Tutorial 3. *More Games. NE in Continuous Strategies. Sequential Games. Solutions.*

Problem 1. There are 2 students taking a game theory class, Zulu and Yva. Their teacher suggests they play the following simultaneous-move game in tutorial: each student writes a nonnegative number on a piece of paper. Denote Zulu's number by Z and Yva's number by Y. Teacher says that the reward is the bonus points they can earn toward their grade, denote U_Z and U_Y respectively. The payoffs are calculated according to the following formulas:

$$U_Z = (4 - Z - Y)Z; \qquad U_Y = (4 - Z - Y)Y$$

This means for example, that if Zulu writes 1 and Yva writes 5, Zulu gets $(4 - 1 - 5)1 = -2$ points and Yva gets $(4 - 1 - 5)5 = -10$ points.

a) What are players' strategies and how many strategies does each player have? Is teacher a player?

In this game a strategy is choosing a number. There are infinitely many strategies available to each player. Teacher is not a player, he does not get to play any strategy, he does not have a payoff, and is not really involved in the game: there is no interaction between him and the students.

b) Find Zulu's best response function.

Zulu's objective is to maximize her payoff GIVEN the number Y that Yva writes. Zulu takes Yva's number as given because she cannot directly affect what Iva chooses. Mathematically this means that we take partial derivative of Zulu's payoff U_Z with respect to Z, treating Y as if it was a constant, set the derivative equal to zero and solve for Z as a function of Y: $\frac{\partial U_Z}{\partial Z} = 4 - Y - 2Z = 0$
To solve for Z, bring Z to left hand side, everything else on right hand side:
$2Z = 4 - Y$,
Zulu's best response function is $Z^{BR} = \frac{4-Y}{2}$**.**
It is very handy because if we know what Yva chooses, we can easily calculate the number that will give the highest payoff to Zulu (we don't have to solve max problem for each possible Y).

c) Find Yva's best response function.

Follow the same steps to find that Yva's best response to a number that Zulu writes is $Y^{BR} = \frac{4-Z}{2}$*. Compare to Zulu's BR: it looks very similar, because they have the same payoff functions. It is called 'symmetric'.*

d) Find Nash equilibrium of this game. Draw BR's on a diagram with Z on horizontal and Y on the vertical axis. How many bonus marks do they earn in equilibrium?

Recall that in NE all players must be playing their best responses to each other. So mathematically you need a combination (Z; Y) that satisfies both best responses simultaneously; solve the system of two BR equations for X and Y
Substitute Y^{BR} for Y into Zulu's BR:
$Z = \frac{4-(4-Z)/2}{2} = 2 - 1 + \frac{1}{4}Z$; *now bring Z to the LHS:*
$.75Z = 1, \quad Z = \frac{1}{.75} = \frac{4}{3}$
Now that we found Zulu's NE strategy, plug $Z = \frac{4}{3}$ into Yva's BR to find that in NE Yva will also write $Y = \frac{4}{3}$. Notice that at this moment you can check whether your calculations are off. When you cross-substitute the numbers you found for the equilibrium into the players' BR's, you should keep getting the same numbers, once again confirming that the numbers are indeed the best responses to each other, so both players are maximizing their payoffs with respect to the choice of the other player and have no incentive to change their strategy given that other players do not change their strategies. Substitute $\frac{4}{3}$ into the payoff functions, $U_Z = U_Y = 1.78$

The NE of this game is ($\frac{4}{3}$, $\frac{4}{3}$).

Remember that the definition of NE starts 'the list of strategies, one for each player...'; it would be a mistake to quote NE as the payoffs they receive.

e) Is Nash equilibrium efficient? Find the efficient outcome in which they receive the same payoffs[1].

To prove that equilibrium is inefficient you have to find an outcome in which at least one person is better off compared to Nash and the other is doing at least as good. Maximize total marks $\pi = U_Z + U_Y$ s.t. Z=Y, sub the constraint into the objective function, take derivative and find that Y=Z=1. Calculate $U_Z = U_Y = (4-1-1) \cdot 1 = 2$. When they both write 1, both get higher payoff compared to NE, NE is inefficient.

f) Show that $Z = 1$ and $Y = 1$ is not a NE.

