Singular Solutions and the Spectrum of a Zeroth-order Pseudo-differential Operator

Cascade RAIN Meeting 2020

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Joint work with

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Motivation: Internal Waves

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Figure: Picture taken from the International Space Station (ISS) in February, 2013. It shows the north coast of the island of Trinidad in the southeastern Caribbean Sea and the huge internal waves that are visible in the top left. Image credit: NASA/JPL.

- Internal waves play an important role in ocean circulation. Generated by tides and winds, they propagate through the oceans and seas, redistributing momentum and energy before dissipating.
- Topography of the ocean floor may play a role in dissipation and generation of internal wave attractors.

Motivation: Internal Waves

- Usual simplification: non-rotating, linearly stratified fluid.
- Extensively studied in 2D: Maas & Lam ’95, Ogilvie ’05, Lam & Maas ’08, Rieutord, Georgeot & Valdettaro ’01.
- And in 3D: Manders & Mass ’03, Drijfhout & Maas ’07, Pillet et. al. ’18.

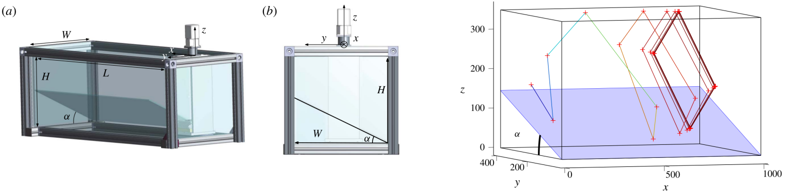

Figure: **L:** Experimental setting. **R:** A internal ray beam path leads to an attractor in a plane transverse to the along-slope, down-canal direction into which rays are initially launched. Source: Pillet et. al., JFM 2018.

Motivation: Internal Waves

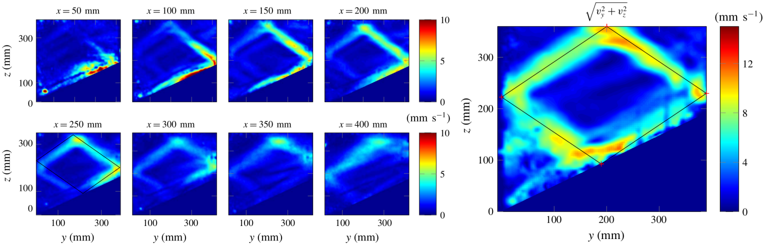

Figure: **L:** (Filtered) Velocity amplitudes at different slices of the tank obtained by Particle Image Velocimetry (PIV). **R:** The attractor at $x = 250$ mm, with the superimposed ray tracing prediction. Source: Pillet et. al., JFM 2018.

Forced Internal Waves in a Periodic Tank

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Main ingredients and simplifications:

- Linearly stratified fluid with density $\overline{\rho}$, small perturbations η,
- Incompressible fluid ($\nabla \cdot \mathbf{u} = 0$) and no-flux BC,
- (Linearized) Conservation of mass:

$$\partial_t \eta + \mathbf{u} \cdot \nabla \overline{\rho} = 0,$$

- (Linearized) Conservation of momentum:

$$\overline{\rho}\, \partial_t \mathbf{u} + \eta g \mathbf{e}_3 + \nabla \Pi = \mathbf{F} e^{-i\omega_0 t},$$

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- Domain: 2-torus
- Symbol: $\widetilde{p}(z, \xi) \to \frac{\xi_3}{|\xi|} - r\beta(y, z)$ (to account for topography).

$$i\partial_t u - P(x,D)u = fe^{-i\omega_0 t} \quad \text{in } \mathbb{T}^2 \times (0,T], \qquad u\big|_{t=0} = 0,$$

- $u = u(x,t)$, periodic BC
- $P(x,D)$ self-adjoint pseudo-differential operator,
- The symbol $p(x,\xi)$ is homogeneous of degree 0, of form

$$p(x,\xi) = \frac{\xi_2}{\sqrt{1+|\xi|^2}} - \text{periodic function.}$$

We would like to figure out:

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The Problems of Interest

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- What is the long-time behaviour of solutions?
- What is the spectrum of P?
- Does the spectrum of P affect the long-term evolution?

