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TWO BINARY SEQUENCE FAMILIES WITH LARGE MERIT FACTOR

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Abstract. We calculate the asymptotic merit factor, under all rotations of sequence elements, of two families of binary sequences derived from Legendre sequences. The rotation is negaperiodic for the first family, and periodic for the second family. In both cases the maximum asymptotic merit factor is 6. As a consequence, we obtain the first two families of skew-symmetric sequences with known asymptotic merit factor, which is also 6 in both cases.

1. Introduction

We consider a *sequence* A of length n to be an n-tuple $(a_0, a_1, \dots, a_{n-1})$ of real numbers. The *aperiodic autocorrelation* of A at shift u is

$$C_A(u) := \begin{cases} \sum_{j=0}^{n-u-1} a_j a_{j+u} & \text{for } 0 \le u < n \\ C_A(-u) & \text{for } -n < u < 0, \end{cases}$$

and its *energy* $E(A)$ is $C_A(0)$. Provided that $\sum_{0<|u|<n}[C_A(u)]^2 > 0$, the *merit factor* of A is defined to be

$$F(A) := \frac{[E(A)]^2}{\sum_{0<|u|<n} [C_A(u)]^2}.$$

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The sequence $A = (a_0, a_1, \dots, a_{n-1})$ of length n is called *binary* if each a_j takes the value $+1$ or -1, in which case $E(A) = n$. Let F_n be the maximum value of the merit factor over the set of all 2^n binary sequences of length n. The Merit Factor Problem, which is to determine the value of $\limsup_{n\to\infty} F_n$, is important not only in digital communications engineering but also in complex analysis, theoretical physics, and theoretical chemistry (see [5] for a survey, and [6] for background on related problems). The current state of knowledge on this problem can be summarised as $6 \le \limsup_{n\to\infty} F_n \le \infty$, where the lower bound arises from an analysis due to Høholdt and Jensen [4] of a periodically rotated Legendre sequence (see Theorem 2). There is also considerable numerical evidence, though currently no proof, that an asymptotic merit factor greater than 6.34 can be achieved for a family of binary sequences related to Legendre sequences [2].

In this paper we determine the asymptotic merit factor, at all rotations of sequence elements, of two families of binary sequences constructed from a Legendre sequence of length n. The first family is derived from a "negaperiodic" construction described by Parker [8], and its sequences have length $2n$; the second family is derived from a "periodic" construction described by Yu and Gong [12], and its sequences have length $4n$. For both families, the maximum asymptotic merit factor over all rotations is 6, equal to the best proven result for $\limsup_{n\to\infty} F_n$. For the negaperiodic construction, the maximum value of 6 was previously proved by Xiong and Hall [11] for two specific negaperiodic rotations, following numerical work in [8], but no asymptotic merit factor values at other negaperiodic rotations were previously known. For the periodic construction, the general form of the merit factor over all periodic rotations was determined numerically for large n by Yu and Gong [12], but no asymptotic values were previously known.

A *skew-symmetric* sequence is a binary sequence $(a_0, a_1, \dots, a_{2m})$ of odd length $2m + 1$ for which

$$a_{m+j} = (-1)^j a_{m-j} \quad \text{for } 0 < j \le m. \tag{1}$$

By slight modification of the negaperiodic and periodic constructions we obtain two families of skew-symmetric sequences, each with asymptotic merit factor 6 (see Corollaries 6 and 9). To our knowledge, these are the first constructions of families of skew-symmetric sequences for which the asymptotic merit factor has been calculated. Historically, skew-symmetric sequences were considered to be good candidates for a large merit factor, in part because half of their aperiodic autocorrelation coefficients are guaranteed to be zero (see [5, Section 3.1] for background). While skew-symmetric sequences of length n that attain the optimal merit factor value F_n are known for

$$n \in \{3, 5, 7, 9, 11, 13, 15, 17, 21, 27, 29, 39, 41, 43, 45, 47, 49, 51, 53, 55, 57, 59\},$$

they are not known for any larger values of n, because the value of F_n itself is not known for $n > 60$ [5]. Golay [3] argued heuristically that the supremum limit of the merit factor of the set of skew-symmetric sequences is equal to $\limsup_{n\to\infty} F_n$, which would imply that nothing is lost by restricting attention to the subset of binary sequences that are skew-symmetric (see [5, Section 4.7] for a discussion). In view of this, it is particularly interesting that the asymptotic merit factor of the skew-symmetric sequence families constructed here is equal to the best proven result for $\limsup_{n\to\infty} F_n$, namely 6.

