# The distribution of the $L_4$ norm of Littlewood polynomials

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**Abstract**

Classical conjectures due to Littlewood, Erdős and Golay concern the asymptotic growth of the $L_p$ norm of a Littlewood polynomial (having all coefficients in $\{1, -1\}$) as its degree increases, for various values of $p$. Attempts over more than fifty years to settle these conjectures have identified certain classes of the Littlewood polynomials as particularly important: skew-symmetric polynomials, reciprocal polynomials, and negative reciprocal polynomials. Using only elementary methods, we find an exact formula for the mean and variance of the $L_4$ norm of polynomials in each of these classes, and in the class of all Littlewood polynomials. A consequence is that, for each of the four classes, the normalized $L_4$ norm of a polynomial drawn uniformly at random from the class converges in probability to a constant as the degree increases.

## 1 Introduction

For real $p > 0$ and a polynomial $f \in \mathbb{C}[z]$, write

$$\|f\|_p = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \right)^{1/p}.$$

For $p \geq 1$, this defines the $L_p$ norm of $f$ on the complex unit circle. The *supremum norm* and the *Mahler measure* of $f$ are respectively defined to be

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p \quad \text{and} \quad \|f\|_0 = \lim_{p\to 0^+} \|f\|_p.$$

We consider the class of *Littlewood polynomials* $\mathcal{L}_n$, namely polynomials of degree $n-1$ (where $n > 1$) having all coefficients in $\{1, -1\}$. Notice that $\|f\|_\infty = \sup_{|z|=1} |f(z)|$, and that $\|f\|_2 = \sqrt{n}$ for each $f \in \mathcal{L}_n$. Various extremal problems concerning the behaviour of Littlewood polynomials on the unit circle have been studied. For example, Erdős [Erd95, NB90] conjectured that there is a positive constant $c_1$ such that $\|f\|_\infty/\sqrt{n} > 1 + c_1$ for all $f \in \mathcal{L}_n$, while Mahler's problem [Bor02] is

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to determine whether there exists a positive constant $c_2$ such that $\|f\|_0/\sqrt{n} < 1-c_2$ for all $f \in \mathcal{L}_n$. Since, for $f \in \mathcal{L}_n$ and real $q, r$ satisfying $q < r$, we have

$$\|f\|_0 < \|f\|_q < \|f\|_r < \|f\|_\infty,$$

these and similar problems have been attacked by studying $\|f\|_p$ for general $p$.

The $L_4$ norm has received particular attention because it is one of the easiest of the $L_p$ norms to calculate. In 1966, Littlewood asked how slowly the $L_4$ norm of a polynomial $f \in \mathcal{L}_n$ can grow with $n$ and conjectured [Lit66, §6] that, for all $\epsilon > 0$, there are infinitely many $n$ for which there exists $f \in \mathcal{L}_n$ such that $\|f\|_4/\sqrt{n} < 1+\epsilon$. To the contrary, Golay [Gol82] conjectured that there is a positive constant $c_3$ such that $\|f\|_4/\sqrt{n} > 1 + c_3$ for all $f \in \mathcal{L}_n$, which would imply that Littlewood's conjecture is false and Erdős's conjecture is true.

Several studies have identified an important subset of $\mathcal{L}_n$: the *skew-symmetric* Littlewood polynomials

$$\mathcal{S}_n := \left\{ f \in \mathcal{L}_n : \; n \text{ is odd and } f(z) = (-1)^{\frac{n-1}{2}} z^{n-1} f(-z^{-1}) \right\}. \tag{1.1}$$

Indeed, Golay [Gol82] conjectured that

$$\liminf_{n\to\infty} \min_{f\in\mathcal{S}_n} \|f\|_4/\sqrt{n} = \liminf_{n\to\infty} \min_{f\in\mathcal{L}_n} \|f\|_4/\sqrt{n},$$

which would imply that it is sufficient to restrict attention from $\mathcal{L}_n$ to the class $\mathcal{S}_n$. For twenty-five years from 1988, the smallest known asymptotic value of $\|f\|_4/\sqrt{n}$ for $f \in \mathcal{L}_n$ was $(7/6)^{1/4}$; this value is also attained by certain polynomials in $\mathcal{S}_n$ when $n$ has the form $2p+1$ or $4p+1$, where $p$ is a prime congruent to 1 modulo 4 [SJP09, Corollaries 6 and 9]. Then, in 2013, the value $(7/6)^{1/4}$ was improved to the smallest root of $27x^3 - 498x^2 + 1164x - 722$, which is less than $(22/19)^{1/4}$ [JKS13b]. It was subsequently shown that this smaller value can also be attained within the class $\mathcal{S}_n$ [JKS13a].

This provides clear motivation to understand the distribution of $\|f\|_4/\sqrt{n}$ for polynomials $f$ in both the classes $\mathcal{L}_n$ and $\mathcal{S}_n$. In this paper, we shall show how to determine the exact mean and variance of $\|f\|_4^4$ for a polynomial $f$ drawn uniformly at random from $\mathcal{L}_n$ and $\mathcal{S}_n$, as set out in Theorems 1 and 2.

**Theorem 1.** *Let $f$ be drawn uniformly at random from $\mathcal{L}_n$. Then*

$$\begin{aligned} \mathbb{E}\big(\|f\|_4^4\big) &= 2n^2 - n, \\ \mathrm{Var}\big(\|f\|_4^4\big) &= \frac{16}{3}n^3 - 20n^2 + \frac{56}{3}n - 2 + 2(-1)^n. \end{aligned}$$

**Theorem 2.** *Let $n$ be odd and let $f$ be drawn uniformly at random from $\mathcal{S}_n$. Then*

$$\begin{aligned} \mathbb{E}\big(\|f\|_4^4\big) &= 2n^2 - 3n + 2, \\ \mathrm{Var}\big(\|f\|_4^4\big) &= \frac{32}{3}n^3 - 88n^2 + \frac{592}{3}n - 512\left\lfloor\frac{n-1}{8}\right\rfloor - 512\left\lfloor\frac{n-1}{12}\right\rfloor - 88 + 16(-1)^{\frac{n-1}{2}}(n-3). \end{aligned}$$

Two other subsets of $\mathcal{L}_n$ arise naturally in the study of Littlewood's and Golay's conjectures: the *reciprocal* Littlewood polynomials

$$\mathcal{R}_n := \left\{ f \in \mathcal{L}_n : f(z) = z^{n-1} f(z^{-1}) \right\}, \tag{1.2}$$

and the *negative reciprocal* Littlewood polynomials

$$\mathcal{N}_n := \left\{ f \in \mathcal{L}_n : \; n \text{ is even and } f(z) = -z^{n-1} f(z^{-1}) \right\}. \tag{1.3}$$

For example, the asymptotic value $(7/6)^{1/4}$ for $\|f\|_4/\sqrt{n}$ is attained not only in $\mathcal{S}_n$ but also by polynomials of the form $(z^{\lfloor n/4 \rfloor}(1 + zg(z))) \bmod (z^n - 1)$, for certain $g \in \mathcal{R}_{n-1}$ when $n$ is a prime congruent to 1 modulo 4 and for certain $g \in \mathcal{N}_{n-1}$ when $n$ is a prime congruent to 3 modulo 4 [HJ88]. Furthermore, every skew-symmetric polynomial can be written in the form $f(z^2) + zg(z^2)$, where one of $f$, $g$ is reciprocal and the other is negative reciprocal. By choosing such polynomials $f$ and $g$ each to have small $L_4$ norm, Golay and Harris [GH90] constructed examples of polynomials in $\mathcal{S}_n$ with small $L_4$ norm for each odd $n$ in the range $71 \le n \le 117$; these values of the $L_4$ norm were later shown to attain the actual minimum over $\mathcal{S}_n$ for all but two values of $n$ in this range [PM16].

We shall provide a counterpart to Theorems 1 and 2 by determining the exact mean and variance of $\|f\|_4^4$ for a polynomial $f$ drawn uniformly at random from either of the classes $\mathcal{R}_n$ and $\mathcal{N}_n$, as set out in Theorem 3 (in which $I[\,\cdot\,]$ denotes the indicator function of an event).

**Theorem 3.** *Let $f$ be drawn uniformly at random from $\mathcal{R}_n$, or (for even $n$) from $\mathcal{N}_n$. Then*

$$\mathbb{E}\big(\|f\|_4^4\big) = 3n^2 - 3n + \frac{1-(-1)^n}{2},$$

$$\operatorname{Var}\big(\|f\|_4^4\big) = \begin{cases} 32n^3 - 216n^2 + 304n + 256\left\lfloor \dfrac{n}{6} \right\rfloor + 256 \cdot I[n \bmod 6 = 4] & \text{for } n \text{ even,} \\[2ex] 32n^3 - 144n^2 + 160n - 576\left\lfloor \dfrac{n-1}{4} \right\rfloor - 512\left\lfloor \dfrac{n-1}{6} \right\rfloor - 48 & \text{for } n \text{ odd.} \end{cases}$$

The expected value of $\|f\|_4^4$, as given in Theorems 1, 2, and 3, was previously calculated for $\mathcal{L}_n$ [Sar84] (often attributed instead to [NB90]), for $\mathcal{S}_n$ [Bor02], and for $\mathcal{R}_n$ and $\mathcal{N}_n$ [BC07]. An expression for $\operatorname{Var}(\|f\|_4^4)$ for the class $\mathcal{L}_n$ was stated in [ALS04], but was incorrectly calculated (see the discussion following the proof of Theorem 1). None of the variance expressions for the classes $\mathcal{S}_n$, $\mathcal{R}_n$, and $\mathcal{N}_n$ was previously known, and their derivation is considerably more difficult than that for $\mathcal{L}_n$. We find it surprising and remarkable that these exact closed form expressions exist, and that they can be determined precisely using only elementary methods. (The expressions in Theorem 1 have been verified numerically for $n \le 40$, and those in Theorems 2 and 3 for $n \le 75$.)