Discretization Techniques

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- Fourth-order explicit method in time,
- Fourier collocation method in space,
- In particular, the operator $Q(x, D)$ of symbol $q(x, \xi) = (1 + |\xi|^2)^{-1/2}\xi_2$ is discretized as

$$Q(x, D)w \approx \mathcal{F}^{-1}\left(\texttt{diag}\left(\frac{k_2}{\sqrt{1 + |k|^2}}\right)\mathcal{F}w_N\right)$$

where $\mathcal{F}$ is the discrete Fourier transform and w_N corresponds to w evaluated at the N^2 grid points in real space.

The Evolution Problem

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Theorem (Colin de Verdière-Saint Raymond '20, Dyatlov-Zworski '20)

Let P be a self-adjoint, pseudo-differential operator of order zero and $\omega_0 \notin Spec_{pp}(P)$. Then, under certain additional assumptions on P, for any $f \in C^\infty(\mathbb{T}^2)$, the solution u to the evolution problem satisfies

$$u(t) = e^{-i\omega_0 t} u_\infty + b(t) + \epsilon(t),$$

where

- $u_\infty = \lim_{\varepsilon \to 0}(P - \omega_0 - i\varepsilon)^{-1} f \in \mathcal{H}^{-1/2-} := \cap_{\delta>0} \mathcal{H}^{-1/2-\delta}(\mathbb{T}^2)$ and **is not** in $L^2(\mathbb{T}^2)$ except if it vanishes,
- $\| b(t) \|_{L^2(\mathbb{T}^2)} < \infty$,
- $\epsilon(t) \to 0$ as $t \to \infty$ in $H^{-1/2-}$.

The Evolution Problem

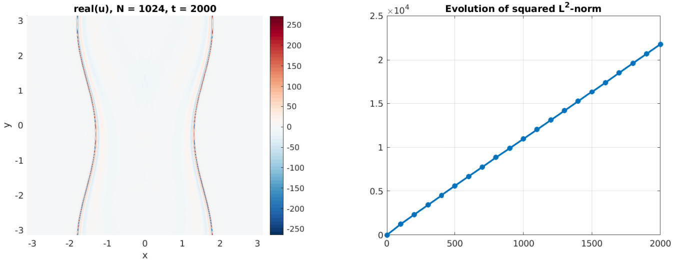

Figure: Attractors and linear evolution of the squared L^2-norm. The forcing is a centered Gaussian, $r = 0.25$ and $\beta(x) = \cos(x_1) + \sin(x_2)$. Video available at youtu.be/3b5UQfRcyEk.

The Evolution Problem

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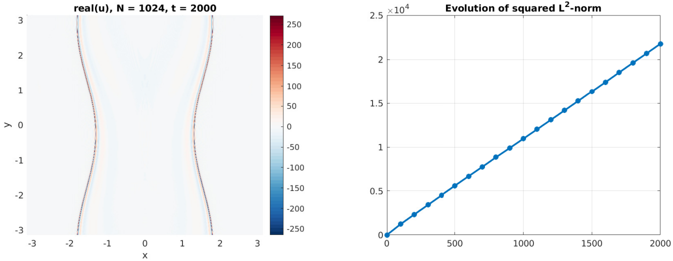


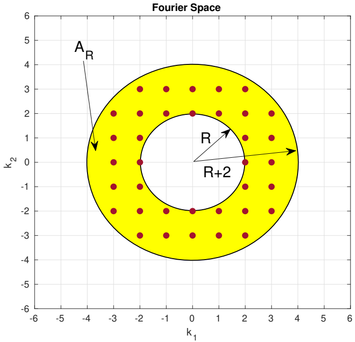
Figure: Attractors and linear evolution of the squared L^2-norm. The forcing is a centered Gaussian, $r = 0.25$ and $\beta(x) = \cos(x_1) + \sin(x_2)$. Video available at youtu.be/3b5UQfRcyEk.

Theorem (cont.)

Moreover, for $t > 0$, $\| u(t) \|^2_{L^2(\mathbb{T}^2)} \sim ct$ except if u_∞ vanishes.

