SFU SIMON FRASER UNIVERSITY

Internal Wave Attractors and Spectra of Zeroth-order Pseudo-differential Operators

M.Sc. Thesis Defence | Mathematics

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Co-supervisor: Weiran Sun

Photo: Romeiser & Graber. IEEE T. Geosci. Remote (2015)

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Motivation: Internal Waves in Stratified Media

Internal Waves in a Stratified Medium

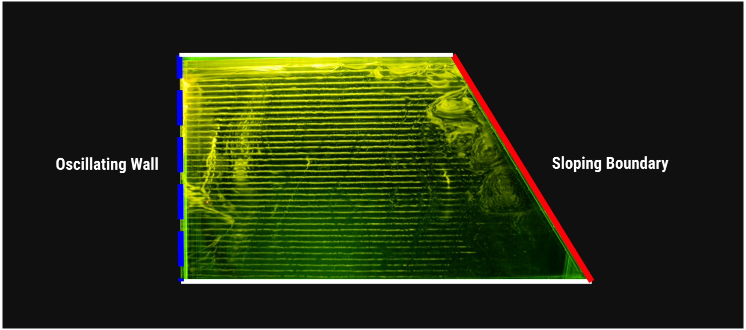

Figure: Benchmark experiment[1] for capturing internal waves. Video: Local | YouTube

[1] G. Davis. et al. *Phys. Rev. Lett.* 124.20 (2020).

A pseudo-differential approach

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- Fluid assumed to be incompressible, stably-stratified, non-rotating.
- Base model: conservation of mass and momentum + Boussinesq approximation.
- Density ρ and pressure Π evolve as perturbations of a hydrostatic state:

$$\rho(x,t) = \overline{\rho}(x_3) + \rho'(x,t),$$
$$\Pi(x,t) = \overline{\Pi}(x_3) + \Pi'(x,t).$$

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_0}\nabla\Pi' - \frac{\rho' g}{\rho_0}\mathbf{e}_3 \quad \Big/ \nabla\cdot$$

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Diagonalization

$$\partial_t \begin{pmatrix} \widehat{\rho'} \\ \widehat{u_3} \end{pmatrix} + \begin{bmatrix} 0 & -\frac{d\overline{\rho}}{dx_3} \\ -\frac{g}{\rho_0}\left(1 - \frac{\xi_3^2}{|\xi|^2}\right) & 0 \end{bmatrix} \begin{pmatrix} \widehat{\rho'} \\ \widehat{u_3} \end{pmatrix} = 0$$

Here $\mathbf{u} = (\mathbf{u}_H, u_3)$. Assume: $\mathbf{u} = u_+\mathbf{e}_+ + u_-\mathbf{e}_-$. Then, $u_\pm$ solve

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$$i\partial_t u_\pm \mp \mathcal{P}(D)u_\pm = 0,$$

where $\mathcal{P}(D)$ is the zeroth-order pseudo-differential operator of symbol

$$p(\xi) = \sqrt{-\frac{g}{\rho_0}\frac{d\bar{\rho}}{dx_3}\frac{\xi_3}{|\xi|}}, \quad \xi \in \mathbb{R}^3.$$

Microlocal Analysis of Linear Forced Waves

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- Physics observations[2] suggest that internal wave attractors make the fluid flow highly singular.
- The fact that they develop for a wide range of frequencies suggests the existence of a continuous spectrum.

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[4] L.R.M. Maas. *Int. J. Bifurcat. Chaos* 15.9 (2005).

[5] Y. Colin de Vèrdiere & L. Saint-Raymond. *Comm. Pure Appl. Math.* 73.2 (2020).

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- Physics observations[2] suggest that internal wave attractors make the fluid flow highly singular.
- The fact that they develop for a wide range of frequencies suggests the existence of a continuous spectrum.

