High-order Discretizations of a Linear Forced Wave Equation

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1. Background

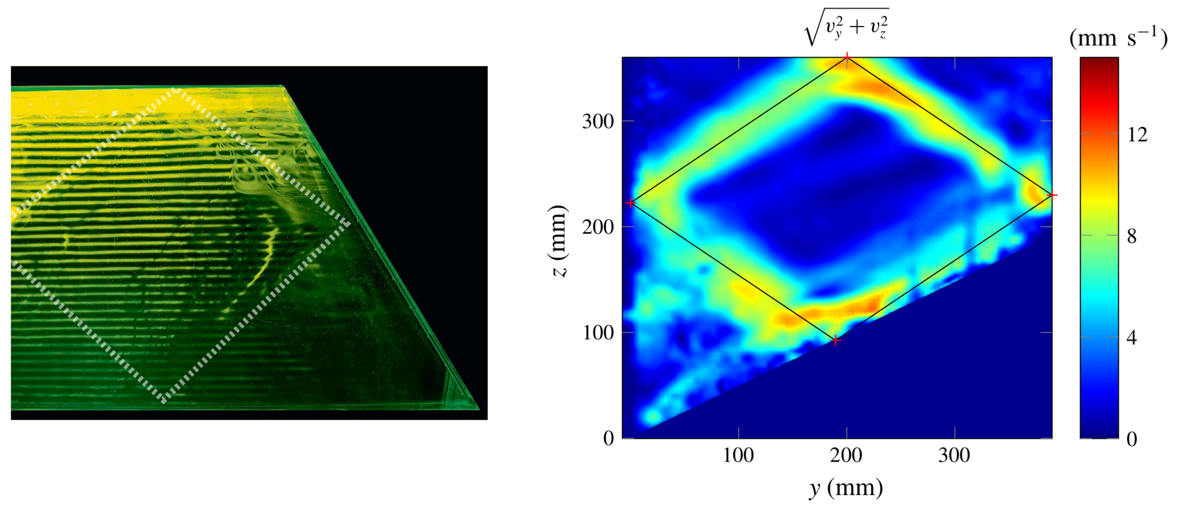

Figure 1: *It has been observed that, in the presence of topography, forcing internal waves lead to the development of singular geometric patterns (called "attractors"). Source: [1, 2].*

A simple model for the propagation of internal waves is to consider a non-rotating, stably stratified fluid with density ρ_0, whose variation can be modeled using the Boussinesq approximation:

$$\frac{\partial \eta}{\partial t} + \mathbf{u}\cdot\nabla\rho_0 = 0, \qquad \nabla\cdot\mathbf{u} = 0, \qquad \rho_0\frac{\partial \mathbf{u}}{\partial t} + \nabla p = -\eta g \mathbf{e}_3 + \mathbf{F}e^{-i\omega_0 t}, \qquad \mathbf{u}\cdot\mathbf{n} = 0, \qquad (1)$$

where η are fluctuations to the density $\rho = \rho_0 + \eta$, $\mathbf{u}$ and p are velocity and pressure fields, g is the gravitational force, $\mathbf{F}$ is a spatial source term and ω_0 is a forcing frequency.

2. Goals

- To numerically approximate solutions to a linear wave equation arising from (1),
- To verify predicted convergence rates,
- To portray the intrinsic characteristics of these solutions.

3. Simplified Mathematical Model

Recent work [3] shows that these attractors can be captured by solving a diagonalized version of (1). In particular, this process involves solving:

$$i\partial_t u - P(x,D)u = fe^{-i\omega_0 t} \quad \text{in } \mathbb{T}^2\times(0,T], \qquad u\big|_{t=0} = u_0 \quad \text{in } \mathbb{T}^2, \qquad (2)$$

where $f \in \mathcal{C}^\infty(\mathbb{T}^2)$ and P is the nonlocal, self-adjoint, zero-th order pseudo-differential operator

$$P(x,D) = (1 + D_{x_1}^2 + D_{x_2}^2)^{-1/2}D_{x_2} - r\beta(x), \qquad (3)$$

where $D = -i\partial$, $\beta \in \mathcal{C}^\infty(\mathbb{T}^2)$ and $r \geq 0$.