In each of Theorems 1, 2, and 3, we find that $\mathbb{E}\big((\|f\|_4/\sqrt{n})^4\big) \to c$ and $\operatorname{Var}\big((\|f\|_4/\sqrt{n})^4\big) \to 0$ as $n \to \infty$, where the expectation and variance are taken over the appropriate class of polynomials and $c$ is a specified constant depending on the class. As $n \to \infty$, it follows by Chebyshev's inequality that $(\|f\|_4/\sqrt{n})^4 \to c$ in probability, and therefore that $\|f\|_4/\sqrt{n} \to c^{1/4}$ in probability. Following [BL01], we can then use the inequality $x \le 1 + x^4$ at $x = \|f\|_4/\sqrt{n}$, and the Dominated Convergence Theorem for convergence in distribution, to conclude that $\mathbb{E}\big(\|f\|_4/\sqrt{n}\big) \to c^{1/4}$. This gives the following corollary.

**Corollary 4.** *Let $f$ be drawn uniformly at random from $\mathcal{L}_n$ or (for odd $n$) from $\mathcal{S}_n$, and let $g$ be drawn uniformly at random from $\mathcal{R}_n$ or (for even $n$) from $\mathcal{N}_n$. Then, as $n \to \infty$,*

$$\frac{\|f\|_4}{\sqrt{n}} \to 2^{1/4} \text{ in probability, and } \mathbb{E}\left(\frac{\|f\|_4}{\sqrt{n}}\right) \to 2^{1/4},$$

$$\frac{\|g\|_4}{\sqrt{n}} \to 3^{1/4} \text{ in probability, and } \mathbb{E}\left(\frac{\|g\|_4}{\sqrt{n}}\right) \to 3^{1/4}.$$

Of the asymptotic results given in Corollary 4, only those for $\mathcal{L}_n$ were previously known [BL01, CE14]; our techniques are very different. (Choi and Mossinghoff [CM11] were able to demonstrate the convergence in probability of $(\|f\|_4/\sqrt{n})^4$ for reciprocal *unimodular* polynomials of degree $n-1$, a class which contains $\mathcal{R}_n$, by calculating sequences of moments of the distribution but reported technical obstacles in attempting to apply their methods to the class $\mathcal{R}_n$.) Corollary 4 highlights the difficulty of trying to determine those polynomials achieving the slowest growth of the $L_4$ norm: they have zero density within their respective class as $n \to \infty$. Indeed, a value of $\liminf_{n\to\infty} \left(\|f\|_4/\sqrt{n}\right)$ smaller than $(22/19)^{1/4}$ can be attained for $f$ in $\mathcal{L}_n$ and $\mathcal{S}_n$ [JKS13b, JKS13a] (as already noted), and a value of $\liminf_{n\to\infty} \left(\|g\|_4/\sqrt{n}\right)$ equal to $(5/3)^{1/4}$ can be attained for $g$ in $\mathcal{R}_n$ and $\mathcal{N}_n$ (by taking the offset fraction in the main result of [HJ88] to be 0). Although each statement in Corollary 4 about convergence in probability can alternatively be derived from the corresponding statement about convergence of $\mathbb{E}(\|f\|_4/\sqrt{n})^4$ in Theorems 1, 2, and 3 using McDiarmid's inequality, the exact variance expressions given in these theorems provide a much deeper understanding of the distribution of the $L_4$ norm in these four classes than was previously available.

We are aware of only one other result on the distribution of $\|f\|_4/\sqrt{n}$ for Littlewood polynomials: Fredman, Saffari and Smith [FSS89] proved that

$$\frac{\|f\|_4}{\sqrt{n}} > 1.025 \quad \text{for all } f \in \mathcal{R}_n,$$

so that the restriction of Littlewood's conjecture from $\mathcal{L}_n$ to the class $\mathcal{R}_n$ is false and the corresponding restrictions of Golay's and Erdős's conjectures are true.

Our results for the $L_4$ norm can be equivalently formulated in terms of the *merit factor* $F(A)$ of the length $n$ binary sequence $A = (a_0, a_1, \ldots, a_{n-1})$ corresponding to a Littlewood polynomial $f(z) = \sum_{j=0}^{n-1} a_j z^j$, where

$$1 + \frac{1}{F(A)} = \left(\frac{\|f\|_4}{\sqrt{n}}\right)^4.$$

In this formulation, $\mathcal{L}_n$ corresponds to the set of length $n$ binary sequences, and $\mathcal{S}_n$, $\mathcal{R}_n$, and $\mathcal{N}_n$ correspond to the subsets that are skew-symmetric, symmetric, and anti-symmetric, respectively (see [Jed05] for background on the merit factor and its importance in practical digital communications, and [Jed08] for a survey of related problems). Application of Chebyshev's inequality to Theorems 1, 2 and 3 gives the following corollary.

**Corollary 5.** *Let $A$ be drawn uniformly at random from the set of all length $n$ binary sequences or (for odd $n$) from the set of skew-symmetric length $n$ binary sequences, and let $B$ be drawn uniformly at random from the set of symmetric length $n$ binary sequences or (for even $n$) from the set of anti-symmetric length $n$ binary sequences. Then, as $n \to \infty$,*

$$F(A) \to 1 \text{ in probability, and } F(B) \to \tfrac{1}{2} \text{ in probability.}$$

In Section 2 we give some preliminary results for use in later calculations. In Sections 3, 4, and 5 we prove Theorems 1, 3, and 2, respectively.

## 2 Summation and calculation identities

Throughout the paper, we denote the indicator function of the event $X$ by $I[X]$, and write $I_u := I[u \text{ odd}]$. We shall make use of the following summation identities, of which (2.1) to (2.5) can readily be verified by considering the cases $n$ even and $n$ odd separately; (2.6) follows from (2.4); (2.7) can be verified by substituting $u = 2U + 1$ and considering the cases $n \bmod 4 = 1$ and $n \bmod 4 = 3$ separately; (2.8) follows from (2.7) by substituting $v = 2V + 1$; (2.9) can be verified by considering the cases $n \bmod 8 = 1, 3, 5, 7$ separately; and (2.10) can be verified by considering each of the congruence classes of $n$ modulo 6 separately:

$$\sum_{u=1}^{n-1} \left\lfloor \frac{u}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{2.1}$$

$$3\sum_{u=1}^{n-1} \left\lfloor \frac{u}{2} \right\rfloor \left\lfloor \frac{u-2}{2} \right\rfloor = 2\left\lfloor \frac{n}{2} \right\rfloor \left(\frac{n-2}{2}\right) \left\lceil \frac{n-4}{2} \right\rceil, \tag{2.2}$$

$$\sum_{u=1}^{n-1} I_u = \left\lfloor \frac{n}{2} \right\rfloor, \tag{2.3}$$

$$2\sum_{u=1}^{n-1} I_u \left(\frac{u-1}{2}\right) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor, \tag{2.4}$$

$$3\sum_{u=1}^{n-1} I_u \left(\frac{u-1}{2}\right)\left(\frac{u-3}{2}\right) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-4}{2} \right\rfloor, \tag{2.5}$$

$$2\sum_{u,v=1}^{n-1} I_u I[u + 2v > 2n] = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor, \tag{2.6}$$

$$\sum_{u=\frac{n+1}{2}}^{n-1} I_u \left(\frac{2u-n-1}{2}\right) = \left\lfloor \frac{n-1}{4} \right\rfloor \left\lfloor \frac{n-3}{4} \right\rfloor \quad \text{for } n \text{ odd}, \tag{2.7}$$

$$\sum_{u,v=1}^{n-1} I_u I_v I[2u + v > 2n] = \left\lfloor \frac{n-1}{4} \right\rfloor \left\lfloor \frac{n-3}{4} \right\rfloor \quad \text{for } n \text{ odd}, \tag{2.8}$$

$$\sum_{u=\lceil \frac{3n+1}{4} \rceil}^{n-1} I_u = \left\lfloor \frac{n-1}{8} \right\rfloor \quad \text{for } n \text{ odd}, \tag{2.9}$$

$$\sum_{u=\lceil \frac{2n+1}{3} \rceil}^{n-1} I_u = \left\lfloor \frac{n}{6} \right\rfloor + I[n \bmod 6 = 4]. \tag{2.10}$$

We next obtain expressions for $\mathbb{E}(\|f\|_4^4)$ and $\mathrm{Var}(\|f\|_4^4)$ for $f \in \mathcal{L}_n$. Given $\sum_{j=0}^{n-1} a_j z^j \in \mathcal{L}_n$, we write

$$C_u := \sum_{j=0}^{u-1} a_j a_{j+n-u} \quad \text{for } 0 \le u < n, \tag{2.11}$$

regarding the polynomial coefficients $a_j$ as random variables each taking values in $\{1, -1\}$.

**Proposition 6.** *For* $f(z) = \sum_{j=0}^{n-1} a_j z^j \in \mathcal{L}_n$,

$$\mathbb{E}\big(\|f\|_4^4\big) = n^2 + 2E, \tag{2.12}$$

$$\mathrm{Var}\big(\|f\|_4^4\big) = 4(V - E^2), \tag{2.13}$$

*where*

$$E = \sum_{u=1}^{n-1} \mathbb{E}\, C_u^2, \tag{2.14}$$

$$V = \sum_{u,v=1}^{n-1} \mathbb{E}\big(C_u^2 C_v^2\big). \tag{2.15}$$

*Proof.* It is easily shown (see [Lit66] or [HJ88], for example) that

$$\|f\|_4^4 = n^2 + 2\sum_{u=1}^{n-1} C_u^2. \tag{2.16}$$

Take the expectation to obtain (2.12). Take the variance and substitute from (2.14) to obtain (2.13). □

# 3 The class $\mathcal{L}_n$

In this section, we use Proposition 6 to prove Theorem 1 for the class $\mathcal{L}_n$ of Littlewood polynomials.