- Colin de Verdière & Saint-Raymond[3] (2018) give—for the first time—a mathematical explanation of these phenomena.
- They analyze equations of the form

$$iu_t - Pu = f \quad \text{in } \mathbb{T}^2 \times (0,\infty),$$

where P is a zeroth-order pseudo-differential operator, e.g.,

$$P(x,D) = \langle D \rangle^{-1} D_{x_2} - \frac{1}{2}\cos(x_1).$$

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Main Questions

Let $D_{x_j} := -i\partial_{x_j}$ and $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$. For a particular class of operators P acting on elements of $\mathrm{L}^2(\mathbb{T}^2)$:

$$P(x, D) = \langle D \rangle^{-1} D_{x_2} - r\beta(x), \quad r > 0, \quad \beta \in C^\infty(\mathbb{T}^2),$$

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- How do attractors relate to the energy surfaces on which the flow takes place?

We will answer these questions using **numerical techniques**.

Up Next | **The Evolution Problem**

The Evolution Problem

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Problem 1

Find a complex function $u = u(x,t)$, $x = (x_1, x_2) \in \mathbb{T}^2$, $t \geq 0$ such that

$$iu_t - Pu = f \quad \text{in } \mathbb{T}^2 \times (0, \infty), \qquad u\big|_{t=0} = 0.$$

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$$P(x,D)v(x) = \langle D \rangle^{-1} D_{x_2} v(x) - r\beta(x)v(x) = \mathcal{F}^{-1}\Big\{\langle \xi \rangle^{-1}\xi_2\,\widehat{v}(\xi)\Big\} - r\beta(x)v(x),$$

where $\mathcal{F}\{v\} \equiv \widehat{v}$ is the Fourier transform.

What to expect?

What to expect?

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Theorem (Colin de Verdière & Saint-Raymond (2018), Dyatlov & Zworski (2019))

Let $f \in C^\infty(\mathbb{T}^2)$, 0 not an eigenvalue of P. Then, the solution to the evolution problem can be uniquely decomposed as

$$u(t) = u_\infty + b(t) + \epsilon(t),$$

where

1. $\lim_{\varepsilon\to 0}(P - i\varepsilon)^{-1}f =: u_\infty \in \mathrm{H}^s(\mathbb{T}^2)$ *for any* $s < -1/2$ *and is not in* $\mathrm{L}^2(\mathbb{T}^2)$ *unless it vanishes,*
2. $b \in \mathrm{L}^2(\mathbb{T}^2)$ *and is bounded,*
3. ϵ *vanishes as* $t \to \infty$ *in the* $\mathrm{H}^s(\mathbb{T}^2)$*-norm, for any* $s < -1/2$.

Moreover, the energy $\| u(t) \|_0^2$ *grows linearly except if* u_∞ *vanishes.*

Discretization

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- Pseudo-spectral method in space: (F = discrete Fourier transform)

$$i\widehat{u}_t - \langle \xi \rangle^{-1}\xi_2\widehat{u} + r\,F\left\{\beta\,F^{-1}\widehat{u}\right\} = \widehat{f}, \quad \xi_j = -\frac{N}{2},\ldots,\frac{N}{2}-1, \quad t > 0.$$

- Runge-Kutta-type schemes in time:
  1. Classical RK4: yields a time-step restriction $\Delta t \leq 2.8$,
  2. ETDRK4: no time-step restriction, but a large Δt will make aliasing errors are noticeable.

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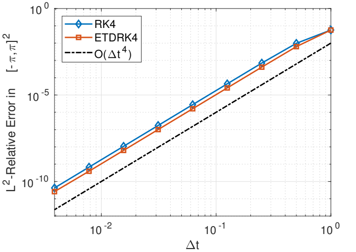
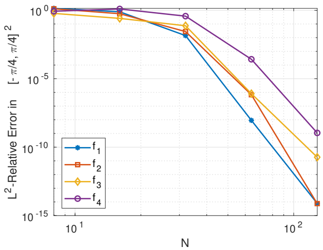
- The presence of attractors makes a study of convergence challenging.
- No numerical methods have been previously constructed for this equation.