2. Definitions and Notation

In this section we introduce further definitions and notation for the paper.

Let $A = (a_0, a_1, \ldots, a_{n-1})$ and $B = (b_0, b_1, \ldots, b_{n-1})$ be sequences of equal length n. The *aperiodic crosscorrelation* between A and B at shift u is

$$C_{A,B}(u) := \begin{cases} \sum_{j=0}^{n-u-1} a_j b_{j+u} & \text{for } 0 \le u < n \\ C_{B,A}(-u) & \text{for } -n < u < 0, \end{cases}$$

and the *periodic crosscorrelation* between A and B at shift u is

$$R_{A,B}(u) := \sum_{j=0}^{n-1} a_j b_{(j+u) \bmod n} \quad \text{for } 0 \le u < n.$$

By adopting the convention that $C_{A,B}(-n) = 0$, we can write

$$R_{A,B}(u) = C_{A,B}(u) + C_{A,B}(u-n) \quad \text{for } 0 \le u < n. \tag{2}$$

The aperiodic autocorrelation $C_A(u)$ defined in Section 1 equals $C_{A,A}(u)$ for $|u| < n$. From the definition of $C_A(u)$ and $F(A)$ we have the relation

$$\frac{1}{F(A)} = -1 + \frac{1}{[E(A)]^2} \sum_{|u|<n} [C_A(u)]^2 \tag{3}$$

for the reciprocal merit factor $1/F(A)$. The *periodic autocorrelation* of A at shift u is

$$R_A(u) := R_{A,A}(u) \quad \text{for } 0 \le u < n.$$

Given a sequence $A = (a_0, a_1, \ldots, a_{n-1})$ of length n, we write $[A]_j$ to denote the sequence element a_j. Let $A = (a_0, a_1, \ldots, a_{n-1})$ and $B = (b_0, b_1, \ldots, b_{m-1})$ be sequences of length n and m, respectively. The *concatenation* $A;B$ of A and B is the length $n+m$ sequence given by

$$[A;B]_j := \begin{cases} a_j & \text{for } 0 \le j < n \\ b_{j-n} & \text{for } n \le j < n+m. \end{cases}$$

Provided $\gcd(m,n) = 1$, the *product* sequence $A \otimes B$ of length mn is defined by

$$[A \otimes B]_j := a_{j \bmod n} b_{j \bmod m} \quad \text{for } 0 \le j < mn,$$

and the *m-decimation* of A is the length n sequence C defined by

$$[C]_j := a_{mj \bmod n} \quad \text{for } 0 \le j < n.$$

The *periodic rotation* A_r of A by a fraction r of its length (for any real r) is the length n sequence given by

$$[A_r]_j := a_{(j + \lfloor nr \rfloor) \bmod n} \quad \text{for } 0 \le j < n, \tag{4}$$

and the *negaperiodic rotation* $A_{\tilde{r}}$ of A by the fraction r is the length n sequence given by

$$[A_{\tilde{r}}]_j := \begin{cases} a_{(j + \lfloor nr \rfloor) \bmod n} & \text{for } 0 \le j < n - \lfloor nr \rfloor \\ -a_{(j + \lfloor nr \rfloor) \bmod n} & \text{for } n - \lfloor nr \rfloor \le j < n. \end{cases} \tag{5}$$

The sequence $A_{\widetilde{r}}$ can be viewed as the first n elements of the length $2n$ sequence $(A;-A)_{\frac{r}{2}}$. For example, take $r=\frac{2}{7}$ and $A=(+,+,+,-,+,-,-)$, where $+$ and $-$ represent sequence elements $+1$ and -1, respectively. Then we have

$$\begin{aligned} A_r &= (+,-,+,-,-,+,+),\\ A_{\widetilde{r}} &= (+,-,+,-,-,-,-).\\ (A;-A)_{\frac{r}{2}} &= (+,-,+,-,-,-,-,-,+,-,+,+,+,+). \end{aligned}$$

3. Legendre Sequences

This section describes Legendre sequences and their asymptotic merit factor properties. (See [10], for example, for background on number-theoretic properties of Legendre sequences.)