Some examples of more recognizable pseudo-differential operators (ΨDO) include

- ΨDO of order 0: $P(x,D) := -3x_2 \implies P(x,D)u = -3x_2 u$,
- ΨDO of order 1: $P(x,D) := iD_{x_1} \implies P(x,D)u = \partial_{x_1}u$,
- ΨDO of order $2s$, $0 < s < 1$: $P(x,D) := (D_{x_1}^2 + D_{x_2}^2)^s \implies P(x,D)u = (-\Delta)^s u$.

4. What to expect

- Attractors are related to singularities in the solution. Indeed, for $r \neq 0$ and $\beta(x) = \cos(x_1)$, we expect the appearance of blow-up singularities along the lines $x_1 = -\pi/2$ and $x_1 = \pi/2$.
- Numerical experiments have been performed in [4] showing these singularities by taking, in particular, $\omega_0 = 0$, $u_0 \equiv 0$ and

$$f(x) = -5\exp\Big(-3\big[(x_1+0.9)^2 + (x_2+0.8)^2\big] + i(2x_1 + x_2)\Big).$$

5. Numerical methods

- Spatial discretization: Fourier collocation method. Equation (2) can be written in Fourier space as

$$\partial_t\widehat{u} = \mathbf{L}(\widehat{u}) + \mathbf{N}(\widehat{u},t), \qquad \widehat{u}\big|_{t=0} = \widehat{u_0},$$

where

$$\mathbf{L}(\widehat{u})(k_1,k_2) := -\frac{ik_2}{\sqrt{1+k_1^2+k_2^2}}\widehat{u}, \qquad \mathbf{N}(\widehat{u},t) = ir\mathscr{F}\Big(\beta(x)\mathscr{F}^{-1}(\widehat{u})\Big) - i\widehat{f}e^{-i\omega_0 t},$$

with $\mathscr{F}$ being the discrete Fourier transform. Notice that now our problem is local and nonlinear (for $r \neq 0$).

- Temporal discretization:
  - Classical fourth-order Runge-Kutta (RK4),
  - RK4-based exponential time-differencing method (ETDRK4) from [5].

6. Convergence studies

First, we perform a spatial convergence study using the ETDRK4 method and several source terms:

$$f_1(x) = \sin(x_1)\cos(2x_2), \quad f_2(x) = f_1(x) + \sin(5x_1)\cos(2x_2) + i\sin(5x_1)\cos(4x_2),$$

$$f_3(x) = 0.5\exp\Big(-2x_1^2 - 2x_2^2\Big),$$

and f_4 as in Section 4. In addition, we consider $\omega_0 = 0.5$, $T = 2$, $u_0 \equiv 0$ and $\beta(x) = \cos(x_1)$. We see in Figure that spectral accuracy is achieved in all scenarios.