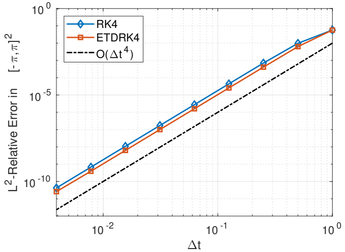
Convergence

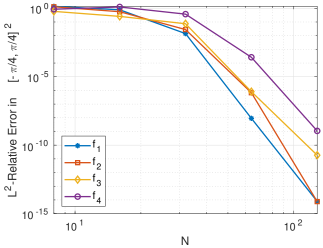




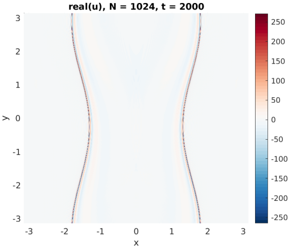
- Numerical methods are fourth-order accurate in time and locally spectrally accurate in space.

Convergence





- Numerical methods are fourth-order accurate in time and locally spectrally accurate in space.
- Global spectral accuracy cannot be expected, since there is no guarantee that u is smoother than $\mathrm{L}^2(\mathbb{T}^2)$.



Solution to the Evolution Problem

$$P(x, D) = \langle D \rangle^{-1} D_{x_2} - 0.25\,(\cos(x_1) + \sin(x_2))$$

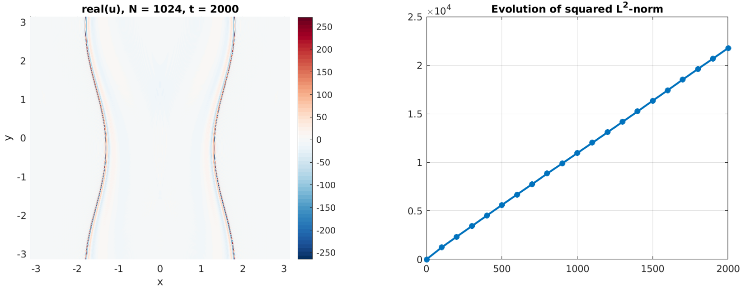


Figure: Development of attractors and linear evolution of the squared L^2-norm. The forcing is a centered Gaussian. Video: Local | YouTube

Regularity of the solution

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Definition

For a grid-function w, define the **radial energy density** (RED) E_s as

$$E_s[w](R) := \frac{1}{N^2} \sum_{k \in A_R \cap \mathbb{Z}^2} \langle k \rangle^{2s} |\widehat{w}(k)|^2,$$

where $A_R := \left\{ x \in \mathbb{R}^2 : R - 2 \leq |x| < R \right\}$.

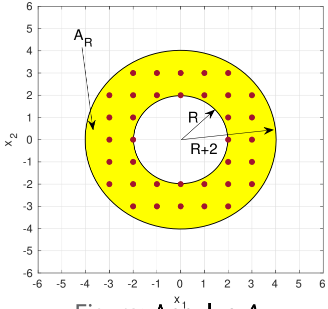

Figure: Annulus A_R

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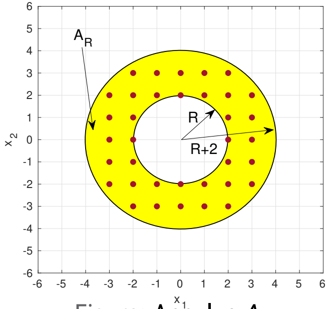

Figure: Annulus A_R

$$w \in H^s(\mathbb{T}^2) \implies \sum_{n=1}^{\infty} E_s[w](2n) < \infty.$$

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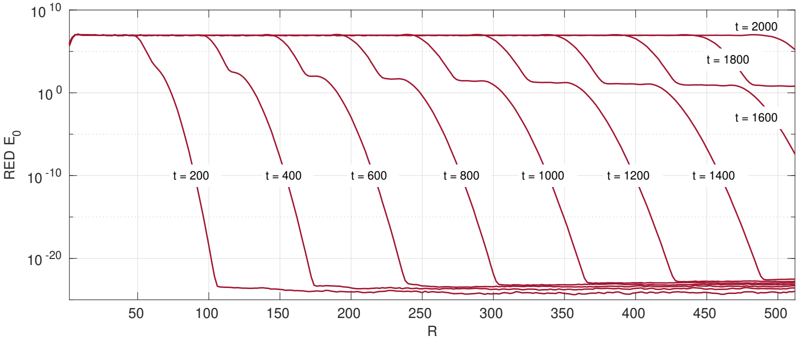

Figure: Radial energy density E_0 at different times. This shows that $u_\infty \notin \mathrm{L}^2(\mathbb{T}^2)$.