Given a prime n and an integer j, the *Legendre symbol* $(j\,|\,n)$ is defined as

$$(j\,|\,n) := \begin{cases} 0 & \text{for } j \equiv 0 \pmod n \\ -1 & \text{for } j \text{ not a square modulo } n \\ +1 & \text{otherwise.} \end{cases}$$

The Legendre symbol is multiplicative, so that

$$(a\,|\,n)\,(b\,|\,n) = (ab\,|\,n), \tag{6}$$

and

$$a \equiv b \pmod n \quad \text{implies} \quad (a\,|\,n) = (b\,|\,n). \tag{7}$$

A *Legendre sequence* $X=(x_0,x_1,\dots,x_{n-1})$ of prime length n is defined by

$$x_j := \begin{cases} 1 & \text{for } j=0 \\ (j\,|\,n) & \text{for } 0<j<n. \end{cases}$$

In our analysis it will often be convenient to change the initial element in a Legendre sequence to be zero, so that

$$x_j = (j\,|\,n) \quad \text{for } 0 \le j < n.$$

Then we will call X a *modified Legendre sequence*. The periodic autocorrelation function of a modified Legendre sequence X is given [10, p. 294] by

$$R_X(u) = -1 \quad \text{for } 0<u<n. \tag{8}$$

We shall make repeated use of the following result, under which up to $o(\sqrt{n})$ elements of a sequence of length n can be changed by a bounded amount without altering the asymptotic reciprocal merit factor:

Proposition 1. *Let $\{A(n)\}$ and $\{B(n)\}$ be sets of sequences, where each of $A(n)$ and $B(n)$ has length n. Suppose that, for each n, all elements of $A(n)$ and $B(n)$ are bounded in magnitude by a constant independent of n. Suppose further that, as $n \longrightarrow \infty$, the number of nonzero elements of $B(n)$ is $o(\sqrt{n})$ and that $F(A(n)) = O(1)$ and $E(A(n)) = \Omega(n)$.*[1] *Then, as $n \longrightarrow \infty$, the elementwise sequence sums*

[1] We use the notation o, O, and Ω to compare the growth rates of functions $f(n)$ and $g(n)$ from $\mathbb{N}$ to $\mathbb{R}^+$ in the following standard way: $f(n)=o(g(n))$ means that $f(n)/g(n)\to 0$ as $n\to\infty$; $f(n)=O(g(n))$ means that there is a constant c, independent of n, for which $f(n)\le cg(n)$ for all sufficiently large n; and $f(n)=\Omega(g(n))$ means that $g(n)=O(f(n))$.

$\{A(n)+B(n)\}$ *satisfy*

$$\frac{1}{F(A(n)+B(n))} = \frac{1}{F(A(n))}(1+o(1)).$$

Proof. For each n, by the definition of aperiodic autocorrelation and crosscorrelation we have

$$C_{A(n)+B(n)}(u) = C_{A(n)}(u) + C_{B(n)}(u) + C_{A(n),B(n)}(u) + C_{B(n),A(n)}(u) \quad \text{for } |u| < n.$$

Since, by assumption, all elements of $A(n)$ and $B(n)$ are bounded in magnitude by a constant independent of n, and the number of nonzero elements of $B(n)$ is $o(\sqrt{n})$ as $n \longrightarrow \infty$, it follows that

$$C_{A(n)+B(n)}(u) = C_{A(n)}(u) + o(\sqrt{n}) \quad \text{as } n \longrightarrow \infty. \tag{9}$$