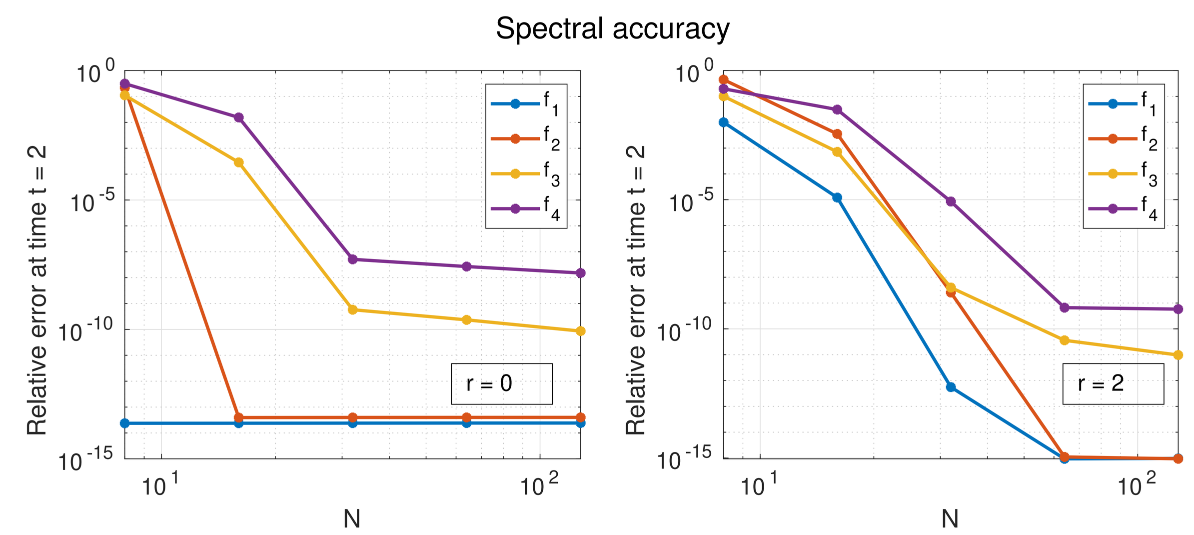

Figure 2: *Relative error v/s number of nodes for the spectral discretization using a time step $\Delta t = 10^{-3}$ and the ETDRK4 method.*

Next, we perform a temporal convergence study using the same parameters as before and source term taken as f_4. We see in Figure 3 that both RK4 and ETDRK4 methods are fourth-order accurate, as expected.

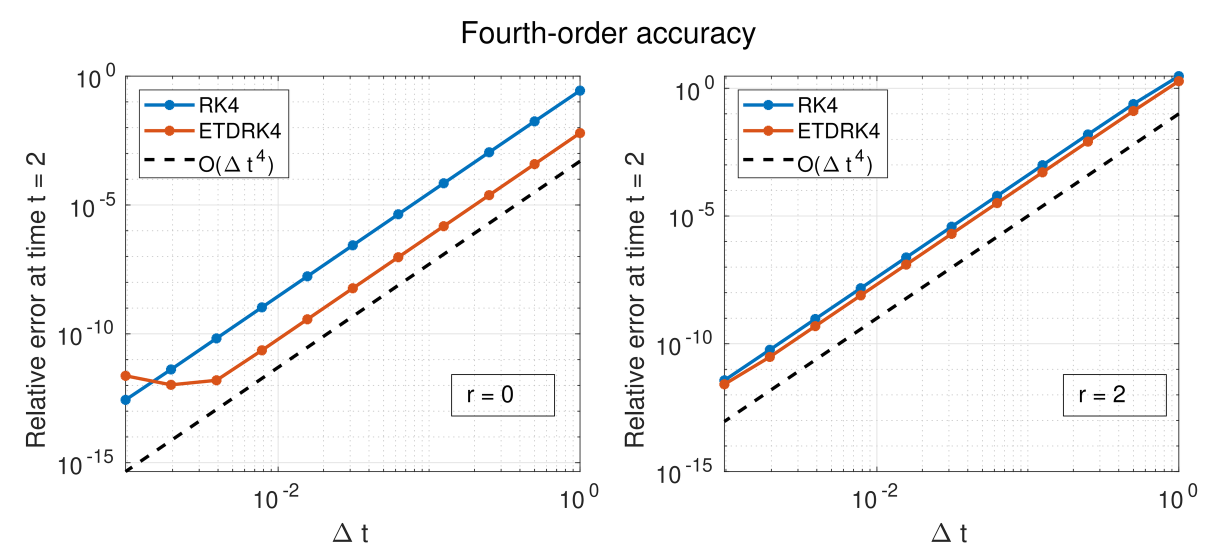

Figure 3: *Relative error v/s time step for the two implemented time discretizations using a spatial discretization of 64 × 64 nodes.*

In both cases, the relative error is computed on $[-\pi,\pi]^2$ with respect to an exact (analytical) solution when $r = 0$, and on $[-\pi/4, -\pi/4]^2$ with respect to a very refined solution when $r \neq 0$.

