Auctions.

What is an auction? When and whhy do we need auctions?

Auction *is a mechanism of allocating a particular object at a certain price.*

'Allocating' part concerns who will get the object and the price part speaks for itself. As we will see, there are several ways to design this mechanism, namely there is more than one way to conduct an auction.

Why one needs an auction to allocate an object and to decide on the price? Recall that in this part of the class we are in the realm of asymmetric information. Some players have private information relevant for the decisions of other players, and as rationality requires them, they use that private information to maximize their payoffs. Before we proceed to the auctions, let's take a look at the nature of the private information in the context of trade among people.

First of all we need more than one person interested in the object: there is more than one potential buyer. The private information takes the form of a person's valuation of the object, which is not known to anybody except that person. ***Valuation*** is buyer's maximum willingness to pay, but actually it is simply *the utility that a buyer will enjoy if he or she obtains the object*. When analyzing auctions we will usually assume that the seller has no value of the object, meaning that he will be satisfied with any price that he can get (which, of course, any rational seller will maximize). When valuations are not observable, if a seller meets a buyer, seller has no clue about what that buyer is willing to pay, the only thing he can be sure of is that buyers will always lie about their willingness to pay hoping to pay low price (this means that the bargaining models are no good). If the seller tries to set a reservation price and make take-it-or-leave-it offers, again he will have a problem: unless the price that he asks is very low, some buyers will reject the offer and there will be no trade. What should a seller do? Get a bunch of buyers in a room and make them compete for the price, which is an auction.

A good auction is

- ***Efficient*** - *of all buyers who are in a room, the one with the highest valuation gets the object.*
- ***Revenue Maximizing*** - *extracts highest possible price from the buyer who gets the object.*

Modeling an auction as a game

Since we are in a game theory class we will analyze auctions using game theory language and game theory tools: in this section we represent an auction as a game.

Like all games we studied so far, we have players who selfishly maximize their payoffs: using all the information available to them, they choose their best responses to other player's strategies.

Any game starts with ***rules***. Recall lecture 1. Rules of the game include: players, strategies available for each player, payoffs for all possible combinations of strategies of all players. Asymmetric information adds one more element: each player can be of more than one type and each player's type is his private information.

Players The players are the people who came to the auction. We call them *buyers*, or *bidders*. We can count them and denote the total number of bidders by N. we can also label each person as $1, 2, ..., i, ...N$. For example, if there are only only two bidders, it is written as $N = 2$, but, of course, we can analyze auction with any number of buyers. Note: the auctioneer is not a player, he is third party or a "black box" who collects bids and determines the outcome of the auction.

Player's Types *'Type' is the private information that a player has.* In our context it is buyer's valuation of the object. Buyer 1's valuation is denoted by V_1, Buyer2's valuation is denoted V_2 etc. up to V_N What is valuation? It is the utility the buyer derives from the object ***if*** he gets it, measured in dollars. At the same time valuation is a buyer's maximum willingness to pay for that object: nobody will pay more than an object is worth to him. Why is it private information? Because by looking at a person we cannot tell how much he or she is willing to pay for the object, but they privately know this information.

Nature and Valuations. Formally economists say: "before buyers come to an auction, nature assigns a valuation for each buyer". How to imagine buyers' valuations? Imagine a huge slot machine. The number of wheels is equal to the number of seats (buyers). All possible valuations are written on each wheel. Before the auction starts, Nature plays the slot machine: presses Start and then Stop buttons. The numbers that appear on the slot machine are the valuations of the buyers who came to the auction. With some probability all valuations will be low, then all buyers who came to the auction are not willing to pay too much for the object. With some probability all numbers will be high: all buyers are willing to pay a lot. With some probability all intermediate combinations will occur: there will be buyers with low and high valuations at the same time. Why am I telling you this? Imagine yourself the auctioneer, for you, when you choose an auction format, all you know is the numbers of wheels on the slot machine (the number of buyers) and *their possible* valuations. *Auctioneer has no clue whether* **actual** *valuations of all buyers who came to his auction are high, low, one high and others low, two high and a bunch of low, bunch of high and bunch of low, all valuations are exactly average etc. He expects that any combination of valuations is possible with some probability.* Also buyers have no clue what are the other buyer's valuations, they can only see what is written on the wheel that corresponds to their seat. This description might seem counterintuitive for you. But think again. Suppose the object is a Chinese antique box. When I come to auction I know my valuation, it depends on: do I have appreciation for Asian art (yes, because my grandmother had a nice collection that fascinated me as a child), do I want a box right now (yes, because I have lots of jewelry and I want a nice storage for it), do I find the shape of this box aesthetically appealing (not, because I think it's too square). So in a way my valuation of this object is determined by nature: my subjective preferences and random experiences that shaped my perception. If my experiences were different, my valuation would be different.