*Proof of Theorem* 1. Let $f(z) = \sum_{j=0}^{n-1} a_j z^j \in \mathcal{L}_n$, and regard the polynomial coefficients $a_j$ as independent random variables that each take the values 1 and $-1$ with probability $\frac{1}{2}$. Substitute the definition (2.11) of $C_u$ into expression (2.14) for $E$ to give

$$E = \sum_{u=1}^{n-1} \sum_{j,k=0}^{u-1} \mathbb{E}(a_j a_{j+n-u} a_k a_{k+n-u}).$$

Since the $a_i$ are independent, the expectation term in the triple sum is nonzero exactly when $\{j, j+n-u, k, k+n-u\} = \{i, i, i', i'\}$ for some indices $i, i'$, or equivalently when $j = k$. Therefore $E = \sum_{u=1}^{n-1} \sum_{j=0}^{u-1} \mathbb{E}(1) = n(n-1)/2$, and then (2.12) gives

$$\mathbb{E}(\|f\|_4^4) = 2n^2 - n,$$

as required.

Substitute the definition (2.11) of $C_u$ into expression (2.15) for $V$ to give

$$V = \sum_{u,v=1}^{n-1} \sum_{j,k=0}^{u-1} \sum_{\ell,m=0}^{v-1} \mathbb{E}\big(a_j a_{j+n-u} a_k a_{k+n-u} a_\ell a_{\ell+n-v} a_m a_{m+n-v}\big).$$

There are four mutually disjoint cases for which the expectation term in the sum $V$ is nonzero.

**Case 1:** $u = v$ and $j = k = \ell = m$. The contribution to $V$ is $\sum_{u=1}^{n-1} \sum_{j=0}^{u-1} 1 = \sum_{u=1}^{n-1} u$.

**Case 2:** $u = v$ and $\{j, k, \ell, m\} = \{i, i, i', i'\}$ for some indices $i \neq i'$. The contribution to $V$ (from 3 symmetrical cases) is

$$3 \sum_{u=1}^{n-1} \sum_{j=0}^{u-1} \sum_{\substack{k=0 \\ k \neq j}}^{u-1} 1 = 3 \sum_{u=1}^{n-1} u(u-1).$$

**Case 3:** $u \neq v$ and $j = k$ and $\ell = m$. The contribution to $V$ is

$$\sum_{u=1}^{n-1}\sum_{\substack{v=1\\v\neq u}}^{n-1}\sum_{j=0}^{u-1}\sum_{\ell=0}^{v-1} 1 = \sum_{u=1}^{n-1}\sum_{\substack{v=1\\v\neq u}}^{n-1} uv = \Big(\sum_{u=1}^{n-1} u\Big)^2 - \sum_{u=1}^{n-1} u^2.$$

**Case 4:** $u \neq v$ and $\{j,k\} = \{i, i+n-v\}$ and $\{\ell, m\} = \{i, i+n-u\}$ for some index $i$ satisfying $0 \le i < u+v-n$. The contribution to $V$ (from 4 symmetrical cases) is

$$4\sum_{u=1}^{n-1}\sum_{\substack{v=1\\v\neq u}}^{n-1}\sum_{j=0}^{u+v-n-1} 1 = 4\sum_{u=1}^{n-1}\sum_{\substack{v=n-u+1\\v\neq u}}^{n-1}(u+v-n) = 4\sum_{u=1}^{n-1}\sum_{\substack{w=1\\w\neq 2u-n}}^{u-1} w,$$

putting $w = u+v-n$. The condition $w \neq 2u-n$ in the inner sum takes effect only when $2u-n \ge 1$, so the contribution for this case equals

$$4\sum_{u=1}^{n-1}\sum_{w=1}^{u-1} w - 4\sum_{u=\lceil\frac{n+1}{2}\rceil}^{n-1}(2u-n) = 2\sum_{u=1}^{n-1} u(u-1) - (n^2 - 2n + I_n)$$

by evaluating the sum involving $2u-n$ separately according to whether $n$ is even or odd.

Sum the contributions to $V$ from the four cases and substitute into (2.13), together with the relation $E = n(n-1)/2$ already calculated, to give

$$\begin{aligned}\mathrm{Var}\big(\|f\|_4^4\big) &= 16\sum_{u=1}^{n-1} u^2 - 16\sum_{u=1}^{n-1} u + 4\Big(\sum_{u=1}^{n-1} u\Big)^2 - 4(n^2-2n+I_n) - n^2(n-1)^2\\ &= \frac{16}{3}n^3 - 20n^2 + \frac{56}{3}n - 2 + 2(-1)^n,\end{aligned}$$

as required. □

The calculations in the proof of Theorem 1 follow the general method of Aupetit, Liardet and Slimane [ALS04, §2], but correct the mistaken conclusion of [ALS04, p.44, l.6] that (in our notation) $\mathrm{Var}\big(\|f\|_4^4\big) = \frac{8}{3}n(n-1)(n+4)$. The mistake arises from failing to apply the condition $r + s < N$ in computing the summation of [ALS04, p.44, l.2], which corresponds in our notation to neglecting the condition $u + v > n$ in the analysis of Case 4.

## 4 The classes $\mathcal{R}_n$ and $\mathcal{N}_n$

In this section, we use Proposition 6 to prove Theorem 3 for the class

$$\mathcal{R}_n = \big\{f \in \mathcal{L}_n : f(z) = z^{n-1}f(z^{-1})\big\}$$

of reciprocal Littlewood polynomials, and for the class

$$\mathcal{N}_n = \big\{f \in \mathcal{L}_n :\ n \text{ is even and } f(z) = -z^{n-1}f(z^{-1})\big\}$$

of negative reciprocal Littlewood polynomials.

*Proof of Theorem* 3. The mapping sending $f(z)$ to $g(z) = f(-z)$ is an involution between $\mathcal{R}_{2m}$ and $\mathcal{N}_{2m}$ satisfying $\|f\|_4 = \|g\|_4$. It is therefore sufficient to consider the class $\mathcal{R}_n$.

Let $f(z) = \sum_{j=0}^{n-1} a_j z^j \in \mathcal{R}_n$. By the definition (1.2) of $\mathcal{R}_n$, the coefficients $a_j$ satisfy the symmetry condition

$$a_j = a_{n-1-j} \quad \text{for } 0 \le j < n. \tag{4.1}$$

We regard the coefficients $a_0, a_1, \dots, a_{\lfloor \frac{n-1}{2} \rfloor}$ as independent random variables that each take the values 1 and $-1$ with probability $\frac{1}{2}$, and the remaining coefficients as being determined by (4.1).

Set

$$D_u := 2 \sum_{0 \le j < \frac{u-1}{2}} a_j a_{j+n-u}, \tag{4.2}$$

and use condition (4.1) to rewrite (2.11) as

$$C_u = I_u + D_u,$$

where the term $I_u$ arises from the product $a^2_{(u-1)/2}$ when $u$ is odd. Substitute for $C_u$ in (2.14) and expand to give

$$E = \left\lfloor \frac{n}{2} \right\rfloor + 2 \sum_{u=1}^{n-1} I_u \, \mathbb{E} D_u + \sum_{u=1}^{n-1} \mathbb{E} D_u^2, \tag{4.3}$$

using the summation identity (2.3). Similarly substitute for $C_u$ and $C_v$ in (2.15) and expand; using symmetry in $u$ and $v$, and the summation identity (2.3), we obtain

$$\begin{aligned} V = \left\lfloor \frac{n}{2} \right\rfloor^2 &+ 4 \left\lfloor \frac{n}{2} \right\rfloor \sum_{u=1}^{n-1} I_u \, \mathbb{E} D_u + 4 \sum_{u,v=1}^{n-1} I_u I_v \, \mathbb{E}\big(D_u D_v\big) + 2 \left\lfloor \frac{n}{2} \right\rfloor \sum_{u=1}^{n-1} \mathbb{E} D_u^2 \\ &+ 4 \sum_{u,v=1}^{n-1} I_u \, \mathbb{E}\big(D_u D_v^2\big) + \sum_{u,v=1}^{n-1} \mathbb{E}\big(D_u^2 D_v^2\big). \end{aligned} \tag{4.4}$$

We shall use (4.2) to express each of $\mathbb{E} D_u$, $\mathbb{E}\big(D_u D_v\big)$, $\mathbb{E} D_u^2$, $\mathbb{E}\big(D_u D_v^2\big)$, and $\mathbb{E}\big(D_u^2 D_v^2\big)$ as a sum of expectation terms of the form $\mathbb{E}(a_{j_1} a_{j_2} \dots a_{j_{2r}})$, where $1 \le r \le 4$. In view of the symmetry condition (4.1), such a term is nonzero exactly when the indices $j_1, j_2, \dots, j_{2r}$ admit a *matching decomposition*, namely a partition into $r$ pairs $\{j, k\}$ such that those pairs which are not *equal* ($j = k$) are *symmetric* ($j + k = n - 1$, written as $j \sim k$). We shall identify the index sets admitting a matching decomposition, multiply the resulting expressions by $I_u$ or $I_v$ and sum over $u$ or $v$ in the range $1 \le u, v \le n - 1$ as appropriate, and then substitute into the forms (4.3) and (4.4) for $E$ and $V$ to calculate $\mathbb{E}\big(\|f\|_4^4\big)$ and $\mathrm{Var}\big(\|f\|_4^4\big)$ from Proposition 6.

We shall use the observations that, for $1 \le u, v \le n - 1$,

a pair (equal or symmetric) cannot be formed from indices $j, j + n - u$ satisfying $0 \le j < \frac{u-1}{2}$, (4.5)

$$j \not\sim k \text{ for indices } j, k \text{ satisfying } 0 \le j < \tfrac{u-1}{2} \text{ and } 0 \le k < \tfrac{v-1}{2}, \tag{4.6}$$

and

the only pairs that can be formed from indices $j, j + n - u, k, k + n - u$ satisfying $0 \le j < k < \frac{u-1}{2}$ are $k = j + n - u$ (equal) and $j + n - u \sim k + n - u$ (symmetric). (4.7)

**The sum $\sum_{u=1}^{n-1} I_u\, \mathbb{E}D_u$.** From (4.2) and (4.5), this sum is

$$\sum_{u=1}^{n-1} I_u\, \mathbb{E}D_u = 2\sum_{u=1}^{n-1} I_u \sum_{0\le j<\frac{u-1}{2}} \mathbb{E}(a_j a_{j+n-u}) = 0. \tag{4.8}$$

**The sum $\sum_{u,v=1}^{n-1} I_u I_v\, \mathbb{E}\big(D_u D_v\big)$.** From (4.2), this sum equals

$$4\sum_{u,v=1}^{n-1} I_u I_v \sum_{0\le j<\frac{u-1}{2}} \sum_{0\le k<\frac{v-1}{2}} \mathbb{E}(a_j a_{j+n-u} a_k a_{k+n-v}).$$

The indices $j, j+n-u, k, k+n-v$ admit a matching decomposition in two ways, represented in Cases 1 and 2 below. All other possible arrangements of these four indices into two pairs are inconsistent with the given ranges for $u, v, j, k$, either because of a single pairing excluded by (4.5) or (4.6), or else by combination of two pairings as summarized in Table 1.

