Tutorial 9. *Auctions.* *Solutions.*

Question 1. Suppose a variable V is uniformly distributed between zero and one. What is the probability that a V drawn from this distribution will be smaller than 0.7? If we draw two V's from this distribution, what is the probability that both will be smaller than 0.7? What is the probability that three V's drawn from the distribution will be smaller than 0.7?

If a variable is uniformly distributed on [0, 1] interval, the probability that a random draw will result in a number smaller than some X such that $0 \leq X \leq 1$ is equal to X. So the probability that V is going to be smaller than 0.7 is 70%. If two events are independent, the probability that both will happen at the same time is equal to the product of their probabilities, so the likelihood that 2 Vs will be smaller than 0.7 is $0.7 \cdot 0.7 = 0.49$. The probability that we will obtain three values smaller than 0.7 is $0.7^3 = 0.343$. The point of this exercise is to demonstrate what happens to each buyer's probability of winning an auction as number of buyers increases if all buyers use similar bidding strategies.

Problem 1. There are three buyers interested in the object, their names are Alfred, Ben, and Chloe. Their actual true valuations are $V_A = 0.8$, $V_B = 0.3$, and $V_C = 0.5$. All three know that the valuations are independent and are drawn from a uniform distribution between zero and 1.

a) Describe optimal bidding strategies in an English auction. What will be the outcome?

   Start price at zero and increase it continuously. B drops out of bidding at 0.3, Chloe stopes bidding at 0.5, at which point the winner - Alfred and the price $P = 0.5$ are determined.

b) What will be the outcome of a second-price sealed-bid auction?
   Weakly dominant strategy is to bid true V's, so the outcome is the same as in (a).

c) Find symmetric linear Baysian-Nash equilibrium of a first-price sealed-bid auction in which all buyers bid a fixed proportion α of their true valuations.
   Following the logic from the lecture, if a buyer bids b_i and all other buyers bid $b_j = \alpha V_j$, the probability of outbidding one of the other buyers is $\frac{b_i}{\alpha}$. In order to win the buyer needs to outbid both at the same time, so probability of winning conditional on the bid is $(\frac{b_i}{\alpha})^2$. then the expected payoff as a function of bid is $U_i = (V_i - b_i)(b_i/\alpha)^2$; find the first-order condition (notice that αs very conveniently cancel) and find optimal bid $b_i = \frac{2}{3}V_i$. So each will bid 2/3 of their valuation. The bids are $b_A = .53$, $b_B = 0.2$ and $b_C = 0.33$. Alfred wins and pays price .53.

d) What will be the outcome of a Dutch auction?

   Same as (c)

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Problem 2 A seller has an object and he knows that there are five buyers interested in the object. The valuations are uniformly distributed on [0; 1]. Buyers' valuations are private information. What auction format would you recommend to use?

If $N = 5$ numbers are drawn randomly from a uniform distribution between zero and one, the i^{th}-highest number can be calculated using the formula

$$\frac{N - i + 1}{N + 1} = \frac{6 - i}{6}$$

Both second-price sealed-bid and English auctions generate expected revenue equal to the expected second-highest valuation $ER = \frac{4}{6}$.
In the symmetric BNE when bids are linear in valuations with $N = 5$ each buyer will bid $\alpha = \frac{N-1}{N} = \frac{4}{5} = 0.8$ of their true valuation. If there are five potential buyers, on average the highest valuation will be $V_1 = \frac{5}{6}$, therefore the expected highest bid is going to be $E(b_1) = 0.8 \cdot \frac{5}{6} = \frac{4}{5}$.