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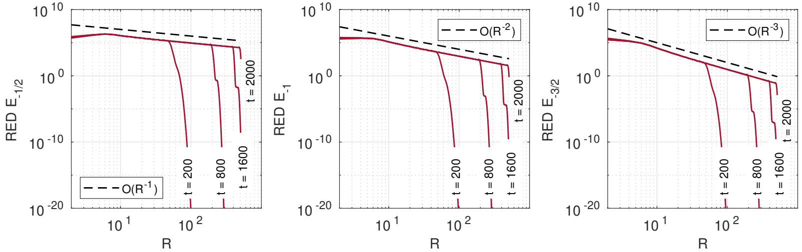

Figure: Radial energy density E_s at different times. If a distribution $v \in \mathrm{H}^s(\mathbb{T}^2)$, then $E_s[v]$ should decay faster than $\mathcal{O}(R^{-1})$.

In this section, we have:

- Constructed pseudo-spectral methods for the evolution problem,
- Developed a criterion to determine their convergence in the presence of non-smooth solutions,
- Determined a method to inspect the regularity of a computed approximation.

Up Next | **The Eigenvalue Problem**

The Eigenvalue Problem

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Problem 2

Find pairs (u, λ) such that

$$P(x, D)u(x) = \lambda u(x), \quad x \in \mathbb{T}^2.$$

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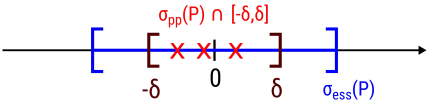
- Eigenvalues can then be embedded in the essential spectrum.



Viscous Approximation

- Can the eigenvalues of $P + i\nu\Delta$ approximate those of P as $\nu \to 0^+$?

Viscous Approximation

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Theorem (Galkowski & Zworski (2019))

There exists an open neighbourhood U of 0 in $\mathbb{C}$, and a set $\mathscr{R}(P) \subset \{z \in \mathbb{C} : \operatorname{Im} z \leq 0\} \cap U$ such that for every $K \Subset U$, $\mathscr{R}(P) \cap K$ is discrete and

$$\sigma_{\mathrm{pp}}(P + i\nu\Delta) \to \mathscr{R}(P) \quad \text{as } \nu \to 0^+,$$

uniformly on K. Furthermore,

$$\mathscr{R}(P) \cap \mathbb{R} = \sigma_{\mathrm{pp}}(P) \cap U.$$

Discretization

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Problem 3

Find pairs $(\lambda^{(\nu)}, u^{(\nu)})$ such that

$$(P + i\nu\Delta)u^{(\nu)} = \lambda^{(\nu)}u^{(\nu)} \quad \text{in } \mathbb{T}^2.$$

- Pseudo-spectral discretization (2D DFT) computes eigenvalues with spectral accuracy.

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- Computation is expensive: taking N grid-points per direction yields a full mass matrix of size $N^2 \times N^2$.
- Tracking trajectories $\nu \mapsto \lambda_j^{(\nu)}$ requires “tweaks” to the default "magnitude-then-phase" sorting of eigenvalues.

Viscous Eigenvalues

$$P = \langle D \rangle^{-1} D_{x_2} - 0.55\,(\cos(x_1 - 2x_2) + \sin(x_2))$$

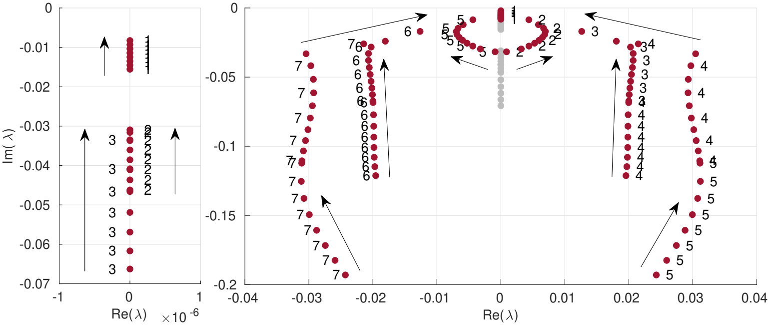


Figure: Eigenvalues of $P + i\nu\Delta$ as $\nu \to 0^+$. **L:** First three eigenvalues. **R:** All the eigenvalues.