7. Capturing blow-up singularities

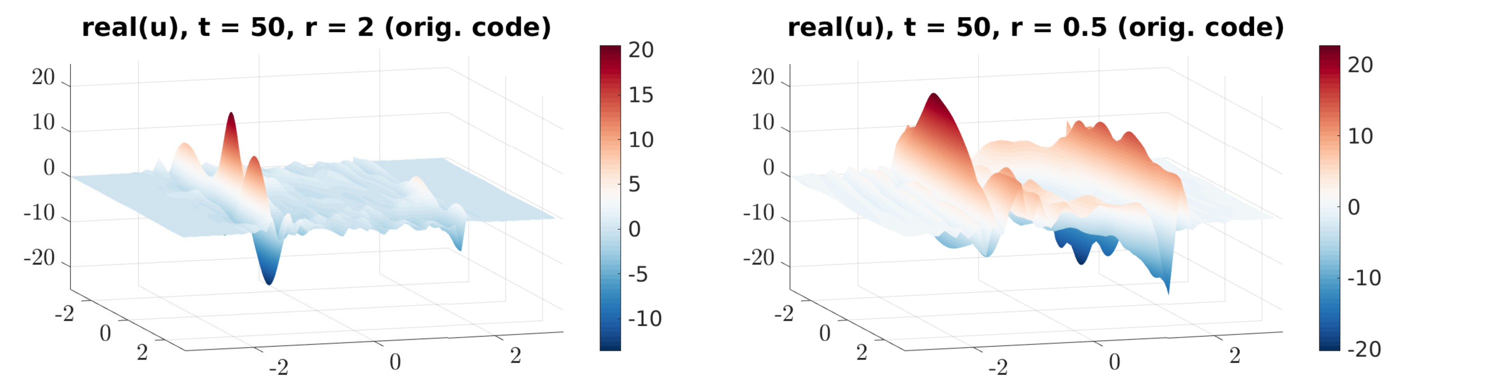

Figure 4: *Figures from [4] depicting the inherent singularities in this problem. Their computation includes the use of matrix exponentials and an Euler time-stepping scheme.*

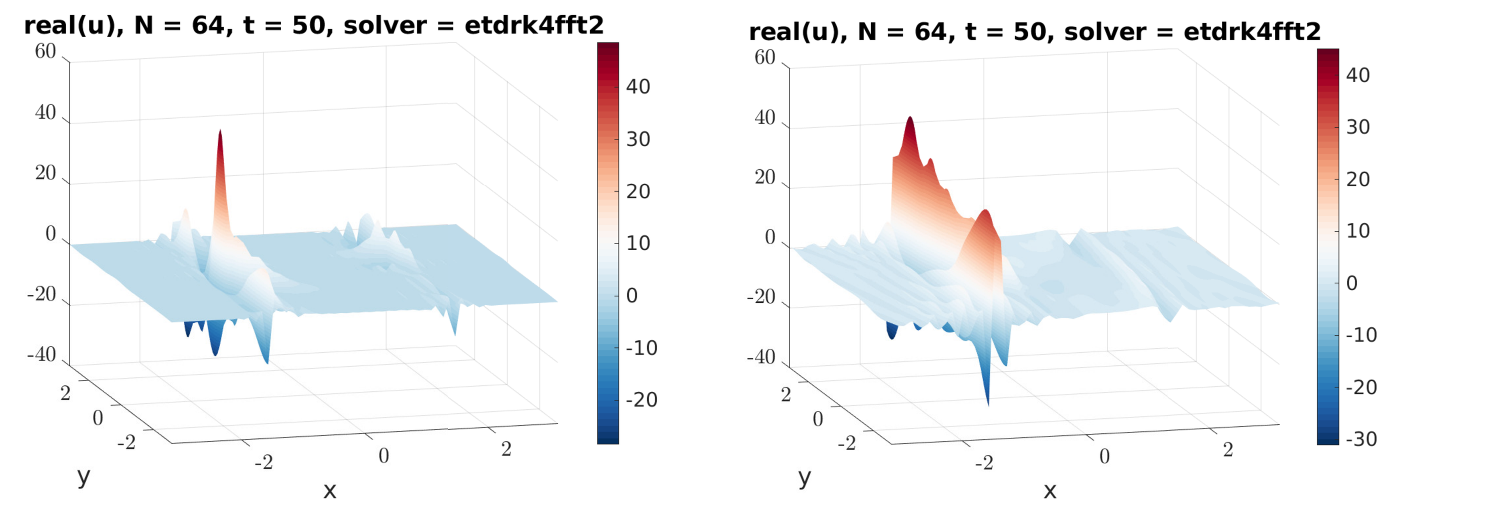

Figure 5: *Solutions to* (2) *with* $r = 2$ *(to the left) and* $r = 0.5$ *(to the right). Here, parameters are taken as in Section 4.*

