 = -35q + 15$; *solve for* $q = \frac{1}{2}$.

What does this $q = \frac{1}{2}$ mean? It means that if Iryna gives quiz with probability .5, students are indifferent between preparing for the quiz, relaxing, and playing any mixed strategy. If $q < \frac{1}{2}$ students prefer to relax: if, for example, Iryna plays a mixed strategy $(0.1, 0.9)$, when she quizzes students with probability 10% each week, $EU^P = -16.5$ and $EU^R = 11.5$ students' BR is to relax, which is what they'll do, but this situation cannot be an equilibrium outcome because eventually Iryna must realize that students never study, so her mixed strategy is a bad one. If $q > \frac{1}{2}$, students' BR is to prepare. In conclusion, if q IS NOT 1/2 students will not play a mixed strategy and the situation is not NE, which means that the only q consistent with equilibrium is 0.5.

*Repeat the same steps for **For Iryna:***
$EU^Q = -10p + 15(1-p) = 15 - 25p$
$EU^N = 0p - 15(1-p) = 15p - 15$
set $EU^Q = EU^N$: $15 - 25p = 15p - 15$, *obtain* $p = \frac{3}{4}$.
Let's look at Iryna's expected payoffs given Students' mix. $EU^Q = .75(-10) + .25 \cdot 15 = -3.75$; $EU^N = .75(0) + .25(-15) = -3.75$, *notice that since both pure strategies give the same EU any weighted average of the two (any mix) will sill give the same payoff.*
Answer: *the mixed strategy NE is:*
Iryna plays strategy $(\frac{1}{2}; \frac{1}{2})$ *(plays Q with probability .5)*
Students play mixed strategy $(\frac{3}{4}; \frac{1}{4})$ *- come prepared to 75% of tutorials.*
To find players' expected payoffs in mixed strategy NE you have to weigh the payoff of each outcome by the probability of it, which is a product of the probabilities with which the strategies are chosen by the players. In NE Students get expected payoff EU=15pq-20p(1-q)-20(1-p)q+15(1-p)(1-q).

(e) Suppose that Iryna has to ensure that students study every week, does this mean that she must give a quiz every single week? In other words, is it possible that $q < 1$ and students still study?

This question helps you understand applications of mixed-strategies NE. Look what happens in this mixed-strategy NE. Students are not always prepared, they are not ready to write the quiz with probability 25%. At the same time giving the quiz with probability 1 is also not good for Iryna: students will always be prepared, and then there is no point to quiz. So if Iryna needs to ensure that students are always prepared without giving a test every week, what she can do is to tilt q slightly over its NE value so that P becomes BR. Check: when q=0.51, $EU^P = -2.15$ *and* $EU^R = -2.85$.

Problem 3. 'Matching Pennies' is an example of a ***zero-sum game***: *a player wins at the expense of the other player, therefore, in each outcome the sum of the payoffs is zero.* Each round of the game Player 1 and Player 2 choose heads H or tail T. If they chose the same, as in outcomes (H, H) and (T, T) Player 1 gets one dollar from Player 2. If they choose different, Player 1 pays one dollar to Player 2. Represent this game in normal form and find all NE (in pure and mixed strategies). *No pure strategy NE; NE mix is (0.5; 0.5) and (0.5; 0.5).*

Problem 4. This is another famous game, called 'Battle of Sexes'. On a night a Man and a Woman can either go to a boxing match or to listen to opera. Man likes boxing more than opera and the Woman prefers the opera to boxing, at the same time the like each other, so they would prefer to be together to doing their favorite thing on their own (for example, Woman would rather attend boxing match with the Man than to be at opera by herself). Construct a payoff matrix that would represent this game. Find all NE. *See textbook. Notice that this is a coordination game, it has 2 pure strategy NE (the ones when they meet) and also a mixed strategy NE.*

Problem 5. Consider a two-player Rock-Paper-Scissors game: rock beats scissors, scissors beat paper, paper beats rock (this is another zero-sum game). This game has no pure strategy NE and you can prove that in the mixed strategies NE each strategy is played with probability $\frac{1}{3}$. Suppose winning gives player payoff of 1 and losing results in -1 payoff, in case of tie both get zero. Lets add one more strategy called Dynamite. Dynamite beats everything, but if both players choose it, they end up in the hospital, payoff -10,000 for both. Discuss how adding this new strategy affects the equilibrium outcomes of the game.

The original game does not have a pure strategy NE, adding Dynamite creates 6 pure strategy NE in the game: one player plays Dynamite and the other player plays something else: (D, R), (D, P), (D, S), (S, D), (R, D), (P, D). You could find these using the following logic: suppose I play D, my opponent's BR is NOT to play dynamite, but play something else. My BR to opponent playing "something else" is D. Then all combinations of 'Dynamite and sth. else' is a combination of BRs which means it is a NE. This game allows you to think about the importance of beliefs: since the game is simultaneous and payoffs of outcome (D, D) are extremely bad the players may think it's better not to play that strategy just in case, but then if you believe that the other person will never play D, you should play it as in that case it will guarantee the victory, but then there is no way for you to know that he is not thinking the same.

More questions: indicate true/false and explain. *Prisoners' Dilemma is a good game to use to answer these questions.*

1. If all players have a dominant strategy, there is a unique NE.
2. Since players seek to maximize their payoffs, NE of a game must be efficient.
3. If a player has a dominant strategy, he is guaranteed to get the highest payoff in the NE.

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