**Case 1:** $j = k$ and $j+n-u = k+n-v$. This gives $j = k$ and $u = v$, and the contribution to the sum is

$$4\sum_{u=1}^{n-1} I_u \sum_{0\le j<\frac{u-1}{2}} 1 = 4\sum_{u=1}^{n-1} I_u \left(\frac{u-1}{2}\right) = 2\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor$$

using the summation identity (2.4).

**Case 2:** $j = k$ and $j+n-u \sim k+n-v$. This gives $j = k = \frac{u+v-n-1}{2}$. These values of $j, k$ have already been counted as part of Case 1 when $u = v$, so we impose the constraint $u \neq v$ and calculate the additional contribution to the sum from Case 2 as

$$4\, I_n \sum_{\substack{u,v=1\\ u\neq v}}^{n-1} I_u I_v\ I[u+v>n].$$

Substitute $u = 2U+1$ and $2V+1$ to evaluate this additional contribution as

$$\begin{aligned}4\, I_n \sum_{\substack{U,V=0\\ U\neq V}}^{\frac{n-3}{2}} I[U+V>\tfrac{n-2}{2}] &= 4\, I_n \sum_{U,V=0}^{\frac{n-3}{2}} I[U+V>\tfrac{n-2}{2}] - 4\, I_n \sum_{U=0}^{\frac{n-3}{2}} I[U>\tfrac{n-2}{4}]\\ &= 4\, I_n \frac{(n-1)(n-3)}{8} - 4\, I_n \left\lfloor\frac{n-1}{4}\right\rfloor.\end{aligned}$$

Combine Cases 1 and 2 to give

$$\sum_{u,v=1}^{n-1} I_u I_v\, \mathbb{E}\big(D_u D_v\big) = 2\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 2\, I_n \left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right) - 4\, I_n \left\lfloor\frac{n-1}{4}\right\rfloor. \tag{4.9}$$

Table 1: Inconsistent index pairings for the sum $\sum_{u,v=1}^{n-1} I_u I_v \, \mathbb{E}\big(D_u D_v\big)$

| Index pairings | Inconsistency from |
|---|---|
| $j = k+n-v, \quad k = j+n-u$ | $u+v=2n$ |
| $j = k+n-v, \quad k \sim j+n-u$ | $j = \frac{u-1}{2} + \frac{n-v}{2} > \frac{u-1}{2}$ |
| $j \sim k+n-v, \quad k = j+n-u$ | $k = \frac{v-1}{2} + \frac{n-u}{2} > \frac{v-1}{2}$ |
| $j \sim k+n-v, \quad k \sim j+n-u$ | $j+k = \frac{u-1}{2} + \frac{v-1}{2}$ |

**The sum $\sum_{u=1}^{n-1} \mathbb{E} D_u^2$.** This sum arises by restricting to $u = v$ in the analysis of the previous sum (with the $I_u I_v$ term absent), and so is

$$\sum_{u=1}^{n-1} \mathbb{E} D_u^2 = 4 \sum_{u=1}^{n-1} \sum_{0 \le j < \frac{u-1}{2}} 1 = 4 \sum_{u=1}^{n-1} \left\lfloor \frac{u}{2} \right\rfloor = 4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \tag{4.10}$$

using the summation identity (2.1).

**The sum $\sum_{u,v=1}^{n-1} I_u \, \mathbb{E}\big(D_u D_v^2\big)$.** From (4.2), this sum equals

$$8 \sum_{u,v=1}^{n-1} I_u \sum_{0 \le j < \frac{u-1}{2}} \sum_{0 \le k, \ell < \frac{v-1}{2}} \mathbb{E}(a_j a_{j+n-u} a_k a_{k+n-v} a_\ell a_{\ell+n-v}).$$

The contributions from indices $k = \ell$ involve terms in $\mathbb{E}(a_j a_{j+n-u})$, which by (4.5) is zero. By symmetry in $k$ and $\ell$, it is therefore sufficient to take twice the contributions from indices $k < \ell$, namely

$$16 \sum_{u,v=1}^{n-1} I_u \sum_{0 \le j < \frac{u-1}{2}} \sum_{0 \le k < \ell < \frac{v-1}{2}} \mathbb{E}(a_j a_{j+n-u} a_k a_{k+n-v} a_\ell a_{\ell+n-v}).$$

In order for the indices $j, j+n-u, k, k+n-v, \ell, \ell+n-v$ to admit a matching decomposition, at least one equal pair or one symmetric pair must be formed from the four indices $k, k+n-v, \ell, \ell+n-v$. In the given range $k < \ell < \frac{v-1}{2}$, from observation (4.7) either $\ell = k+n-v$ (represented in Cases 2 and 3 below) or $k+n-v \sim \ell+n-v$ (represented in Cases 1 and 4 below). We can now determine all matching decompositions by starting with each of these two pairs in turn, and identifying all possible arrangements of the four remaining unpaired indices into two further pairs consistent with the given ranges for $u, v, j, k, \ell$. These arrangements are listed in Cases 1 to 4 below; all other arrangements are inconsistent, either because of a single pairing excluded by (4.5) or (4.6), or else by combination of two or more pairings as summarized in Table 2.

**Case 1:** $k+n-v \sim \ell+n-v$ and $j = k$ and $j+n-u \sim \ell$. This gives $2v-u = n$ and $j = k$ and $\ell = 2v-n-1-j$, and the contribution to the sum is

$$16 \, I_n \sum_{v=\frac{n+1}{2}}^{n-1} \sum_{\substack{0 \le j < \frac{2v-n-1}{2} \\ j > \frac{3v-2n-1}{2}}} 1.$$

**Case 2:** $\ell = k + n - v$ and $j = k$ and $j + n - u = \ell + n - v$. This gives $2v - u = n$ and $j = k$ and $\ell = j + n - v$, and the contribution to the sum is

$$16\, I_n \sum_{v=\frac{n+1}{2}}^{n-1} \sum_{0 \le j < \frac{3v-2n-1}{2}} 1.$$

**Case 3:** $\ell = k + n - v$ and $j = k$ and $j + n - u \sim \ell + n - v$. This gives $j = k = \frac{u+2v-2n-1}{2}$ and $\ell = \frac{u-1}{2}$. These values of $j, k, \ell$ have already been counted as part of Case 2 when $2v - u = n$, so we impose the constraint $2v - u \neq n$ and evaluate the additional contribution to the sum from Case 3 as

$$16 \sum_{\substack{u,v=1 \\ 2v-u\neq n}}^{n-1} I_u \; I[u + 2v > 2n] \; I[u < v].$$

**Case 4:** $k + n - v \sim \ell + n - v$ and $j = k$ and $j + n - u = \ell$. This gives $j = k = \frac{u+2v-2n-1}{2}$ and $\ell = \frac{2v-u-1}{2}$, and the contribution to the sum is

$$16 \sum_{u,v=1}^{n-1} I_u \; I[u + 2v > 2n] \; I[u > v].$$

Table 2: Inconsistent index pairings for the sum $\sum_{u,v=1}^{n-1} I_u\, \mathbb{E}\big(D_u D_v^2\big)$

| Index pairings | | | Inconsistency from |
|---|---|---|---|
| $\ell = k + n - v$, | $j = \ell + n - v$, | $k = j + n - u$ | $u + 2v = 3n$ |
| $\ell = k + n - v$, | $j = \ell + n - v$, | $k \sim j + n - u$ | $j = \frac{u-1}{2} + n - v > \frac{u-1}{2}$ |
| $\ell = k + n - v$, | $j \sim \ell + n - v$, | $k = j + n - u$ | $\ell = \frac{v-1}{2} + \frac{2n-u-v}{2} > \frac{v-1}{2}$ |
| $\ell = k + n - v$, | $j \sim \ell + n - v$, | $k \sim j + n - u$ | $j + \ell = \frac{u+n-2}{2} > \frac{u-1}{2} + \frac{v-1}{2}$ |
| $k + n - v \sim \ell + n - v$, | $j = \ell$, | $k = j + n - u$ | $k = \ell + n - u > \ell$ |
| $k + n - v \sim \ell + n - v$, | $j = \ell$, | $k \sim j + n - u$ | $k + \ell = u - 1$ |

The combined contribution to the sum from Cases 1 and 2 is

$$16\, I_n \sum_{v=\frac{n+1}{2}}^{n-1} \sum_{\substack{0 \le j < \frac{2v-n-1}{2} \\ j \neq \frac{3v-2n-1}{2}}} 1,$$

in which the condition $j \neq \frac{3v-2n-1}{2}$ in the inner sum takes effect only when $\frac{3v-2n-1}{2}$ is a non-negative integer. The combined contribution from Cases 1 and 2 therefore equals

$$16\, I_n \sum_{v=\frac{n+1}{2}}^{n-1} \frac{2v-n-1}{2} - 16\, I_n \sum_{v=\lceil \frac{2n+1}{3} \rceil}^{n-1} I_v = 8\, I_n \left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right) - 16\, I_n \left\lfloor \frac{n-1}{6} \right\rfloor$$

using the summation identity (2.10).