This implies by the definition of $F(A(n)+B(n))$ that, as $n \longrightarrow \infty$,

$$\begin{aligned}\frac{1}{F(A(n)+B(n))} &= \frac{1}{[E(A(n)+B(n))]^2}\sum_{0<|u|<n}[C_{A(n)}(u)+o(\sqrt{n})]^2\\ &= \frac{1}{[E(A(n)+B(n))]^2}\Bigg(\sum_{0<|u|<n}[C_{A(n)}(u)]^2\\ &\qquad + o(n)\sqrt{\sum_{0<|u|<n}[C_{A(n)}(u)]^2} + o(n^2)\Bigg)\end{aligned} \tag{10}$$

since, by the Cauchy–Schwarz inequality,

$$\sum_{0<|u|<n} o(\sqrt{n})C_{A(n)}(u) \le \sqrt{\Bigg(\sum_{0<|u|<n} o(n)\Bigg)\Bigg(\sum_{0<|u|<n}[C_{A(n)}(u)]^2\Bigg)}.$$

Now by setting $u=0$ in (9) and dividing by $E(A(n))$ we obtain, as $n \longrightarrow \infty$,

$$\begin{aligned}\frac{E(A(n)+B(n))}{E(A(n))} &= 1 + \frac{o(\sqrt{n})}{E(A(n))}\\ &= 1+o(1),\end{aligned}$$

since $E(A(n)) = \Omega(n)$ by assumption. The square of the reciprocal of this relation is

$$\frac{[E(A(n))]^2}{[E(A(n)+B(n))]^2} = 1 + o(1) \quad \text{as } n \longrightarrow \infty.$$

Substitute in (10) and use the definition of $F(A(n))$ to show that, as $n \longrightarrow \infty$,

$$\begin{aligned}\frac{1}{F(A(n)+B(n))} &= \frac{1}{F(A(n))}(1+o(1))\left(1 + o(n)\frac{\sqrt{F(A(n))}}{E(A(n))} + o(n^2)\frac{F(A(n))}{[E(A(n))]^2}\right)\\ &= \frac{1}{F(A(n))}(1+o(1)),\end{aligned}$$

since $F(A(n)) = O(1)$ and $E(A(n)) = \Omega(n)$ by assumption. $\square$

A result similar to Proposition 1 was stated in Corollary 6.3 of [2], but with conditions on the asymptotic growth of the sequence elements $A(n)$ and their merit factor $F(A(n))$ mistakenly omitted.

The asymptotic merit factor of a Legendre sequence has been calculated at all periodic rotations:

Theorem 2 (Høholdt and Jensen [4]). *Let X be a Legendre sequence of prime length $n > 2$, and let r be a real number satisfying $|r| \le \frac{1}{2}$. Then*

$$\frac{1}{\lim\limits_{n\longrightarrow\infty} F(X_r)} = \tfrac{1}{6} + 8\left(|r| - \tfrac{1}{4}\right)^2.$$

The constraint $|r| \le \frac{1}{2}$ in Theorem 2 is for notational convenience only, since by definition X_r is the same as X_{r+1} for any real r. By Proposition 1, Theorem 2 also holds for the modified Legendre sequence X' of length n, since X'_r differs from X_r in exactly one element for each n. The exact, rather than the asymptotic, value of $F(X_r)$ in Theorem 2 has been calculated [1] by refining the analysis of [4], but the exact value is not required here.

The following generalisation of Theorem 2 is a key tool of this paper:

Theorem 3. *Let X be a modified Legendre sequence of prime length $n > 2$, and let r, s, and t be real numbers satisfying $|r + \frac{s+t}{2}| \le \frac{1}{2}$, $|s| \le \frac{1}{2}$, and $|t| \le \frac{1}{2}$. Let $\{r(n)\}$, $\{s(n)\}$, and $\{t(n)\}$ be sets of real numbers such that $ns(n)$ and $nt(n)$ are integer for each n, and such that, as $n \longrightarrow \infty$, $r(n) = r + O(n^{-1})$, $s(n) = s + O(n^{-1})$, and $t(n) = t + O(n^{-1})$. Then, as $n \longrightarrow \infty$,*

$$\begin{aligned}\frac{1}{n^2}\sum_{|u|<n} & C_{X_{r(n)},\,X_{r(n)+s(n)}}(u)\, C_{X_{r(n)+t(n)},\,X_{r(n)+s(n)+t(n)}}(u)\\ &= \tfrac{1}{6} + 8(|r + \tfrac{s+t}{2}| - \tfrac{1}{4})^2 + 2(|s| - \tfrac{1}{2})^2 + 2(|t| - \tfrac{1}{2})^2 + O(n^{-1}(\log n)^2).\end{aligned}$$