- How the Fourier coefficients of $u_N(t)$ decay? $\longrightarrow$ Radial Energy Density

$$G_s(R) = \frac{1}{N^2} \sum_{k \in A_R \cap \mathbb{Z}^2} (1+|k|^2)^s \, |\widehat{u_N}(k)|^2, \quad A_R = \left\{ x \in \mathbb{R}^2 : R \leq |x| < R+2 \right\}.$$



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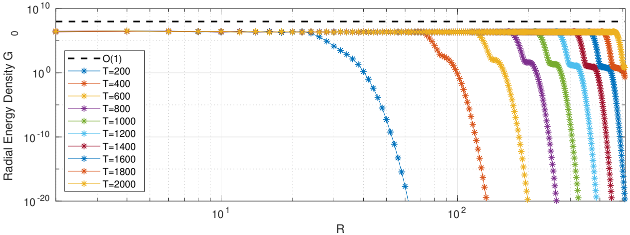

Figure: If a distribution $v \in \mathcal{H}^s$, then the radial energy density G_s should decay faster than $\mathcal{O}(R^{-1})$ as $R \to \infty$.

The Evolution Problem

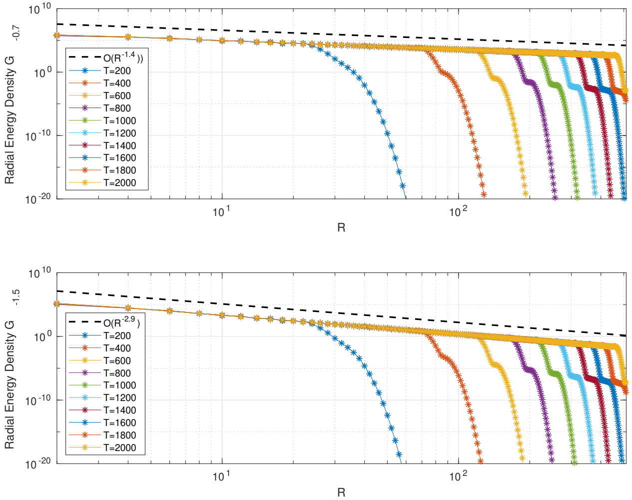

Figure: Radial energy density as $T \to \infty$. Top: $s = -0.7$. Bottom: $s = -1.5$.

Spectrum of P

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Theorem (Colin de Verdière '20)

Let $J := [\min p, \max p]$, where p is the symbol of P. Then,

1. $\sigma_{\text{ess}}(P) = J$,
2. P has a finite number of eigenvalues in J and they have finite multiplicity.

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Theorem (Galkowski-Zworski '19)

Under certain assumptions on P, there exists an open neighbourhood of 0 in $\mathbb{C}$, U and a set

$$\mathcal{R}(P) \subset \{\text{Im}(z) \leq 0\} \cap U$$

such that for every compact set $K \subset U$, $\mathcal{R}(P) \cap K$ is discrete, and

$$\text{spec}_{L^2}(P + i\nu\Delta) \longrightarrow \mathcal{R}(P), \quad \nu \to 0+,$$

uniformly on K. Moreover,

$$\mathcal{R}(P) \cap \mathbb{R} = \text{spec}_{pp,L^2}(P) \cap U.$$

Spectrum of P

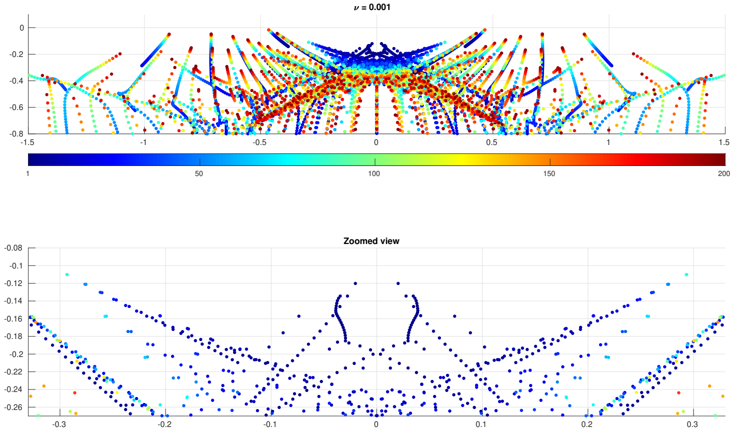

Figure: Superposition of the first 200 eigenvalues (ordered by magnitude, then phase) for several values of ν decreasing until $\nu = 10^{-3}$. Each color represents a position in the vector of eigenvalues. As ν decreases, eigenvalues move up to the real axis. Video available at youtu.be/qeNRxWSptuo.