In this section, we have:

- Described the spectrum P,
- Determined a numerical strategy to approximate the embedded eigenvalues.
- Found proper ways to track the trajectories drawn by the viscous eigenvalues.

Up Next | **Eigenmodes, Long-term Evolution and Attractors**

Influence of Eigenmodes in the Long-term Evolution

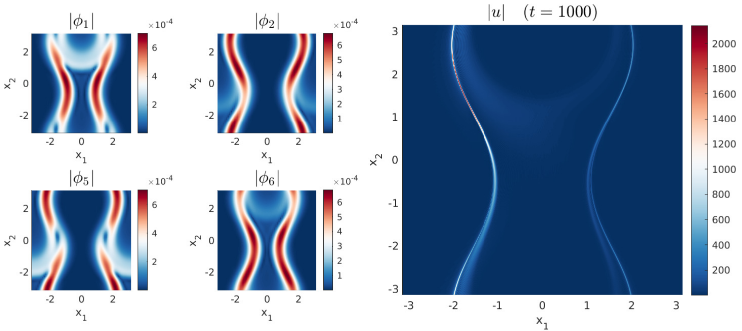


Figure: $P(x, D) = \langle D \rangle^{-1} D_{x_2} - 0.5\,(\cos(x_1) + \sin(x_2))$. **L:** Low-viscosity approximation of some of the embedded eigenmodes of P. **R:** Long-term evolution.

The Picture in Fourier Space

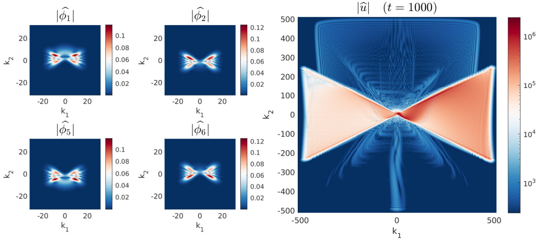
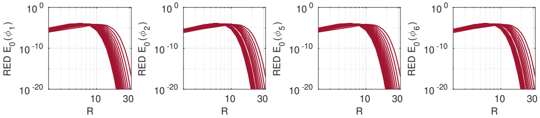


Figure: Magnitude of Fourier coefficients. **L:** Approximated embedded eigenmodes. **R:** Long-term evolution.



$$P(x,D) = \langle D \rangle^{-1} D_{x_2} - 0.5(\cos(x_1) + \sin(x_2))$$

Figure: Radial energy density E_0 of the viscous eigenmodes. As $\nu \to 0^+$, the curves move to the right.

Dynamical Aspects

- For $P(x, D) = \langle D \rangle^{-1} D_{x_2} - r\beta(x)$, its principal symbol is given by

$$p(x,\xi) = |\xi|^{-1}\xi_2 - r\beta(x), \quad (x,\xi) \in \overline{T}^* \mathbb{T}^2.$$

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$$X := \kappa_*(|\xi| H_p) \quad \text{on} \quad \Sigma_0 := \kappa\big(p^{-1}(\{0\})\big) \subset \mathbb{T}^3,$$

where H_p is the Hamiltonian vector field, and κ is the quotient map of the $\mathbb{R}^+$ action

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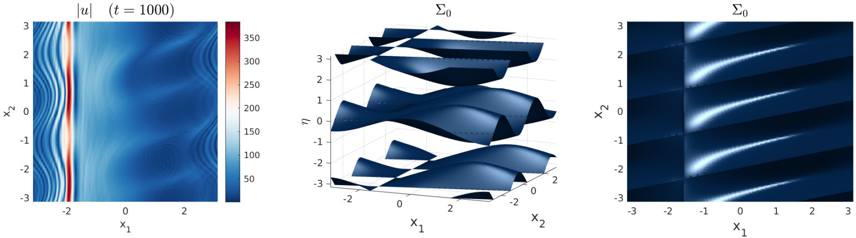
- It is assumed that the flow is Morse-Smale with no fixed points (CdV-SR 18', D-Z 19'), and that Σ covers $\mathbb{T}^2$ (CdV-SR 18').