Knowledge *Each buyer knows for sure his own valuation and the number of buyers who participate* *in the auction, but not the exact valuations of the other buyers.* What else does he know? Each buyer *knows the distribution of valuations: range of all possible valuations for the object and the probability with which each valuation happens.* Why does he need to know the distribution of the valuations? Because he has to form some expectations about bids of other buyers to choose his own optimal bid. An example of a distribution is: all possible valuations range from 1 to 100 and each valuation is equally likely; then when each buyer looks at a person who sits next to him and only knows that: 'my neighbor's valuation is 1 with probability .01, his valuation is 2 with probability 0.01, etc, his valuation is 100 with probability .01 and the expected (on average) valuation for this buyer is 50' (recall that this is a uniform distribution: all values in the range are equally likely, so we'd say that V's are uniformly distributed on the interval [1, 100]).

Strategies Each buyer i will place his bid b_i hoping to get the object at price below V_i. *so, in this context the strategies are the bids.* 'Bid zero' is one strategy, that we can write as $b_i = 0$ 'bid 1 cent', is another strategy, etc. Out of all possible bids the buyer will choose the one that will maximize his expected payoff given the expectations about strategies (bids) of the other buyers (remember in equilibrium all players play best responses to each other strategies, same here). Keep in mind that although optimal bid depends on buyer's valuation, it does not have to be equal to the true willingness to pay. If a buyer wants to bid more or less than his valuation, there is nothing that prevents him from doing so.

Payoffs After the game is over the players receive their payoffs that are determined by the strategies that all players chose to play. Same thing here: depending on all bids submitted (strategies played by all players), the auctioneer will determine the winner and announce some price, P. One buyer will win the auction and get the object. What is his utility? His utility is his private valuation that he will get to enjoy with the object minus the price he has to pay. *If* buyer indexed i *wins* the auction, then his utility, denoted U_i is

$$U_i = V_i - P$$

It is a very simple linear thing and *it will be exactly the same equation for all auction formats.*

For all other buyers, the losers, the utility is equal to zero: they don't get to enjoy having the object but also do not pay anything (the only exception is all-pay auction in which all buyers pay regardless of whether they win or lose).

Outcome What do we care about? Who got the object and how much he paid, the *winner and the price.* The outcome *depends on the bids placed by all buyers.* In all auctions that we will look at in this class the winner is the buyer who submitted the highest bid. The price is determined differently in different auction formats.

Bayesian Nash Equilibrium. What happens when all players maximize their payoffs? We get some kind of equilibrium, it is our familiar Nash Equilibrium (all player choose strategies that maximize their payoffs given the strategies of other players). When information is asymmetric, the equilibrium is called Bayesian Nash Equilibrium (BNE).

Not very formal definition of BNE. *A **Bayesian Nash Equilibrium** is a combination of players' strategies such that*

- *All players* (all N buyers)
- *All types* for each player, (for any possible valuation that each buyer has)
- *Maximize their expected payoffs* (expected because they are not sure about what other players will do and what payoff they will actually get in the end)
- When maximizing expected payoffs, *each player forms beliefs* about what are the strategies of other players (each buyer does not know what other buyers will bid, and therefore has to make some projections about that).

Classification of auctions.

There are several ways to auction an object off that are actually used. One can classify auctions according to the *bidding rules* and by the *nature of the object* for sale.

Nature of object: there are 2 types of objects. Objects that have private value and objects that have common value.