The combined additional contribution to the sum from Cases 3 and 4 is

$$16 \sum_{\substack{u,v=1 \\ u\neq v \\ 2v-u\neq n}}^{n-1} I_u\ I[u+2v>2n] = 16 \sum_{u,v=1}^{n-1} I_u\ I[u+2v>2n] - 16 \sum_{u=\lceil \frac{2n+1}{3} \rceil}^{n-1} I_u - 16\, I_n \sum_{v=\lceil \frac{3n+1}{4} \rceil}^{n-1} 1$$

$$= 16 \cdot \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor - 16 \left( \left\lfloor \frac{n}{6} \right\rfloor + I[n \bmod 6 = 4] \right) - 16 I_n \left\lfloor \frac{n-1}{4} \right\rfloor,$$

where the first summation is evaluated using the identity (2.6), the second using the identity (2.10), and the third by considering the cases $n \bmod 4 = 1$ and $n \bmod 4 = 3$ separately.

Add the contribution from Cases 1 and 2 to the additional contribution from Cases 3 and 4 to obtain

$$\sum_{u,v=1}^{n-1} I_u\, \mathbb{E}\big(D_u D_v^2\big) = 8\, I_n \left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right) + 8 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor$$
$$- 16 \left\lfloor \frac{n}{6} \right\rfloor - 16 I_n \left\lfloor \frac{n-1}{6} \right\rfloor - 16 I_n \left\lfloor \frac{n-1}{4} \right\rfloor - 16 \cdot I[n \bmod 6 = 4]. \tag{4.11}$$

**The sum $\sum_{u,v=1}^{n-1} \mathbb{E}\big(D_u^2 D_v^2\big)$.** From (4.2), this sum equals

$$16 \sum_{u,v=1}^{n-1} \sum_{0\le j,k<\frac{u-1}{2}} \sum_{0\le \ell,m<\frac{v-1}{2}} \mathbb{E}(a_j a_{j+n-u} a_k a_{k+n-u} a_\ell a_{\ell+n-v} a_m a_{m+n-v}).$$

We distinguish five mutually disjoint cases.

**Case 1:** $j = k$ and $\ell = m$. The contribution to the sum is

$$16 \sum_{u,v=1}^{n-1} \sum_{0\le j<\frac{u-1}{2}} \sum_{0\le \ell<\frac{v-1}{2}} 1 = 16 \left( \sum_{u=1}^{n-1} \left\lfloor \frac{u}{2} \right\rfloor \right)^2 = 16 \left\lfloor \frac{n}{2} \right\rfloor^2 \left\lfloor \frac{n-1}{2} \right\rfloor^2$$

using the summation identity (2.1).

**Case 2:** $j = k$ and $\ell \neq m$. By symmetry in $\ell$ and $m$, the contribution to the sum is

$$2 \cdot 16 \sum_{u,v=1}^{n-1} \sum_{0\le j<\frac{u-1}{2}} \sum_{0\le \ell<m<\frac{v-1}{2}} \mathbb{E}(a_\ell a_{\ell+n-v} a_m a_{m+n-v}),$$

and the inner sum over $\ell < m$ is zero because by (4.7) the index $\ell$ cannot form an equal or symmetric pair with any of the other three indices $\ell+n-v$, $m$, $m+n-v$ in the given range $\ell < m < \frac{v-1}{2}$.

**Case 3:** $j \neq k$ and $\ell = m$. Similarly to Case 2, the contribution to the sum is zero.

**Case 4:** $j \neq k$ and $\ell \neq m$ and $u = v$. By symmetry in $j, k$ and in $\ell, m$, the contribution to the sum is

$$4 \cdot 16 \sum_{u=1}^{n-1} \sum_{0 \le j < k < \frac{u-1}{2}} \sum_{0 \le \ell < m < \frac{u-1}{2}} \mathbb{E}(a_j a_{j+n-u} a_k a_{k+n-u} a_\ell a_{\ell+n-u} a_m a_{m+n-u}).$$

If $j \neq \ell$, then by (4.7) the expectation term is zero because the smaller of $j, \ell$ cannot form a pair with any of the other seven indices. We may therefore take $j = \ell$, and then by similar reasoning take $k = m$, so that the contribution to the sum is

$$64 \sum_{u=1}^{n-1} \sum_{0 \le j < k < \frac{u-1}{2}} 1 = 32 \sum_{u=1}^{n-1} \left\lfloor \frac{u}{2} \right\rfloor \left\lfloor \frac{u-2}{2} \right\rfloor = \frac{64}{3} \left\lfloor \frac{n}{2} \right\rfloor \binom{n-2}{2} \left\lceil \frac{n-4}{2} \right\rceil$$

using the summation identity (2.2).

**Case 5:** $j \neq k$ and $\ell \neq m$ and $u \neq v$. By symmetry in $u, v$ and in $j, k$, the contribution to the sum is

$$4 \cdot 16 \sum_{\substack{u,v=1 \\ u<v}}^{n-1} \sum_{0 \le j < k < \frac{u-1}{2}} \sum_{\substack{0 \le \ell, m < \frac{v-1}{2} \\ \ell \neq m}} \mathbb{E}(a_j a_{j+n-u} a_k a_{k+n-u} a_\ell a_{\ell+n-v} a_m a_{m+n-v}).$$

By (4.7), the index $j$ cannot form an equal or symmetric pair with any of the indices $j+n-u, k, k+n-u$ in the given range $j < k < \frac{u-1}{2}$. Furthermore, $j$ cannot form a symmetric pair with any of the indices $\ell, \ell+n-v, m, m+n-v$ in the given ranges $\ell, m < \frac{v-1}{2}$ and $j < \frac{u-1}{2}$ and $u < v$. A matching decomposition for the eight indices of the expectation term therefore requires that $j$ form an equal pair with one of the four indices $\ell, \ell+n-v, m, m+n-v$. By symmetry in $\ell, m$, we may replace the resulting four contributions to the sum by twice the contribution from $j = \ell$ and twice the contribution from $j = \ell + n - v$.

We claim that the contribution from $j = \ell + n - v$ is zero. To prove the claim, set $j = \ell + n - v$, so that $\ell$ and $k$ are now constrained via $\ell + n - v < k < \frac{u-1}{2}$ and the remaining six unpaired indices are $\ell + 2n - u - v, k, k+n-u, \ell, m, m+n-v$. It is straightforward to check that $\ell$ cannot form an equal or symmetric pair with any of the three indices $\ell+2n-u-v, k, k+n-u$ subject to the given constraint $\ell+n-v < k < \frac{u-1}{2}$, and therefore by (4.7) the only possible pairing involving $\ell$ is $\ell = m+n-v$. We therefore set $\ell = m + n - v$, so that $m$ and $k$ are now constrained via $m + 2n - 2v < k < \frac{u-1}{2}$ and the remaining four unpaired indices are $m + 3n - u - 2v, k, k+n-u, m$. By (4.5), the indices $k, k+n-u$ cannot form an equal or symmetric pair and so, for a matching decomposition, $m$ must form an equal or symmetric pair with $k$ or $k+n-u$. This is not possible subject to the given constraint $m + 2n - 2v < k < \frac{u-1}{2}$, proving the claim.

The contribution to the sum from Case 5 is therefore twice the contribution from $j = \ell$, namely

$$128 \sum_{\substack{u,v=1 \\ u<v}}^{n-1} \sum_{0 \le j < k < \frac{u-1}{2}} \sum_{\substack{0 \le m < \frac{v-1}{2} \\ m \neq j}} \mathbb{E}(a_k a_{k+n-u} a_{j+n-u} a_{j+n-v} a_m a_{m+n-v}).$$

The index $k$ cannot form a symmetric pair with any of the indices $k+n-u, j+n-u, j+n-v, m, m+n-v$ in the given ranges $j < k < \frac{u-1}{2}$ and $m < \frac{v-1}{2}$ and $u < v$. A matching decomposition for the six indices of the expectation term therefore requires that $k$ form an equal pair with one of the four indices $j+n-u, j+n-v, m, m+n-v$. We now determine all matching decompositions by starting with each of these four equal pairs in turn, identifying all possible arrangements of the four remaining unpaired indices into two further pairs consistent with the given ranges for $u, v, j, k, m$. These arrangements are listed in Cases 5a to 5e below. Inconsistent arrangements of indices arising from combinations of two or more pairings are summarized in Table 3.

**Case 5a:** $k = j+n-u$ and $m = j+n-v$ and $k+n-u \sim m+n-v$. This gives $j = \frac{2u+2v-3n-1}{2}$ and $k = \frac{2v-n-1}{2}$ and $m = \frac{2u-n-1}{2}$, and the contribution to the sum is

$$128\, I_n \sum_{\substack{u,v=1 \\ u<v}}^{n-1} I[2u+2v > 3n]\; I[2v-u < n].$$

**Case 5b:** $k = m+n-v$ and $k+n-u \sim j+n-u$ and $j+n-v = m$. This gives $j = \frac{2u+2v-3n-1}{2}$ and $k = \frac{2u-2v+n-1}{2}$ and $m = \frac{2u-n-1}{2}$, and the contribution to the sum is

$$128\, I_n \sum_{\substack{u,v=1 \\ u<v}}^{n-1} I[2u+2v > 3n]\; I[2v-u > n].$$

**Case 5c:** $k = j+n-v$ and $m = j+n-u$ and $k+n-u, m+n-v$ form an equal or symmetric pair. This gives $k = j+n-v$ and $m = j+n-u$ when the third pair (formed by $k+n-u, m+n-v$) is equal; the special case $j = \frac{2u+2v-3n-1}{2}$, $k = \frac{2u-n-1}{2}$, $m = \frac{2v-n-1}{2}$ is obtained when this pair is symmetric, and so is not counted again. The contribution to the sum is

$$128 \sum_{\substack{u,v=1 \\ u<v}}^{n-1} \sum_{j\geq 0} I[j < \tfrac{2u+v-2n-1}{2}].$$

**Case 5d:** $k = j+n-v$ and $m \sim k+n-u$ and $j+n-u, m+n-v$ form an equal or symmetric pair. This gives $k = j+n-v$ and $m = u+v-n-1-j$ when the third pair is symmetric; the special case $j = \frac{2u-n-1}{2}$, $k = \frac{2u-2v+n-1}{2}$, $m = \frac{2v-n-1}{2}$ is obtained when this pair is equal, and is not counted again. The contribution to the sum is

$$128 \sum_{\substack{u,v=1 \\ u<v}}^{n-1} \sum_{j\geq 0} I[\tfrac{2u+v-2n-1}{2} < j < \tfrac{u+2v-2n-1}{2}].$$