Energy surfaces and development of attractors



$$P(x, D) = \langle D \rangle^{-1} D_{x_2} - 0.45\,(\cos(x_1 - 2x_2) + \sin(x_2))\,,$$
$$p(x, \xi) = |\xi|^{-1} \xi_2 - 0.45\,(\cos(x_1 - 2x_2) + \sin(x_2))\,,$$

$$(x_1, x_2, \xi_1, \xi_2) \overset{\kappa}{\mapsto} (x_1, x_2, \eta),$$
$$\Sigma_0 = \left\{ (x_1, x_2, \eta) \in \mathbb{T}^3 : 0.45\,(\cos(x_1 - 2x_2) + \sin(x_2)) = \sin(\eta) \right\}$$

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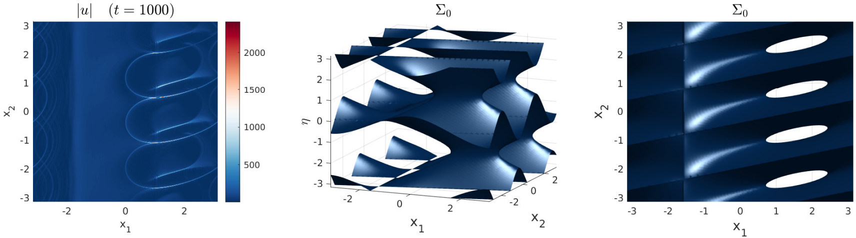

$$P(x,D) = \langle D \rangle^{-1} D_{x_2} - 0.55\,(\cos(x_1 - 2x_2) + \sin(x_2))\,,$$

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In this section, we have:

- Applied the developed numerical methods to help shed insight into this phenomenon. This gives tools to analysts to study this problem in depth.

Up Next | **Conclusions and Future Work**

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SFU

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- We developed convergent, high-order numerical methods for the evolution and eigenvalue problems,
- We showed how the methods reflect the expected analytical behaviour of the solution,
- We provided numerical evidence of the influence of the embedded eigenmodes in the evolution at long times.

Future Work

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- Develop de-aliasing techniques.

References

- **Y. Colin de Vèrdiere.** Spectral theory of pseudo-differential operators of degree 0 and application to forced linear waves. *Anal. PDE* 13 (2020), no. 5, 1521–1537
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Thank you!

Addendum

Discretization of the Eigenvalue Problem

- Let $\mathbf{F}_1$ denote the 1-dimensional DFT Matrix. Then, the 2-dimensional DFT matrix is given by $\mathbf{F}_2 = \mathbf{F}_1 \otimes \mathbf{F}_1$.
- Denote by $\mathbf{D}_\nu$ the diagonal matrix of evaluations of the modified symbol

$$\widetilde{p}(k;\nu) := \langle \xi \rangle^{-1} \xi_2 - i\nu|\xi|^2,$$

at wave numbers $k_j = -\frac{N}{2}, \dots, \frac{N}{2} - 1$. Similarly, denote by $\mathbf{B}$ the diagonal matrix of evaluations of $\beta(x)$ at the grid points.

Discrete Eigenvalue Problem

$$\left(\mathbf{D}_\nu - \frac{r}{N^2} \mathbf{F}_2 \mathbf{B} \mathbf{F}_2^* \right) \widehat{\mathbf{u}}_N^{(\nu)} = \lambda^{(\nu)} \widehat{\mathbf{u}}_N^{(\nu)}.$$

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Propagation of Round-off Errors: Example

- The function $\beta(x) = \cos(x_1) + \sin(x_2)$ sampled at $x_1, x_2 = -\pi, -\frac{pi}{2}, 0, \frac{\pi}{2}$ can be written as:

$$B_n = \begin{bmatrix} -1 & -6.12 \cdot 10^{-17} & 1 & -6.12 \cdot 10^{-17} \\ -2 & -1 & 0 & -1 \\ -1 & 6.12 \cdot 10^{-17} & 1 & 6.12 \cdot 10^{-17} \\ 0 & 1 & 2 & 1 \end{bmatrix}, \quad B_e = \begin{bmatrix} -1 & 0 & 1 & 0 \\ -2 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix},$$

where the computations in B_n have been done numerically, and in exact arithmetic for B_e.