Proof. See Appendix. □

We can recover Theorem 2 from Theorem 3 by setting $r(n) = r$ and $s(n) = t(n) = 0$ for each n, applying (3), using Proposition 1 to alter the initial element and thereby change a modified Legendre sequence into a Legendre sequence, and then taking the limit as $n \longrightarrow \infty$. There is no loss of generality in Theorem 3 from the restrictions $|r + \frac{s+t}{2}| \le \frac{1}{2}$, $|s| \le \frac{1}{2}$, and $|t| \le \frac{1}{2}$, since $X_{r+1} = X_r$ for all r. Nonetheless, we will find it useful to define, for real r, s, t, c,

$$\phi(r,c) := \begin{cases} (|r| - c)^2 & \text{for } |r| \le \frac{1}{2} \\ \phi(r+1,c) & \text{for all } r,\ c \end{cases} \tag{11}$$

and

$$\psi(r,s,t) := \tfrac{1}{6} + 8\phi(r + \tfrac{s+t}{2}, \tfrac{1}{4}) + 2\phi(s, \tfrac{1}{2}) + 2\phi(t, \tfrac{1}{2}), \tag{12}$$

so that the conclusion of Theorem 3 can be rephrased to hold for unrestricted r, s, t as $n \longrightarrow \infty$:

$$\begin{aligned}\frac{1}{n^2}\sum_{|u|<n} & C_{X_{r(n)},\,X_{r(n)+s(n)}}(u)\, C_{X_{r(n)+t(n)},\,X_{r(n)+s(n)+t(n)}}(u)\\ &= \psi(r,s,t) + O(n^{-1}(\log n)^2).\end{aligned} \tag{13}$$

It is easily verified that $\phi(r + \frac{1}{2}, \frac{1}{4}) = \phi(r, \frac{1}{4})$ for all r, which implies that

$$\phi(r, \tfrac{1}{4}) = \phi(r', \tfrac{1}{4}) \quad \text{provided } r' = r + \tfrac{a}{2} \text{ for some integer } a. \tag{14}$$

4. The Negaperiodic Construction

In this section we present the first of our two constructions, involving a negaperiodic rotation $Y_{\widetilde{r}}$ (as defined in (5)) of a sequence Y of length $2n$ that is in turn derived from a sequence X of odd length n. Lemma 4 analyses this construction for the case $r = \rho$, where ρ is subject to the constraint that $n\rho$ is integer (which forces the number $\lfloor 2nr \rfloor$ of rotated elements in the negaperiodic rotation $Y_{\widetilde{r}}$ to be even). We then show in Theorem 5 that, in the specific case that X is a Legendre sequence, Lemma 4 can be used to determine the asymptotic merit factor of $Y_{\widetilde{r}}$ for all real r, regardless of whether $\lfloor 2nr \rfloor$ is even or odd.

Lemma 4. *Let X be a sequence of odd length n, each of whose elements is bounded in magnitude by a constant independent of n, and let Z be the 2-decimation of X. Let r be a real number, and write $\rho := \lfloor nr \rfloor / n$ and $\delta := \frac{n+1}{2n}$. Define Y to be the first $2n$ elements of the sequence $X \otimes (+,+,-,-)$. Then, as $n \longrightarrow \infty$,*

$$\sum_{|u|<2n} [C_{Y_{\widetilde{\rho}}}(u)]^2$$
$$= \sum_{|u|<n} \left([C_{Z_r}(u) + C_{Z_{r+\delta}}(u)]^2 + [C_{Z_r, Z_{r+\delta}}(u) - C_{Z_{r+\delta}, Z_{r+2\delta}}(u) + O(1)]^2 \right).$$

Proof. We firstly determine an explicit expression for $[Y_{\widetilde{\rho}}]_j$, and deduce an expression for $C_{Y_{\widetilde{\rho}}}(u)$. By analysis of the cases u even and u odd, we then express $|C_{Y_{\widetilde{\rho}}}(u)|$ in terms of crosscorrelations of periodically rotated versions of the sequence Z_r, for both positive and negative values of u. Squaring and summing over u then gives the required result.