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$$p(x,\xi) = \frac{\xi_2}{\sqrt{1+|\xi|^2}} - 0.5\left(\cos(x_1)+\sin(x_2)\right)$$

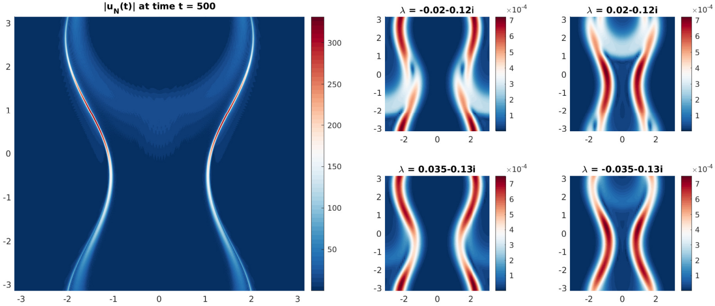

Figure: **L:** Solution to the evolution problem. **R:** First 4 eigenfunctions of the viscous approximation, $\nu = 10^{-3}$.

Viscous Approximations

$$p(x,\xi) = \frac{\xi_2}{\sqrt{1+|\xi|^2}} - 2\cos(x_1)$$

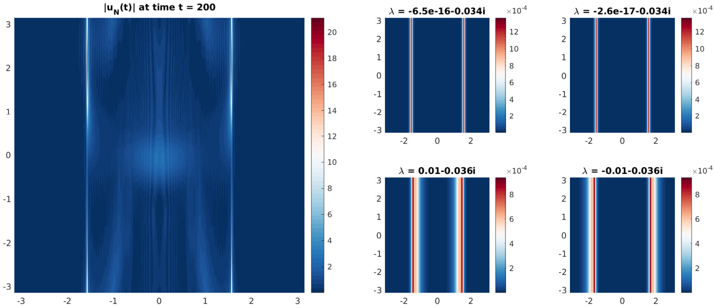

Figure: **L:** Solution to the evolution problem. **R:** First 4 eigenfunctions of the viscous approximation, $\nu = 10^{-4}$.

Viscous Approximations

$$p(x,\xi) = \frac{\xi_2}{\sqrt{1+|\xi|^2}} - 0.55\left(\cos(x_1 - 2x_2) + \sin(2x_2)\right)$$

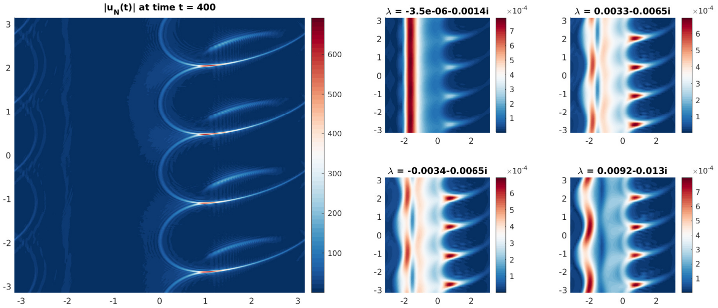

Figure: **L:** Solution to the evolution problem. **R:** First 4 eigenfunctions of the viscous approximation, $\nu = 1.5 \cdot 10^{-4}$.

Conclusions

- An evolution problem with solutions that live in Sobolev spaces of negative exponent,
- Numerical evidence that some of the eigenmodes in the small-viscosity limit converge to embedded eigenmodes, which impact long-term evolution,
- A lot remains to be said at a theoretical level!

References

- Y. Colin de Verdière and L. Saint-Raymond, *Attractors for two dimensional waves with homogeneous Hamiltonians of degree 0*. Commun. Pure Appl. Anal. 73 (2020), no. 2, 421–462.
- Y. Colin de Verdière, *Spectral theory of pseudo-differential operators of degree 0 and application to linear forced waves*. arXiv:1804.03367, to appear in Anal. PDE (2020).
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- G. Pillet, E.V. Ermanyuk, L.R.M. Mass, I.N. Sibgatullin and T. Dauxois, *Internal wave attractors in three-dimensional geometries: trapping by oblique reflection*. J. Fluid Mech. 845 (2018), 203–225.

References

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Thank you!