Propagation of Round-off Errors: Example

SFU

- The construction of **B** in the eigenvalue problem requires the computation of the 1D DFT of the columns of B_n (or B_e).
- Using the 1D FFT for B_n and 1D DFT (exactly) for B_e, yields

$$\widehat{B_n} = \begin{bmatrix} -4 & 1.11 \cdot 10^{-16} & 4 & 1.11 \cdot 10^{-16} \\ -2.22 \cdot 10^{-16} + 2i & -1.22 \cdot 10^{-16} + 2i & -1.11 \cdot 10^{-16} + 2i & -1.22 \cdot 10^{-16} + 2i \\ 0 & -1.11 \cdot 10^{-16} & 0 & -1.11 \cdot 10^{-16} \\ -2.22 \cdot 10^{-16} - 2i & -1.22 \cdot 10^{-16} - 2i & -1.11 \cdot 10^{-16} - 2i & -1.22 \cdot 10^{-16} - 2i \end{bmatrix},$$

$$\widehat{B_e} = \begin{bmatrix} -4 & 0 & -4 & 0 \\ 2i & 2i & 2i & 2i \\ 0 & 0 & 0 & 0 \\ -2i & -2i & -2i & -2i \end{bmatrix}.$$

Propagation of Round-off Errors: Lost Sparsity

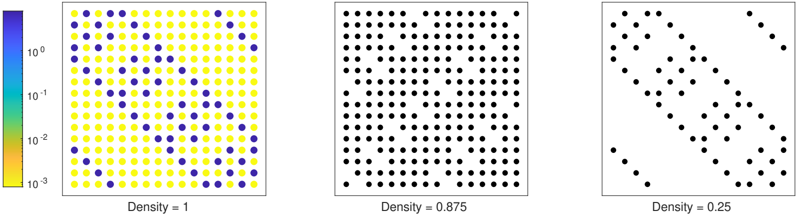

Figure: Visualization of the density of $\mathbf{F}_2\mathbf{B}\mathbf{F}_2^* \in \mathbb{C}^{16\times 16}$ when computed using the two-dimensional DFT (left), one-dimensional FFTs (center) and one-dimensional DFTs but computed in exact arithmetic (right).

Default Sorting Strategy of Eigenvalues

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1. Magnitude-then-phase: eigenvalues are ordered in increasing magnitude. If two eigenvalues have the same magnitude, the one with smallest phase (in $[-\pi, \pi)$) appears first.
2. Reorder so that eigenvalues with non-negative real part appear first.

Default Sorting Strategy of Eigenvalues

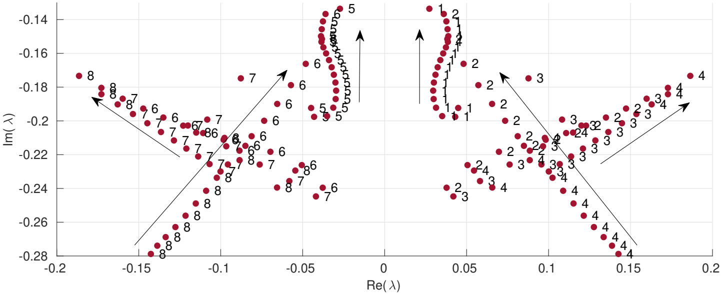

Figure: Evolution of the first 8 eigenvalues for $r = 0.5, \beta(x) = \cos(x_1) + \sin(x_2)$ as ν decreases.