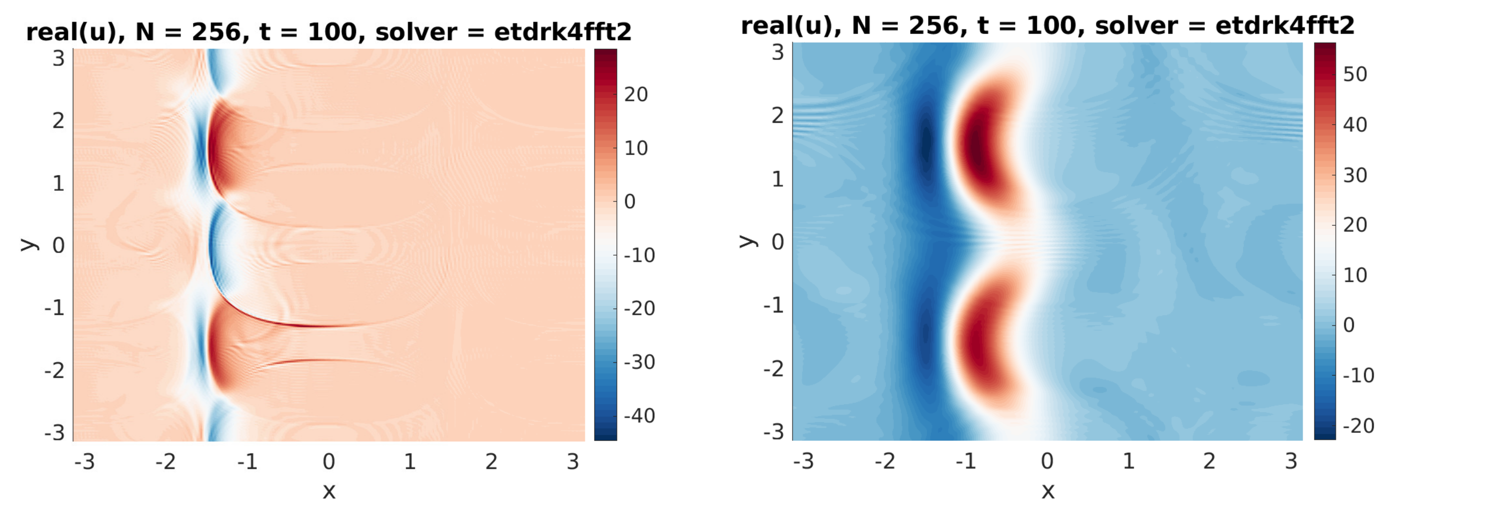

Figure 6: *Solutions to* (2) *with* $r = 2$ *(to the left) and* $r = 0.5$ *(to the right) using the same parameters as in Section 4, but this time with* $\beta(x) = \cos(x_1)\sin(2x_2)$.

8. Concluding Remarks

- The numerical methods are shown to be fourth-order accurate in time and spectrally accurate in space, both in the linear and nonlinear cases,
- Blow-up singularities are captured accordingly.

References

[1] D. Larousserie. *Des vagues qui tournent en losange.* Le Monde Science & Médecine (February 6, 2019), 3.

[2] G. Pillet, E. V. Ermanyuk, L. R. M. Maas, I. N. Sibgatullin & T. Dauxois, *Internal wave attractors in three-dimensional geometries: trapping by oblique reflection.* J. Fluid Mech. 845 (2018), 203–225.

[3] Y. Colin de Verdière & L. Saint-Raymond, *Attractors for two-dimensional waves with homogeneous Hamiltonians of degree 0.* Com. Pure Appl. Math. (2019), doi:10.1002/cpa.21485.

[4] S. Dyatlov & M. Zworski, *Microlocal analysis of forced waves.* Preprint, 2019. arXiv:1806.00809.

[5] A.-K. Kassam & L. N. Trefethen, Fourth-order time-stepping for stiff PDEs. SIAM J. Sci. Comput 26 (2005), no. 4, 1214–1233.