Write $X = (x_0, x_1, \ldots, x_{n-1})$, $Y = (y_0, y_1, \ldots, y_{2n-1})$, and $Z = (z_0, z_1, \ldots, z_{n-1})$. Now element $j \bmod 4$ of the sequence $(+,+,-,-)$ can be written as $(-1)^{(j^2-j)/2}$, for any integer j. By the definition of Y we therefore have

$$y_j = (-1)^{\frac{j^2-j}{2}} x_{j \bmod n} \quad \text{for } 0 \le j < 2n. \tag{15}$$

Write $k := \lfloor \frac{j+2n\rho}{2n} \rfloor$. Since $n\rho = \lfloor nr \rfloor$ is integer, the definition (5) of $Y_{\widetilde{\rho}}$ gives, for $0 \le j < 2n$,

$$\begin{aligned}
[Y_{\widetilde{\rho}}]_j &= \begin{cases} y_{(j+2n\rho) \bmod 2n} & \text{for } 0 \le j < 2n - 2n\rho \\ -y_{(j+2n\rho) \bmod 2n} & \text{for } 2n - 2n\rho \le j < 2n \end{cases} \\
&= (-1)^k y_{(j+2n\rho) \bmod 2n} \\
&= (-1)^k y_{(j+2n\rho-2nk) \bmod 2n} \\
&= (-1)^{k+\frac{(j+2n\rho-2nk)^2-(j+2n\rho-2nk)}{2}} x_{(j+2n\rho) \bmod n}
\end{aligned}$$

from (15). Since n is odd, this implies that

$$[Y_{\widetilde{\rho}}]_j = (-1)^{\frac{j^2-j}{2} - n\rho} x_{(j+2n\rho) \bmod n} \quad \text{for } 0 \le j < 2n.$$

Therefore, for $0 \le u < 2n$,

$$\begin{aligned}C_{Y_{\widetilde{\rho}}}(u) &= (-1)^{\frac{u^2-u}{2}} \sum_{j=0}^{2n-u-1} (-1)^{ju} x_{(j+2n\rho) \bmod n}\, x_{(j+u+2n\rho) \bmod n}\\
&= (-1)^{\frac{u^2-u}{2}} \sum_{j=0}^{2n-u-1} (-1)^{ju} [X;X]_{(j+2n\rho) \bmod 2n} [X;X]_{(j+u+2n\rho) \bmod 2n}\\
&= (-1)^{\frac{u^2-u}{2}} \sum_{j=0}^{2n-u-1} (-1)^{ju} w_j w_{j+u},\end{aligned}$$

where $W = (w_0, w_1, \ldots, w_{2n-1}) := (X;X)_\rho$. Take the absolute value, and separate both the summation index j and the argument u into even and odd values, giving

$$(16)\quad \left|C_{Y_{\widetilde{\rho}}}(2u)\right| = \left|\sum_{j=0}^{n-u-1} w_{2j} w_{2(j+u)} + \sum_{j=0}^{n-u-1} w_{2j+1} w_{2(j+u)+1}\right| \qquad \text{for } 0 \le u < n,$$

(17)

$$\left|C_{Y_{\widetilde{\rho}}}(2u+1)\right| = \left|\sum_{j=0}^{n-u-1} w_{2j} w_{2(j+u)+1} - \sum_{j=0}^{n-u-2} w_{2j+1} w_{2(j+u)+2}\right| \quad \text{for } 0 \le u < n.$$

We next express (16) and (17) in terms of crosscorrelations of periodically rotated versions of the sequence Z_r. Now, for all integer i, j satisfying $0 \le 2j + i < 2n$ we have

$$\begin{aligned}w_{2j+i} &= [X;X]_{(2j+i+2n\rho) \bmod 2n}\\
&= x_{(2j+i+2n\rho) \bmod n}\\
&= x_{(2j+(n+1)i+2n\rho) \bmod n}\\
&= x_{2(j+ni\delta+\lfloor nr \rfloor) \bmod n},\end{aligned}$$

by definition of δ and ρ. Since $n\delta = \frac{n+1}{2}$ is integer because n is odd, we therefore have, for all integer i, j satisfying $0 \le 2j + i < 2n$,

$$\begin{aligned}w_{2j+i} &= z_{(j+ni\delta+\lfloor nr \rfloor) \bmod n}\\
&= [Z_{r+i\delta}]_j.\end{aligned}$$