- An object has a ***private value*** *if each buyer's valuation is independent of how other buyers value the object.* Example: artwork by not so famous artists. When you look at a painting of that sort, your willingness to pay for it is determined solely by how much it aesthetically pleases you and to what extent you want to see it on your wall every day. Your valuation is 'private and independent' because other buyers can value it quite differently, they might not like it as much as you do, or they might really like it a lot. Also from how much you value it yourself, you absolutely cannot predict how much other people value it, because values are subjective.
- An object is ***common value*** *if the value of the object is exactly the same for all buyers.* A perfect example of a common value object is an envelope with money, whichever buyer gets the envelope, his value of the object will be exactly the same and equal to the number of monetary units contained in the envelope. A more realistic example is an oil field, the amount of oil in the field is fixed and all buyers' value of the field amounts to the profits they can extract from it. Why do we have an auction then? Because usually in case of objects like oil field or a firm, buyers cannot know the exact value of the object, they do some research about how much oil is in the field (how potentially profitable is the firm) and form their expectation about the value of the object and place bids according to their expectations. Since buyers' estimates are not 100% accurate, they will make mistakes. When object is common value, it can be the case that the winner pays more than the object is worth $P > V$ and gets negative utility $U = V - P < 0$, this is what is called ***winner's curse***[1].
- When you see a problem, how do you know it's common or private value object? Very easy. Look at buyers' valuations: if all buyers have same (common) valuation, then it's a common value object. If all buyers have different valuations, then it's private value object.

As for the bidding procedure, there are four standard auction formats (Note: any private/common value object can be auctioned off using any auction format). The system works

[1]For more detail on how it works, see tutorial problem

as follows:

Can buyers observe each other's bids?

Yes. ***Open*** outcry (oral) auction. In open auctions buyers bid by raising hand or sending some other kind of observable signal that not only the auctioneer, but also all other buyers can perfectly see.

No. ***Sealed-bid***auctions: buyers submit their bids to auctioneer in sealed envelopes, so that buyers cannot observe each other's bids whatsoever.

In all actions the buyer who submits the *highest bid gets the object.* But auctions differ in terms of the price that is charged for the object. What is the price that the winner has to pay?

Own bid. If the winner (the buyer who bid the highest) has to pay price equal to his bid, then it is called ***first-price*** auction: $P =$*winning bid.*

Second highest bid (not winner's own bid). If price is equal to second-highest bid, then it is called ***second-price*** auction. $P = 2^{nd}$ *highest bid*, below the winning bid.

In their search for optimal auction format, people used all possible combinations of the above. The four standard auction formats are:

English auction is second-price oral auction.

Second -price sealed-bid auction, speaks for itself.

First -price sealed-bid auction.

Dutch is oral first-price auction.

Let's look at detailed rules, find optimal bidding strategies and outcome for each auction format. Keep in mind that when we decide on the optimal bid we imagine ourselves as one of the buyers who only knows his own valuation, number of buyers, and the distribution of V's.

English Auction

Is the most popular and most widely used auction format.

Bidding Rules. A standard textbook description goes as follows. The auctioneer starts at a very low price and calls out successively higher prices. That is, he calls a low price and several buyers raise hands to show that they are willing to pay that price. Then the auctioneer raises price by a certain small amount (fixed increment), if several buyers accept this new, higher price, auctioneer raises price again etc. The process goes on until auctioneer increases a price and does not receive any more bids. Alternative procedure applied in US used cars auctions is: buyers call out prices without any guidance from the auctioneer, the auction ends when no higher prices are called. In any case, auctioneer gives the object to the buyer who placed the last bid (*highest bid wins*), gives object to that buyer and charges that last accepted price. Why is it called second-price? I think because technically the buyer with the highest valuation in the room is waiting for other buyers to drop-out and gets the object when the person with the second-highest valuation (places second-highest bid) stops bidding, and that is how the price is determined: *the final price is the price at which the second-highest bidder stops bidding.*