If Zulu expects that Yva will write 1, her BR is $Z^{BR} = \frac{4-1}{2} = 1.5$, if she does that she will receive $U_Z = (4-1-1.5) \cdot 1.5 = 2.25$, which is better than what she gets in part (e), therefore (1, 1) is not a NE because each player has an incentive to change her strategy given the choice that the other players are making.
To conclude, you could also observe that is is a game of strategic substitutes and NE of such games are inefficient and if players lower their actions (in this case write lower number all are better off.)

[1] Any combination of Z and Y s.t. $Z + Y = 2$ will maximize joint payoff and will be efficient. If you try to max joint payoff $\pi = U_Z + U_Y$, both FOC's will be $\frac{\partial \pi}{\partial Z} = \frac{\partial \pi}{\partial Y} = 4 - 2Y - 2Z = 0$, meaning the efficient outcome is not unique.

Problem 2. Consider the following game. 'Incumbent' is currently a monopoly that makes profits in its market. Another firm, Entrant, contemplates entering the market. If the new firm enters, the market will be shared and the incumbent firm will earn lower profits if it accommodates the entrant. Another possibility is that when the new firm enters the market, the incumbent will engage in price war which will result in negative profits for both firms. The payoffs are summarized in the matrix below (obviously if Entrant stays out it does not matter what Incumbent's strategy would be in case of entry).

Entrant

		Enter	*Stay Out*
Incumbent	Accommodate	50; *50*	100; *0*
	Fight	-50; *-100*	100; *0*

a) Find all Nash equilibria in this game.

There are 2 pure strategy NE (Accommodate; Enter) and (Fight, Stay Out); there is also a NE in mixed strategies.

b) Represent this game in extensive form and find SPNE

When you draw the game tree it makes more sense to put the Entrant at the root of the tree, so for this part of the question let's re-label Entrant as player one, so his strategies and payoffs will go first into the brackets. For SPNE we have to give list a strategies, remembering that a strategy is a ***complete plan of action*** *- for both players we need to explain what they do he would do at each of his decision nodes.*
SPNE is (Enter if Accommodate, Out if Fight; Accommodate if enter, Accommodate if Out). Which means that Incumbent's strategy is to accommodate no matter what entrant does and Entrant's strategy is to enter.
Notice that the other NE outcome of part (a) is still a NE in a sequential game, but it is not subgame perfect. This NE is (Stay Out if Fight, Enter if Accommodate; Fight if enter, Fight if Out) - this is NE because it is a combination of the BRs: none of the players has an incentive to unilaterally change his strategy, but it's not subgame perfect, because in the subgame when Entrant chooses to enter the market and Incumbent decides to Fight, the latter is not choosing his BR, which does not prevent this situation from being an equilibrium, because the game path goes not go to that decision node, so the Incumbent will never have to carry out this self-damaging strategy (REMEMBER: strategy is just a PLAN of action, not something that will necessarily happen). From the Entrant's perspective, given Incumbent's strategy to Fight in case of entry, the BR is to stay out.

Do **Problem 9** from Ch. 15 p. 542 in Eaton, Eaton and Allen textbook.

Additional Questions

1. Stag Hunt is another famous game. Two hunters decide whether to go hunting a stag or a hare. Stag is much larger than hare, and joint effort is required to kill it. The payoffs are summarized in matrix below.

		Hunter 2	
		Stag	*Hare*
Hunter 1	Stag	5; *5*	0; *3*
	Hare	3; *0*	2; *2*

   a) Find all NE if the game is played simultaneously.

      (Stag, Stag), (Hare, Hare) and one mixed-strategy NE (0.5, 05; 0.5, 05)

   b) Draw the extensive form and find SPNE if the game is played sequentially. Does it matter which Hunter moves first?

      (Stag if Stag, Hare if Hare; Stag if Stag, Hare if Hare) is the SPNE combination of strategies that results in both Hunters going for the stag; it does not matter who moves first because strategies and payoffs are symmetric, so the outcome will be exactly the same. Notice that this is not a game of conflict, so when the game is played sequentially hunters achieve efficient equilibrium.