**Case 5e:** $k = m$ and $k+n-u \sim j+n-v$ and $j+n-u, m+n-v$ form an equal or symmetric pair. This gives $k = m = u+v-n-1-j$ when the third pair is symmetric; the special case $j = \frac{2u-n-1}{2}$, $k = m = \frac{2v-n-1}{2}$ is obtained when this pair is equal, and is not counted again. The contribution to the sum is

$$128 \sum_{\substack{u,v=1 \\ u<v}}^{n-1} \sum_{j\geq 0} I[\tfrac{u+2v-2n-1}{2} < j < \tfrac{u+v-n-1}{2}].$$

Table 3: Inconsistent index pairings for Case 5 of the sum $\sum_{u,v=1}^{n-1} \mathbb{E}\big(D_u^2 D_v^2\big)$

| Index pairings | | | Inconsistency from |
|---|---|---|---|
| $k = j + n - u,$ | $m = k + n - u,$ | $j + n - v \sim m + n - v$ | $m = \frac{v-1}{2} + \frac{v-u}{2} + \frac{n-u}{2} > \frac{v-1}{2}$ |
| $k = j + n - u,$ | $m \sim k + n - u,$ | $j + n - v \sim m + n - v$ | $u = v$ |
| $k = j + n - u,$ | $m = j + n - v,$ | $k + n - u = m + n - v$ | $u = v$ |
| $k = j + n - u,$ | $m \sim j + n - v,$ | $k + n - u = m + n - v$ | $k = \frac{n-1}{2}$ |
| $k = j + n - u,$ | $m \sim j + n - v,$ | $k + n - u \sim m + n - v$ | $u = n$ |
| $k = m + n - v,$ | $k + n - u \sim j + n - u,$ | $j + n - v \sim m$ | $u = n$ |
| $k = m + n - v,$ | $k + n - u \sim j + n - v,$ | $j + n - u = m$ | $k = \frac{n-1}{2}$ |
| $k = m + n - v,$ | $k + n - u \sim j + n - v,$ | $j + n - u \sim m$ | $v = n$ |
| $k = m + n - v,$ | $k + n - u = m$ | | $u + v = 2n$ |
| $k = m + n - v,$ | $k + n - u \sim m$ | | $k = \frac{u-1}{2} + \frac{n-v}{2} > \frac{u-1}{2}$ |
| $k = j + n - v,$ | $m \sim j + n - u,$ | $k + n - u = m + n - v$ | $m = \frac{n-1}{2}$ |
| $k = j + n - v,$ | $m \sim j + n - u,$ | $k + n - u \sim m + n - v$ | $v = n$ |
| $k = j + n - v,$ | $m = k + n - u,$ | $j + n - u = m + n - v$ | $v = n$ |
| $k = j + n - v,$ | $m = k + n - u,$ | $j + n - u \sim m + n - v$ | $m = \frac{n-1}{2}$ |
| $k = m,$ | $k + n - u \sim j + n - u,$ | $j + n - v \sim m + n - v$ | $u = v$ |
| $k = m,$ | $k + n - u = m + n - v$ | | $u = v$ |
| $k = m,$ | $k + n - u \sim m + n - v,$ | $j + n - u \sim j + n - v$ | $j = k$ |

We now calculate the total contribution to the sum from Cases 1 to 5. The combined contribution from Cases 5a and 5b is

$$128\, I_n \sum_{\substack{u,v=1 \\ u<v}}^{n-1} I[2u + 2v > 3n]\; I[2v - u \neq n] = 128\, I_n \sum_{\substack{u,v=1 \\ u<v}}^{n-1} I[2u + 2v > 3n] - 128\, I_n \sum_{v=\lceil \frac{5n+1}{6} \rceil}^{n-1} 1.$$

Since each summand of the sum over $u, v$ is symmetric in $u$ and $v$, this combined contribution is

$$\begin{aligned} 128 \cdot \frac{1}{2} I_n \sum_{u,v=1}^{n-1} I[2u + 2v > 3n] - 64\, I_n \sum_{u=\lceil \frac{3n+1}{4} \rceil}^{n-1} 1 - 128\, I_n \sum_{v=\lceil \frac{5n+1}{6} \rceil}^{n-1} 1 \\ = 32\, I_n \left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right) - 64\, I_n \left\lfloor \frac{n-1}{4} \right\rfloor - 128\, I_n \left\lfloor \frac{n-1}{6} \right\rfloor. \end{aligned}$$

The combined contribution to the sum from Cases 5c, 5d, and 5e is

$$128 \sum_{\substack{u,v=1 \\ u<v}}^{n-1} \sum_{j\geq 0} I[j < \tfrac{u+v-n-1}{2},\, j \neq \tfrac{2u+v-2n-1}{2},\, j \neq \tfrac{u+2v-2n-1}{2}]$$

$$= 128 \sum_{\substack{u,v=1 \\ u<v}}^{n-1} \left( \left\lfloor \frac{u+v-n}{2} \right\rfloor I[u+v>n] - I_v\, I[2u+v>2n] - I_u\, I[u+2v>2n] \right),$$

which is of the form $128 \sum_{\substack{u,v=1 \\ u<v}}^{n-1} s_{u,v}$ where $s_{u,v}$ is symmetric in $u$ and $v$. We calculate this combined contribution as $64 \sum_{u,v=1}^{n-1} s_{u,v} - 64 \sum_{u=1}^{n-1} s_{u,u}$. We have

$$64 \sum_{u,v=1}^{n-1} s_{u,v} = 64 \sum_{u,v=1}^{n-1} \left\lfloor \frac{u+v-n}{2} \right\rfloor I[u+v>n] - 128 \sum_{u,v=1}^{n-1} I_u\, I[u+2v>2n]$$

$$= 64 \left( \frac{2}{3} \left\lfloor \frac{n}{2} \right\rfloor \left( \frac{n-2}{2} \right) \left\lceil \frac{n-4}{2} \right\rceil + \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \right) - 128 \cdot \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor,$$

where the second summation is evaluated using the identity (2.6), and the first by substituting $w = u+v-n$ to obtain $64 \sum_{u=1}^{n-1} \sum_{w=1}^{u-1} \lfloor \frac{w}{2} \rfloor = 64 \sum_{u=1}^{n-1} \lfloor \frac{u}{2} \rfloor \lfloor \frac{u-1}{2} \rfloor$ from the identity (2.1), then applying the identity

$$\left\lfloor \frac{u}{2} \right\rfloor \left\lfloor \frac{u-1}{2} \right\rfloor = \left\lfloor \frac{u}{2} \right\rfloor \left\lfloor \frac{u-2}{2} \right\rfloor + I_u \left( \frac{u-1}{2} \right),$$

and finally using the identities (2.2) and (2.4). We also have

$$64 \sum_{u=1}^{n-1} s_{u,u} = 64 \sum_{u=\lceil \frac{n+1}{2} \rceil}^{n-1} \left\lfloor \frac{2u-n}{2} \right\rfloor - 128 \sum_{u=\lceil \frac{2n+1}{3} \rceil}^{n-1} I_u$$

$$= 32 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor - 128 \left( \left\lfloor \frac{n}{6} \right\rfloor + I[n \bmod 6 = 4] \right),$$

where the first summation is evaluated by considering the cases $n$ even and $n$ odd separately, and the second using the identity (2.10).

The combined contribution to the sum from Cases 5c, 5d, and 5e is therefore

$$64 \sum_{u,v=1}^{n-1} s_{u,v} - 64 \sum_{u=1}^{n-1} s_{u,u}$$

$$= \frac{128}{3} \left\lfloor \frac{n}{2} \right\rfloor \left( \frac{n-2}{2} \right) \left\lceil \frac{n-4}{2} \right\rceil - 64 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 128 \left\lfloor \frac{n}{6} \right\rfloor + 128 \cdot I[n \bmod 6 = 4].$$

Add the contributions from Case 1, Case 4, Cases 5a/5b, and Cases 5c/5d/5e to obtain

$$\sum_{u,v=1}^{n-1} \mathbb{E}\big(D_u^2 D_v^2\big) = 16\left\lfloor\frac{n}{2}\right\rfloor^2 \left\lfloor\frac{n-1}{2}\right\rfloor^2 + 64\left\lfloor\frac{n}{2}\right\rfloor \left(\frac{n-2}{2}\right) \left\lceil\frac{n-4}{2}\right\rceil - 64\left\lfloor\frac{n}{2}\right\rfloor \left\lfloor\frac{n-2}{2}\right\rfloor$$
$$+ 32\, I_n \left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right) - 64\, I_n \left\lfloor\frac{n-1}{4}\right\rfloor - 128\, I_n \left\lfloor\frac{n-1}{6}\right\rfloor + 128\left\lfloor\frac{n}{6}\right\rfloor + 128 \cdot I[n \bmod 6 = 4]. \tag{4.12}$$