Second Step in the Sorting Process: Example

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- Consider $N = 40, \nu = 0.015, r = 0.5, \beta(x) = \cos(x_1) + \sin(x_2)$.
- The first 8 eigenvalues (ordered using Step 1 only) and their magnitudes are:

$$\left(\lambda_j\right)_{j=1}^{8} = \begin{pmatrix} 0.0166 - 0.1905i \\ -0.0166 - 0.1905i \\ -0.0369 - 0.2734i \\ 0.0369 - 0.2734i \\ -0.0401 - 0.3252i \\ 0.0401 - 0.3252i \\ 0.2518 - 0.2424i \\ -0.2518 - 0.2424i \end{pmatrix}, \quad \left(|\lambda_j|\right)_{j=1}^{8} = \begin{pmatrix} 0.191246980677908 \\ 0.191246980677910 \\ 0.275887857562819 \\ 0.275887857562819 \\ 0.327685703948980 \\ 0.327685703948985 \\ 0.349551258136058 \\ 0.349551258136059 \end{pmatrix}.$$

- It appears that both λ and $-\overline{\lambda}$ are eigenvalues. The second step in the sorting process fixes the order in which they appear.

Fibre-radial Compatification

- The bundle $\overline{T}^*\mathbb{T}^2$ is a manifold with interior $T^*\mathbb{T}^2$ and a boundary $\partial\overline{T}^*\mathbb{T}^2$.
- What is $\partial\overline{T}^*\mathbb{T}^2$? Consider an equivalence relation on $T * \mathbb{T}^2\backslash\{0\}$:

$$(x,\xi) \sim (y,\varphi) \quad \iff \quad x = y \quad \text{and} \quad \xi = s\varphi, \text{ for some } s > 0.$$

Then, $\partial\overline{T}^*\mathbb{T}^2$ is the set of equivalence classes arising from this relation.

- The natural quotient map is

$$\kappa : T^*\mathbb{T}^2\backslash\{0\} \to \partial\overline{T}^*\mathbb{T}^2,$$

defined as follows: for each $(x,\xi) \in T^*\mathbb{T}^2\backslash\{0\}$, the ray $\{(x,s\xi) : s > 0\}$ intersects $\partial\overline{T}^*\mathbb{T}^2$ at $\kappa(x,\xi)$, i.e.,

$$\lim_{s\to\infty}(x,s\xi) = \kappa(x,\xi) \in \partial\overline{T}^*\mathbb{T}^2.$$

Description of the Energy Surface

SFU

- $\Sigma_0 = \kappa(p^{-1}(\{0\}))$.
- $p(x,\xi) = 0 \iff r\beta(x) = |\xi|^{-1}\xi_2$.
- Since $(x,\xi) \in \overline{T}^*\mathbb{T}^2$, we may parametrize ξ as $(s\cos\eta, s\sin\eta)$ for $\eta \in [-\pi,\pi)$ and some $s > 0$.
- Replacing this into the previous equation, we have

$$\Sigma_0 = \{\, (x,\eta) \in \mathbb{T}^3 \,:\, r\,\beta(x) = \sin(\eta) \,\}.$$

Morse-Smale Flow

A flow f^t is called "Morse-Smale" if the following properties hold:

1. There are only a finite number of equilibria, each one of them hyperbolic with smooth stable and unstable manifolds.
2. There are only a finite number of periodic orbits, each of them hyperbolic with smooth stable and unstable manifolds.
3. Stable and unstable manifolds of equilibria and periodic orbits intersect transversally, that is, for any $x \in W^u(\Lambda_1) \cap W^s(\Lambda_2)$ (with Λ_1, Λ_2 being either an equilibrium point or a periodic orbit), there holds
$$T_xM = T_xW^u(\Lambda_1) + T_xW^s(\Lambda_2),$$
where T_xM denotes the tangent space of M at x.
4. The union of equilibria and periodic orbits coincides with the non-wandering set $\Omega(f)$, defined as
$$\Omega(f) := \Big\{ x \in M : \forall \text{ neighbourhood } V \text{ of } x,\ \forall\, t_0 \in \mathbb{R}^+, \\ \exists\, t \in \mathbb{R}^+ \cap (t_0, +\infty) \text{ with } f^t(V) \cap V \neq \emptyset \Big\}.$$