2. If a game is played sequentially it is always better to be the first to make the decision. Is this true or false? *F. For example, matching pennies or any game of conflicting interest: when the second player picks his BR it results in lower payoff to the first player, who would actually prefer to move second.*

3. If each of the players has a DS, then if the game is played sequentially the SPNE will be the same as NE of a simultaneous game. True or false? *True, DS is a BR to all strategies, so it will be played in each subgame.*

4. Before each class teacher gives students a question to work on. In the beginning of the class the teacher asks whether any one of the students wants to answer the question. If one of the students volunteers to answer, the teacher is happy and everyone enjoys the class - all students get value of $V = 10$. If no one is prepared to answer the question, the teacher gets upset and gives students a quiz, all students get payoff $V = -3$. The time cost of preparing the answer for the question is $C = 5$. Being prepared does not affect the payoff in case there is a quiz - the teacher will ask a different question. There are $N = 5$ students in class. How many pure strategy NE are in this game?

   There are 5 asymmetric NE in which one of the students is prepared and the other are not. It will be extremely time-consuming to build a 5-dimensional payoff matrix in which you'll check every cell to find all NE's it is better to use logic. It makes sense that one *and only one student is prepared in the NE and it does not matter which one out of the five, so there are 5 possible outcomes. Check. The student who is prepared has payoff* $10 - 5 = 5$*, if he changes the strategy his payoff will become -3, so he has no incentive to do that. All other students who are not prepared get payoff 10 and also have no incentive to change strategy to prepare which would lower their payoff to 5. An outcome when all are unprepared is not a NE because given that others are unprepared each person has an incentive to prepare which will increase their payoff from -3 to 5. Any outcome in which more than one student is prepared is not a NE because each of the students who are prepared has an incentive to change his strategy provided the others don't, because in that case their payoff will increase from 5 to 10.*

Just for fun let's try to find a symmetric NE is mixed strategies. Denote probability of being prepared by p, for convenience let's label the probability of being unprepared $x = 1-p$*. In mixed strategy NE p should be such that a student is indifferent between playing any of the pure strategies or a mix. If a student is prepared, there is no uncertainty for him, he gets his payoff* $U_{prep} = 10 - 5 = 5$*. If a student is unprepared, his payoff depends on what other 4 students did. Each of the other students has not studied with probability x; since their choices are independent, the probability that all four other students are not prepared is the product of individual probabilities* $= x^4$*. Probability that at least one other student is prepared is* $1 - x^4$ *(since the payoffs do not depend on how many are prepared, we do not need to look at the probabilities of all possible combinations).* $EU_{unprep} = 10 \cdot (1 - x^4) + (-3) \cdot x^4$.

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\begin{aligned}
EU_{prep} &= EU_{unprep} \\
5 &= 10 \cdot (1 - x^4) + (-3) \cdot x^4 &(1)\\
5 &= 10 - 10x^4 - (-3) \cdot x^4 &(2)\\
5 &= 13 \cdot x^4 &(3)
\end{aligned}
$$

Which solves for $x = (5/13)^{0.25} = 0.7875$*, giving us* $p = 0.2125$*. In a symmetric mixed-strategy NE each student is prepared with probability 21.25 percent.*

5. By now you all know that your grades in ECON classes are curved, so the letter grade you will get in the end of this course depends not only how much you study, but also on how much your classmates study. Do you think that the amount of studying students do in NE is efficient? To answer the question assume an average student cares more about grades than learning[2], so *Payoff = Utility(grade) - Cost(studying).* Which game does this look like? *Prisoners' dilemma; in NE students get the same grades as in the situation when noone studies, but incur the time cost. In the efficient outcome nobody studies - receive the same grades as in equilibrium and save the time. This outcome, however, is not a NE, because if nobody studies each student has an incentive to study and get ahead of the curve.*

6. Problem 7 from Ch. 15 on p.542 in Eaton, Eaton, Allen textbook.

[2]Adding a term that captures utility from studying would complicate the efficiency analysis, that's why I left it out. I do know that many students enjoy learning, and I admit that the assumption is wrong.