We are now ready to determine $\mathbb{E}\big(\|f\|_4^4\big)$ and $\mathrm{Var}\big(\|f\|_4^4\big)$ using Proposition 6, separating the calculation according to whether $n$ is even or odd. Substitution of (4.8) and (4.10) into (4.3) gives

$$E = \frac{1}{2}(2n^2 - 3n + I_n), \tag{4.13}$$

and then from (2.12) we obtain

$$\mathbb{E}\big(\|f\|_4^4\big) = 3n^2 - 3n + \frac{1-(-1)^n}{2},$$

as required. Substitution of (4.8), (4.9), (4.10), (4.11), and (4.12) into (4.4) gives, after simplification,

$$V = \begin{cases} n^4 + 5n^3 - \dfrac{207}{4}n^2 + 76n + 64\left\lfloor\dfrac{n}{6}\right\rfloor + 64 \cdot I[n \bmod 6 = 4] & \text{for } n \text{ even,} \\ n^4 + 5n^3 - \dfrac{131}{4}n^2 + \dfrac{77}{2}n - 128\left\lfloor\dfrac{n-1}{6}\right\rfloor - 144\left\lfloor\dfrac{n-1}{4}\right\rfloor - \dfrac{47}{4} & \text{for } n \text{ odd,} \end{cases}$$

and then from (2.13) and (4.13) we find that

$$\mathrm{Var}\big(\|f\|_4^4\big) = \begin{cases} 32n^3 - 216n^2 + 304n + 256\left\lfloor\dfrac{n}{6}\right\rfloor + 256 \cdot I[n \bmod 6 = 4] & \text{for } n \text{ even,} \\ 32n^3 - 144n^2 + 160n - 576\left\lfloor\dfrac{n-1}{4}\right\rfloor - 512\left\lfloor\dfrac{n-1}{6}\right\rfloor - 48 & \text{for } n \text{ odd,} \end{cases}$$

as required. □

## 5 The class $\mathcal{S}_n$

In this section, we use Proposition 6 to prove Theorem 2 for the class of skew-symmetric Littlewood polynomials

$$\mathcal{S}_n = \Big\{ f \in \mathcal{L}_n :\ n \text{ is odd and } f(z) = (-1)^{\frac{n-1}{2}} z^{n-1} f(-z^{-1}) \Big\}.$$

*Proof of Theorem* 2. We obtain this result by modifying the proof of Theorem 3. Let $f(z) = \sum_{j=0}^{n-1} a_j z^j \in \mathcal{S}_n$. By the definition (1.1) of $\mathcal{S}_n$, the coefficients $a_j$ satisfy the skew-symmetry condition

$$a_j = (-1)^{j+\frac{n-1}{2}} a_{n-1-j} \quad \text{for } 0 \le j < n. \tag{5.1}$$

We regard the coefficients $a_0, a_1, \ldots, a_{\frac{n-1}{2}}$ as independent random variables that each take the values 1 and $-1$ with probability $\frac{1}{2}$, and the remaining coefficients as being determined by (5.1). Use condition (5.1) to rewrite (2.11) as

$$C_u = I_u\big((-1)^{\frac{n-u}{2}} + D_u\big).$$

Substitute for $C_u$ in (2.14), and for $C_u$ and $C_v$ in (2.15), to obtain the expressions

$$E = \frac{n-1}{2} + 2\sum_{u=1}^{n-1}(-1)^{\frac{n-u}{2}} I_u\,\mathbb{E}D_u + \sum_{u=1}^{n-1} I_u\,\mathbb{E}D_u^2, \tag{5.2}$$

$$\begin{aligned} V = \left(\frac{n-1}{2}\right)^2 &+ 4\left(\frac{n-1}{2}\right)\sum_{u=1}^{n-1}(-1)^{\frac{n-u}{2}} I_u\,\mathbb{E}D_u + 4\sum_{u,v=1}^{n-1}(-1)^{\frac{2n-u-v}{2}} I_u I_v\,\mathbb{E}\big(D_u D_v\big) \\ &+ 2\left(\frac{n-1}{2}\right)\sum_{u=1}^{n-1} I_u\,\mathbb{E}D_u^2 + 4\sum_{u,v=1}^{n-1}(-1)^{\frac{n-u}{2}} I_u I_v\,\mathbb{E}\big(D_u D_v^2\big) + \sum_{u,v=1}^{n-1} I_u I_v\,\mathbb{E}\big(D_u^2 D_v^2\big), \end{aligned} \tag{5.3}$$

noting that $(-1)^{n-u} I_u = I_u$ because $n$ is odd here.

We express each of $\mathbb{E}D_u$, $\mathbb{E}\big(D_u D_v\big)$, $\mathbb{E}D_u^2$, $\mathbb{E}\big(D_u D_v^2\big)$, and $\mathbb{E}\big(D_u^2 D_v^2\big)$ as a sum of expectation terms of the form $\mathbb{E}(a_{j_1} a_{j_2} \ldots a_{j_{2r}})$, where $1 \le r \le 4$, and then calculate $\mathbb{E}\big(\|f\|_4^4\big)$ and $\mathrm{Var}\big(\|f\|_4^4\big)$ from Proposition 6 by substitution into the forms (5.2) and (5.3). In view of the skew-symmetry condition (5.1), the expectation term $\mathbb{E}(a_{j_1} a_{j_2} \ldots a_{j_{2r}})$ is nonzero exactly when the indices $j_1, j_2, \ldots, j_{2r}$ admit a matching decomposition. The index sets admitting a matching decomposition are identical to those in the proof of Theorem 3; each symmetric index pair $\{j, k\}$ in the resulting expectation term introduces an additional multiplicative factor $(-1)^{j+\frac{n-1}{2}}$, by (5.1).

We use the same case analyses in the following calculations as in the proof of Theorem 3, inserting additional factors $I_u$, $I_v$, $(-1)^{\frac{n-u}{2}}$, and $(-1)^{\frac{2n-u-v}{2}}$ as appropriate.

**The sum $\sum_{u=1}^{n-1}(-1)^{\frac{n-u}{2}} I_u\,\mathbb{E}D_u$.** Similarly to (4.8), this sum is zero.

**The sum $\sum_{u,v=1}^{n-1}(-1)^{\frac{2n-u-v}{2}} I_u I_v\,\mathbb{E}\big(D_u D_v\big)$.** We modify the calculation of $\sum_{u,v=1}^{n-1} I_u I_v\,\mathbb{E}\big(D_u D_v\big)$ in the proof of Theorem 3. Cases 1 and 2 both receive a multiplicative factor $(-1)^{\frac{2n-u-v}{2}} I_u I_v$ in place of $I_u I_v$. Case 2 receives a further multiplicative factor $(-1)^{j+n-u+\frac{n-1}{2}} = (-1)^{\frac{v-u}{2}}$ because of the symmetric index pair $\{j+n-u,\, k+n-v\}$ with $j = \frac{u+v-n-1}{2}$. The resulting multiplicative factor in both cases is $I_u I_v$ (using the relation $u = v$ for Case 1), and the sum is therefore unchanged from (4.9). Since $n$ is odd, expression (4.9) simplifies to give

$$\sum_{u,v=1}^{n-1}(-1)^{\frac{2n-u-v}{2}} I_u I_v\,\mathbb{E}\big(D_u D_v\big) = 4\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right) - 4\left\lfloor \frac{n-1}{4} \right\rfloor.$$

**The sum $\sum_{u=1}^{n-1} I_u\,\mathbb{E}D_u^2$.** We modify the calculation of $\sum_{u=1}^{n-1}\mathbb{E}D_u^2$ in the proof of Theorem 3 by introducing an additional factor $I_u$, giving

$$\sum_{u=1}^{n-1} I_u\,\mathbb{E}D_u^2 = 4\sum_{u=1}^{n-1} I_u \binom{u-1}{2} = 2\binom{n-1}{2}\binom{n-3}{2}$$

using the summation identity (2.4).

**The sum $\sum_{u,v=1}^{n-1}(-1)^{\frac{n-u}{2}} I_u I_v\, \mathbb{E}\big(D_u D_v^2\big)$.** We modify the calculation of $\sum_{u,v=1}^{n-1} I_u\, \mathbb{E}\big(D_u D_v^2\big)$ in the proof of Theorem 3. Cases 1 to 4 all receive a multiplicative factor $(-1)^{\frac{n-u}{2}} I_u I_v$ in place of $I_u$. The presence of symmetric index pairs introduces further multiplicative factors: in Case 1, a factor of $(-1)^{(k+n-v)+\frac{n-1}{2}}(-1)^{(j+n-u)+\frac{n-1}{2}} = (-1)^{\frac{n-u}{2}}$ because of index pairs $\{k+n-v,\, \ell+n-v\}$ and $\{j+n-u,\, \ell\}$ with $j=k$ and $2v-u=n$; in Case 3, a factor of $(-1)^{(j+n-u)+\frac{n-1}{2}} = (-1)^{\frac{2v-u+n-2}{2}}$ because of index pair $\{j+n-u,\, \ell+n-v\}$ with $j = \frac{u+2v-2n-1}{2}$; and in Case 4, a factor of $(-1)^{(k+n-v)+\frac{n-1}{2}} = (-1)^{\frac{u+n-2}{2}}$ because of index pair $\{k+n-v,\, \ell+n-v\}$ with $k = \frac{u+2v-2n-1}{2}$.

The resulting multiplicative factors are given in Table 4 (using the relation $2v-u=n$ to evaluate Case 2). We see that the calculation differs from that of $\sum_{u,v=1}^{n-1} I_u\, \mathbb{E}\big(D_u D_v^2\big)$ in the proof of Theorem 3 only via the introduction of a factor $I_v$ in all four cases.

Table 4: Multiplicative factors in calculation of $\sum_{u,v=1}^{n-1}(-1)^{\frac{n-u}{2}} I_u I_v\, \mathbb{E}\big(D_u D_v^2\big)$

| Case | Multiplicative factor | Evaluates to |
|---|---|---|
| 1 | $(-1)^{\frac{n-u}{2}} I_u I_v \cdot (-1)^{\frac{n-u}{2}}$ | $I_u I_v$ |
| 2 | $(-1)^{\frac{n-u}{2}} I_u I_v$ | $I_u I_v$ |
| 3 | $(-1)^{\frac{n-u}{2}} I_u I_v \cdot (-1)^{\frac{2v-u+n-2}{2}}$ | $I_u I_v$ |
| 4 | $(-1)^{\frac{n-u}{2}} I_u I_v \cdot (-1)^{\frac{u+n-2}{2}}$ | $I_u I_v$ |

The combined contribution from Cases 1 and 2 is therefore

$$16 \sum_{v=\frac{n+1}{2}}^{n-1} I_v \left(\frac{2v-n-1}{2}\right) - 16 \sum_{v=\lceil \frac{2n+1}{3} \rceil}^{n-1} I_v = 16 \left\lfloor \frac{n-1}{4} \right\rfloor \left\lfloor \frac{n-3}{4} \right\rfloor - 16 \left\lfloor \frac{n-1}{6} \right\rfloor$$

using the summation identities (2.7) and (2.10). The combined additional contribution from Cases 3 and 4 is

$$\begin{aligned} 16 \sum_{\substack{u,v=1 \\ u\neq v \\ 2v-u\neq n}}^{n-1} I_u I_v\; I[u+2v>2n] &= 16 \sum_{u,v=1}^{n-1} I_u I_v\; I[u+2v>2n] - 16 \sum_{u=\lceil \frac{2n+1}{3} \rceil}^{n-1} I_u - 16 \sum_{v=\lceil \frac{3n+1}{4} \rceil}^{n-1} I_v \\ &= 16 \left\lfloor \frac{n-1}{4} \right\rfloor \left\lfloor \frac{n-3}{4} \right\rfloor - 16 \left\lfloor \frac{n-1}{6} \right\rfloor - 16 \left\lfloor \frac{n-1}{8} \right\rfloor \end{aligned}$$

using the summation identities (2.8), (2.10), and (2.9).