Substitution in (16) and (17) then shows that

$$(18)\qquad \left|C_{Y_{\widetilde{\rho}}}(2u)\right| = \left|C_{Z_r}(u) + C_{Z_{r+\delta}}(u)\right| \quad \text{for } 0 \le u < n,$$

and

$$\begin{aligned}\left|C_{Y_{\widetilde{\rho}}}(2u+1)\right| &= \left|C_{Z_r, Z_{r+\delta}}(u) - \left(C_{Z_{r+\delta}, Z_{r+2\delta}}(u) - [Z_{r+\delta}]_{n-u-1} [Z_{r+2\delta}]_{n-1}\right)\right|\\
(19)\qquad &= \left|C_{Z_r, Z_{r+\delta}}(u) - C_{Z_{r+\delta}, Z_{r+2\delta}}(u) + O(1)\right| \quad \text{for } 0 \le u < n,\end{aligned}$$

since, by assumption, the elements of X are bounded in magnitude by a constant independent of n and so the term $[Z_{r+\delta}]_{n-u-1}[Z_{r+2\delta}]_{n-1}$ is $O(1)$ as $n \longrightarrow \infty$.

We lastly find expressions corresponding to (18) and (19) for negative values of the argument. For any sequence A, by definition

$$(20)\qquad C_A(u) = C_A(-u) \quad \text{for } 0 < u < n,$$

and so (18) implies that

$$\big|C_{Y_{\widetilde{\rho}}}(-2u)\big| = \big|C_{Z_r}(-u) + C_{Z_{r+\delta}}(-u)\big| \quad \text{for } 0 < u < n.$$

By combining this with (18), we obtain

$$\sum_{|u|<n} \big[C_{Y_{\widetilde{\rho}}}(2u)\big]^2 = \sum_{|u|<n} \big[C_{Z_r}(u) + C_{Z_{r+\delta}}(u)\big]^2. \tag{21}$$

Furthermore, by applying (20) to (19) we also have, for $0 \le u < n$,

$$\begin{aligned} \big|C_{Y_{\widetilde{\rho}}}(-2u-1)\big| &= \big|C_{Z_r,Z_{r+\delta}}(u) - C_{Z_{r+\delta},Z_{r+2\delta}}(u) + O(1)\big| \\ &= \big|\big(R_{Z_r,Z_{r+\delta}}(u) - C_{Z_r,Z_{r+\delta}}(u-n)\big) \\ &\quad - \big(R_{Z_{r+\delta},Z_{r+2\delta}}(u) - C_{Z_{r+\delta},Z_{r+2\delta}}(u-n)\big) + O(1)\big| \end{aligned}$$

by (2), using the convention that $C_{A,B}(-n) = 0$ for sequences A and B of length n. But rotation of each of the sequences Z_r and $Z_{r+\delta}$ by $n\delta = \frac{n+1}{2}$ elements (to produce sequences $Z_{r+\delta}$ and $Z_{r+2\delta}$) does not change their periodic crosscorrelation function, so $R_{Z_r,Z_{r+\delta}}(u) - R_{Z_{r+\delta},Z_{r+2\delta}}(u)$ is identically zero. Therefore

$$\big|C_{Y_{\widetilde{\rho}}}(-2u-1)\big| = \big|C_{Z_r,Z_{r+\delta}}(u-n) - C_{Z_{r+\delta},Z_{r+2\delta}}(u-n) + O(1)\big| \quad \text{for } 0 \le u < n.$$

By combining this with (19), we obtain

$$\sum_{u=-n}^{n-1} \big[C_{Y_{\widetilde{\rho}}}(2u+1)\big]^2 = \sum_{|u|<n} \big[C_{Z_r,Z_{r+\delta}}(u) - C_{Z_{r+\delta},Z_{r+2\delta}}(u) + O(1)\big]^2,$$

which, after addition of (21), gives the required result. □

We now take the sequence X of Lemma 4 to be a Legendre sequence and, using Theorem 3, derive the asymptotic merit factor of the resulting sequence Y at all negaperiodic rotations.