Optimal Bid. We have to find optimal bids for all N buyers, each of whom has his own private valuation V_i. In this auction format optimal bidding behavior is quite straightforward. Imagine Alfred is one of the N buyers and his valuation is $V_A = 50$. Suppose price starts at \$1 and then is increased by one dollar as long as auctioneer receives bids. Alfred can see the price and comppare it to his willingness to pay, 50 dollars. As long as the price does not exceed his valuation, he wil place bids. As soon as price of 51 dollars is announced, Alfred stops bidding and drops out. Why is it simple: because number of other buyers does not matter for Alfred and also he does not have to know other buyer's valuations to find his optimal bid. ***In English auction each buyer places bids as long as price (current bid which is observable) is below his valuation. As soon as price exceeds buyer's valuation, he stops bidding and drops out.*** *Notice that this strategy is dominant.*

Outcome. *Winner is the buyer with the highest valuation, price is determined by the second-highest valuation and is equal to the second-highest valuation).* [2].

[2] Some textbooks show that the price will be second-highest valuation plus the fixed increment. On the exams you can choose whatever reasoning is more appealing to you as long as you are consistent, you will get full marks

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Second-price Sealed Bid Auction.

Bidding Rules. All buyers take a piece of paper and write down their willingness to pay for the object and submit the sealed envelopes to the auctioneer. Notice: buyers can write down numbers as they please, nobody can make them write their true willingness to pay and any buyer can bid below or above his true valuation. After all buyers submit the envelopes with the bids, auctioneer opens all envelopes. The winner is the buyer who placed the highest bid. The price the winner has to pay is the second-highest bid.

Optimal Bid. Again imagine Alfred is one of the N buyers and his valuation is $V_A = 50$. Can we advise him how much to bid? In general Alfred has three strategies available to him: bid something below 50, bid something above 50, and bid exactly 50. Let's carefully examine whether one of these strategies is better than the others.

Bid below 50. Is the first possible strategy. What can happen if Alfred bids something below his true valuation? Suppose he bids 25 dollars. There are 3 possible scenarios that depend on bids of other buyers.

- All other buyers submitted bids below 25. Then Alfred's *bid* is the highest, he wins the auction and pays some price below 25 (because all other bids, including second-highest were below 25). Alfred gets positive utility and is very happy with bidding 25.
- One of the other buyers submitted a bid above 50, suppose 75. It means that there was a bid not only higher that Alfred's bid, but also higher than Alfred's *valuation*. Does Alfred want to outbid those buyers if he could? No. Because then he would have to place a bid above 75, and pay 75 dollars, his utility would have been negative, so Alfred is still happy with bid of 25 (does not wish he placed a higher bid).
- The highest bid submitted by other buyers was between 25 and 50. Suppose highest bid was 30. Does Alfred get the object? No, because his bid is not the highest. Is Alfred happy with his bid of 25? No. If he placed a bid above 30, he could win the object and still pay 30 dollars and get positive utility. *Therefore: If Alfred bids below his true valuation, there is a possibility that he will not get the object at a price below his valuation (forgone positive utility).*

Bid above 50. is the second possible strategy. What can happen if Alfred bids something above his true valuation, suppose 75 dollars? Again, there are 3 possible scenarios that depend on bids of other buyers.

- All other buyers submitted bids below 50. Then Alfred's *bid* is the highest, he wins the auction and pays some price below 50 (because all other bids, including second-highest were below 50). Alfred gets positive utility and is very happy with bidding 75.
- One of the other buyers submitted a bid above 75, suppose 95. It means that there was a bid not only higher than Alfred's bid, but also higher than Alfred's *valuation*. Does Alfred want to outbid 95 if he could? No. Because then he would have to place a bid above 95, and pay 95 dollars, his utility would have been negative, so Alfred is still happy with bidding 75 and losing the auction.

- The highest bid submitted by other buyers was between 50 and 75, suppose 65. Does Alfred get the object? Yes, because his bid is the highest. Is Alfred happy with his bid of 75? No. Because the price is equal to 65 and is above Alfred's valuation, Alfred gets negative utility. *Therefore: If Alfred bids above his true valuation, there is a possibility that he will get the object at a price above his valuation and receive negative utility.*

Bid exactly 50. is the third possible strategy. What can happen if Alfred bids 50 dollars? Again, there are 3 possible scenarios that depend on bids of other buyers.