Add the contributions from Cases 1 and 2 to the additional contribution from Cases 3 and 4 to obtain

$$\sum_{u,v=1}^{n-1} (-1)^{\frac{n-u}{2}} I_u I_v\, \mathbb{E}\big(D_u D_v^2\big) = 32 \left\lfloor \frac{n-1}{4} \right\rfloor \left\lfloor \frac{n-3}{4} \right\rfloor - 32 \left\lfloor \frac{n-1}{6} \right\rfloor - 16 \left\lfloor \frac{n-1}{8} \right\rfloor.$$

**The sum $\sum_{u,v=1}^{n-1} I_u I_v \, \mathbb{E}\big(D_u^2 D_v^2\big)$.** We modify the calculation of $\sum_{u,v=1}^{n-1} \mathbb{E}\big(D_u^2 D_v^2\big)$ in the proof of Theorem 3. All cases receive a multiplicative factor $I_u I_v$ in place of 1. The presence of symmetric index pairs introduces further multiplicative factors: in Case 5a, a factor of $(-1)^{(k+n-u)+\frac{n-1}{2}} = 1$ because of index pair $\{k+n-u,\, m+n-v\}$ with $k = \frac{2v-n-1}{2}$; in Case 5b, a factor of $(-1)^{(k+n-u)+\frac{n-1}{2}} = 1$ because of index pair $\{k+n-u,\, j+n-u\}$ with $k = \frac{2u-2v+n-1}{2}$; in Case 5d, a factor of $(-1)^m(-1)^{j+n-u} = 1$ because of index pairs $\{m,\, k+n-u\}$ and $\{j+n-u,\, m+n-v\}$ with $m = u+v-n-1-j$; and in Case 5e, a factor of $(-1)^{k+n-u}(-1)^{j+n-u} = 1$ because of index pairs $\{k+n-u,\, j+n-v\}$ and $\{j+n-u,\, m+n-v\}$ with $k = u+v-n-1-j$. Since these further factors all equal 1, we see that the calculation differs from that of $\sum_{u,v=1}^{n-1} \mathbb{E}\big(D_u D_v^2\big)$ in the proof of Theorem 3 only via the introduction of a factor $I_u I_v$ in all cases.

The contribution from Case 1 is therefore

$$16\left(\sum_{u=1}^{n-1} I_u \binom{u-1}{2}\right)^2 = 4\binom{n-1}{2}^2\binom{n-3}{2}^2$$

using the summation identity (2.4). The contribution from Cases 2 and 3 is zero. The contribution from Case 4 is

$$32\sum_{u=1}^{n-1} I_u \binom{u-1}{2}\binom{u-3}{2} = \frac{32}{3}\binom{n-1}{2}\binom{n-3}{2}\binom{n-5}{2}$$

using the summation identity (2.5). The contribution from Cases 5a and 5b is

$$\begin{aligned}
&128 \sum_{\substack{u,v=1\\u<v}}^{n-1} I_u I_v \, I[2u+2v>3n] \; I[2v-u \neq n] \\
&\quad = 128 \cdot \frac{1}{2} \sum_{u,v=1}^{n-1} I_u I_v \, I[2u+2v>3n] - 64 \sum_{u=\lceil \frac{3n+1}{4} \rceil}^{n-1} I_u - 128 \sum_{v=\lceil \frac{5n+1}{6} \rceil}^{n-1} I_v \\
&\quad = 32 \left\lfloor \frac{n-1}{4} \right\rfloor \left\lfloor \frac{n-5}{4} \right\rfloor - 64 \left\lfloor \frac{n-1}{8} \right\rfloor - 128 \left\lfloor \frac{n-1}{12} \right\rfloor,
\end{aligned}$$

where the first summation is evaluated by substituting $u = 2U+1$ and $v = 2V+1$ and considering the cases $n \bmod 4 = 1$ and $n \bmod 4 = 3$ separately, the second using identity (2.9), and the third using an identity analogous to (2.9). The combined contribution to the sum from Cases 5c, 5d, and 5e is

$$128 \sum_{\substack{u,v=1\\u<v}}^{n-1} \left( I_u I_v \binom{\frac{u+v-n-1}{2}}{} I[u+v>n] - I_u I_v \, I[2u+v>2n] - I_u I_v \, I[u+2v>2n] \right),$$

which is of the form $128 \sum_{\substack{u,v=1\\u<v}}^{n-1} s'_{u,v}$ where $s'_{u,v}$ is symmetric in $u$ and $v$. We calculate

$$
\begin{aligned}
64 \sum_{u,v=1}^{n-1} s'_{u,v} &= 64 \sum_{u,v=1}^{n-1} I_u I_v \binom{\frac{u+v-n-1}{2}}{} I[u+v>n] - 128 \sum_{u,v=1}^{n-1} I_u I_v \, I[u+2v>2n] \\
&= 64 \cdot \frac{1}{6}\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\left(\frac{n-5}{2}\right) - 128 \left\lfloor \frac{n-1}{4} \right\rfloor \left\lfloor \frac{n-3}{4} \right\rfloor,
\end{aligned}
$$

where the first summation is evaluated by substituting $w = u+v-n$ to obtain the expression $64 \sum_{u=1}^{n-1} I_u \sum_{w=1}^{u-1} I_w \left(\frac{w-1}{2}\right)$ and then using the identities (2.4) and (2.5), and the second using the identity (2.8); and

$$
\begin{aligned}
64 \sum_{u=1}^{n-1} s'_{u,u} &= 64 \sum_{u=\frac{n+1}{2}}^{n-1} I_u \left(\frac{2u-n-1}{2}\right) - 128 \sum_{u=\lceil \frac{2n+1}{3} \rceil}^{n-1} I_u \\
&= 64 \left\lfloor \frac{n-1}{4} \right\rfloor \left\lfloor \frac{n-3}{4} \right\rfloor - 128 \left\lfloor \frac{n-1}{6} \right\rfloor,
\end{aligned}
$$

using the summation identities (2.7) and (2.10). The combined contribution to the sum from Cases 5c, 5d, and 5e is

$$
\begin{aligned}
&64 \sum_{u,v=1}^{n-1} s'_{u,v} - 64 \sum_{u=1}^{n-1} s'_{u,u} \\
&\quad = \frac{32}{3}\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\left(\frac{n-5}{2}\right) - 192 \left\lfloor \frac{n-1}{4} \right\rfloor \left\lfloor \frac{n-3}{4} \right\rfloor + 128 \left\lfloor \frac{n-1}{6} \right\rfloor.
\end{aligned}
$$

Add the contributions from Case 1, Case 4, Cases 5a/5b, and Cases 5c/5d/5e to obtain

$$
\begin{aligned}
&\sum_{u,v=1}^{n-1} I_u I_v \, \mathbb{E}\big(D_u^2 D_v^2\big) \\
&\quad = 4 \left(\frac{n-1}{2}\right)^2 \left(\frac{n-3}{2}\right)^2 + \frac{64}{3}\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\left(\frac{n-5}{2}\right) - 192 \left\lfloor \frac{n-1}{4} \right\rfloor \left\lfloor \frac{n-3}{4} \right\rfloor \\
&\qquad + 32 \left\lfloor \frac{n-1}{4} \right\rfloor \left\lfloor \frac{n-5}{4} \right\rfloor + 128 \left\lfloor \frac{n-1}{6} \right\rfloor - 64 \left\lfloor \frac{n-1}{8} \right\rfloor - 128 \left\lfloor \frac{n-1}{12} \right\rfloor.
\end{aligned}
$$

We now use the calculated expectation expressions to determine $\mathbb{E}\big(\|f\|_4^4\big)$ and $\mathrm{Var}\big(\|f\|_4^4\big)$ using Proposition 6. Substitution into (5.2) gives

$$
E = \frac{1}{2}(n^2 - 3n + 2), \tag{5.4}
$$

and then from (2.12) we obtain

$$
\mathbb{E}\big(\|f\|_4^4\big) = 2n^2 - 3n + 2,
$$

as required. Substitution into (5.3), after separation according to whether $n \bmod 4 = 1$ or $n \bmod 4 = 3$, gives

$$V = \begin{cases} \frac{1}{4}n^4 + \frac{7}{6}n^3 - \frac{75}{4}n^2 + \frac{151}{3}n - 128\left\lfloor \frac{n-1}{8} \right\rfloor - 128\left\lfloor \frac{n-1}{12} \right\rfloor - 33 & \text{for } n \bmod 4 = 1, \\ \frac{1}{4}n^4 + \frac{7}{6}n^3 - \frac{75}{4}n^2 + \frac{127}{3}n - 128\left\lfloor \frac{n-1}{8} \right\rfloor - 128\left\lfloor \frac{n-1}{12} \right\rfloor - 9 & \text{for } n \bmod 4 = 3, \end{cases}$$

and then from (2.13) and (5.4) we find that

$$\mathrm{Var}\big(\|f\|_4^4\big) = \frac{32}{3}n^3 - 88n^2 + \frac{592}{3}n - 512\left\lfloor \frac{n-1}{8} \right\rfloor - 512\left\lfloor \frac{n-1}{12} \right\rfloor - 88 + 16(-1)^{\frac{n-1}{2}}(n-3),$$

as required.

□

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