- All other buyers submitted bids below 50. Then Alfred's bid is the highest, he wins the auction and pays some price below 50 (because all other bids, including second-highest were below 50). Alfred gets positive utility and is very happy with bidding 50.
- One of the other buyers submitted a bid above 50, suppose 75. It means that there was a bid higher than Alfred's *valuation*. Does Alfred want to outbid 75 if he could? No. Because then he would have to place a bid above 75, and would pay 75 dollars, his utility would have been negative, so Alfred is still happy with bid of 50.
- If the highest bid of all other buyers was exactly fifty - then there are two highest bids of 50. One of the buyers will get the object and pay price of 50, get zero utility and be indifferent between winning and not winning. Is Alfred happy with bidding 50 dollars in this case and does he want to bid 51 dollar and win? Even if Alfred placed bid of 51 dollar, he would get the object and still pay 50, and get zero utility, wining in this case does not make him any happier than losing. But also Alfred does not know that the second highest bid is fifty, could be 50.5 and then Alfred would win and overpay by 50 cents. So in this case, when the second highest bid is also 50, there is no other bid that makes Alfred any happier than bidding his true valuation.

Finally, compare the how well all of the three strategies do compared to each other: bidding below true valuation some times is not a good idea; bidding above the true valuation is also sometimes not a good idea. Bidding exactly the true valuation is always a good idea because whatever are all other bids, Alfred never pays too much and always gets the object when the price (second-highest bid) is below his true valuation. ***Therefore, in the second-price sealed-bid auction each buyer has a weakly dominant strategy to bid exactly their true valuation, whatever it is.*** To be totally precise, it is a weakly dominant strategy: bidding truthfully is never worse and sometimes is better than bidding above/below the true valuation.

Outcome. *Winner is the buyer with the highest valuation, price is equal to the second-highest valuation.* How do we know? Because buyers have a dominant strategy, we know that if buyers are smart, then the bids that auctioneer actually collected are the same as the true actual valuations of the buyers. Since the highest bid is equal to the highest valuation, then the object goes to the person with the highest valuation. Since the price is equal to the second highest bid and second highest bid is equal to the second highest valuation, by simple logic price is equal to the actual true second-highest valuation.

So far we analyzed two auctions. Let's compare them. If you look at the outcomes, the outcomes are the same (winner and price in terms of buyers' valuations). What does it mean for you? It means that if you take a room with buyers with private valuations, you will get exactly the same winner and exactly the same price regardless whether you conduct English or second-price auction. The only difference is that in English auction you will never know what was the highest valuation, but in the second-price sealed bid auction you can see what was the highest valuation among all the buyer who were participating in the auction. A few more comments. Both these auctions are easy on buyers: notice that when we derived optimal bids we did not use the number of buyers, buyers optimal bids would be exactly the same regardless there are 2 or 10 000 potential buyers. Also when we derived each buyer's optimal bid we did not use any information about possible valuations of other buyers. The buyers did not have to form beliefs about what would be the bids of other buyers.

First-price Sealed Bid Auction.

Bidding Rules. All buyers take a piece of paper and write down their bids, or how much they are ready to pay for the object and submit the sealed envelopes to the auctioneer. Notice: buyers can write down whatever they want any buyer can bid below or above his true valuation. After all buyers submit the envelopes with the bids, auctioneer opens all envelopes. The winner is the buyer who placed the highest bid. The price the winner has to pay is equal to his bid.

Optimal Bid. Now buyers have no clue what to do. It is clear what they should not do: they should not bid their true valuations. Why not? Because if a buyer wins, the price is equal to his own bid, if he bids his true valuation his is sure to get zero utility. So buyers should bid something lower than their true valuation, but how much lower? The problem is that *there is a trade-off that buyers face when choosing their bid in a first-price auction. The lower is the bid, the higher is the difference between buyers valuation and the price he will pay if he wins. But at the same time decreasing the bid gives a buyer smaller chance of winning the auction at all. The optimal bid balances these two effects.* In terms of math, the optimal bid maximizes buyer's expected utility:

$$EU = (Valuation - bid)(probability - to - win)$$

How does a buyer know his probability to win? He has to form beliefs of what kind of bids and with what probabilities will be placed in the auction. It also depends on the number of buyers, because to win you have to place a bid higher than all other $N-1$ bids.

For example, if we have $N = 2$ named Alfred and Ben with V's uniformly distributed on [0,1]. Suppose for each of them his bid is a linear function of the his valuation of the object $b_i = \alpha V_i$, or the buyer bids α percent of his reservation price. Also each buyer believes that all buyers will choose exactly the same α. Let's find Alfred's optimal bid. Alfred wins if $b_a > b_B$, notice that $b_B = \alpha V_B$. Alfred wins if $b_A > \alpha V_B$, the probability of which is b_A/α. Then Alfred' expected utility is $U_A = (V_A - b_A)(b_A/\alpha)$. To maximize, take the first-order conditions and find that $b_A^* = \frac{V_A}{2}$. Similarly, $b_B^* = \frac{V_B}{2}$. which is the BNE. Recall the definition of BNE, what we just found is:

- if there are only two buyers, Alfred and Ben (for all players)
- whatever are Alfred's and Ben's true valuations, V_A and V_B (for all types of each player)
- given their beliefs about each other's strategies (given their beliefs of what the other guy will bid depending on his valuation) It is an equilibrium (situation when all players maximize their expected payoffs given the strategies of other players) for them to bid exactly 1/2 of their valuations. Of course, when Alfred and Ben come to an auction they have a certain valuation and place a certain bid. If there are N bidders $U_i = (V_i - b_i)(b_i/\alpha)^{(N-1)}$, where $(b_i/\alpha)^{(N-1)}$ is the probability that all other buyers' valuations are smaller than V_i and therefore the submitted bids are lower. This solves for optimal bid $b_i = \frac{N-1}{N}V_i$. Notice, that the more buyers participate in the auction, the closer is the optimal bid to the true valuation.

In first-price sealed bid auction each buyer's optimal bid is equal to some fraction of the buyer's true valuation (means that optimal bid is always below the valuation). As the number of buyers goes up, chances of winning for each buyer go down, therefore, buyers should bid a higher proportion of their valuation. Buyers' bids depend on buyers beliefs about other buyers' bids and also on the number of buyers.

Outcome. *Winner is the buyer with the highest valuation, price is equal to some fraction of the highest valuation.* How do we know? Because bids are proportional to buyers' valuations and the higher is the valuation, the higher is the bid. Therefore, buyer with the highest valuation will submit the highest bid and will win. Price is equal to the highest bid and it is some proportion of the highest valuation.

Dutch Auction.

Bidding Rules. Price starts really high, hopefully above the highest possible valuation. All buyers observe the price going down. The auction ends when one of the buyers accepts the price and wins the object at that price. Why is it a first-price auction - because you can think of the price as your own bid that you will have to pay once you accept it.

Optimal Bid. Again each buyer knows his own valuation, but not the valuations of the other buyers. The thing not to do is not to take the object as soon as price reaches your valuation, because then you will pay the price equal to your valuation and get zero utility. You should wait. How long to wait? Same as in the first-price auction it depends on your beliefs about when will somebody else decide to take the object. The optimal price at which you should place your bid balances the difference between your valuation and the price and the risk of loosing the object. Technically, you can think about this price same way you think about the bid in the first-price sealed bid auction. Math is exactly the same and logic is exactly the same.

Outcome. *Winner is the buyer with the highest valuation, price is equal to some fraction of the highest valuation.* How do we know? Because the buyer with the highest valuation will be the most impatient since he has the most to lose in the case he does not get the object. The price at which he will take the object depends on his (the highest) valuation and the number of buyers. Why does the number of buyers matter? Because the more buyers who are interested in the object the higher are the chances of losing the object if you wait too long.

Compare first-price sealed-bid auction and Dutch auction. They are strategically identical in the sense that the buyer with the highest valuation will win, his bid will be equal to some proportion of his valuation. The optimal bids of buyers depend on their beliefs about the behavior of other buyers and also the number of buyers participating in the auction.