Theorem 5. *Let X be a Legendre sequence of prime length $n > 2$, and let r be a real number satisfying $|r| \le \frac{1}{2}$. Define Y to be the first $2n$ elements of the sequence $X \otimes (+,+,-,-)$. Then*

$$\frac{1}{\lim\limits_{n\longrightarrow\infty} F(Y_{\widetilde{r}})} = \begin{cases} \frac{1}{6} + 8r^2 & \text{for } |r| \le \frac{1}{4} \\ \frac{1}{6} + 8(|r| - \frac{1}{2})^2 & \text{for } \frac{1}{4} \le |r| \le \frac{1}{2}. \end{cases}$$

Proof. Write $X = (x_0, x_1, \ldots, x_{n-1})$ and let X' be the modified Legendre sequence $(0, x_1, \ldots, x_{n-1})$. The sequence Y', obtained using X' instead of X in the definition of Y, differs from Y in exactly two elements. By Proposition 1, it is therefore sufficient to show that $1/F(Y'_{\widetilde{r}})$ has the asymptotic behaviour claimed for $1/F(Y_{\widetilde{r}})$. To simplify notation, we continue to work with X and Y but set $x_0 := 0$.

Write $\rho := \lfloor nr \rfloor / n$ so that, by the definition (5),

$$[Y_{\widetilde{r}}]_j = \begin{cases} y_{(j+\lfloor 2nr \rfloor) \bmod 2n} & \text{for } 0 \le j < 2n - \lfloor 2nr \rfloor \\ -y_{(j+\lfloor 2nr \rfloor) \bmod 2n} & \text{for } 2n - \lfloor 2nr \rfloor \le j < 2n, \end{cases}$$

$$[Y_{\widetilde{\rho}}]_j = \begin{cases} y_{(j+2\lfloor nr \rfloor) \bmod 2n} & \text{for } 0 \le j < 2n - 2\lfloor nr \rfloor \\ -y_{(j+2\lfloor nr \rfloor) \bmod 2n} & \text{for } 2n - 2\lfloor nr \rfloor \le j < 2n. \end{cases}$$

For each n, either $\lfloor 2nr \rfloor = 2\lfloor nr \rfloor$, in which case the length $2n$ sequences $Y_{\widetilde{r}}$ and $Y_{\widetilde{\rho}}$ are identical, or else $\lfloor 2nr \rfloor = 2\lfloor nr \rfloor + 1$, in which case $Y_{\widetilde{r}}$ and $Y_{\widetilde{\rho}}$ share a common subsequence of length $2n-1$. By Proposition 1, it is therefore sufficient to show that $1/F(Y_{\widetilde{\rho}})$ has the asymptotic behaviour claimed for $1/F(Y_{\widetilde{r}})$.

Let Z be the 2-decimation of X. For $0 \le j < n$, this gives

$$\begin{aligned}[Z]_j &= x_{2j \bmod n} \\ &= (2j \,|\, n)\end{aligned}$$

by (7). Then by (6) and the definition of a modified Legendre sequence,

$$[Z]_j = (2\,|\,n)\,[X]_j \quad \text{for } 0 \le j < n,$$

so that $Z = (2\,|\,n)\,X$. Therefore

$$C_{Z_s,Z_t}(u) = C_{X_s,X_t}(u) \quad \text{for all integer } u \text{ satisfying } |u| < n \text{ and all real } s,t.$$

Application of Lemma 4 to the modified Legendre sequence X then gives

$$\begin{aligned}\frac{1}{n^2}\sum_{|u|<2n}\big[C_{Y_{\widetilde{\rho}}}(u)\big]^2 = \frac{1}{n^2}\sum_{|u|<n}\Big(&\big[C_{X_r}(u) + C_{X_{r+\delta}}(u)\big]^2 \\ &+ \big[C_{X_r,X_{r+\delta}}(u) - C_{X_{r+\delta},X_{r+2\delta}}(u) + O(1)\big]^2\Big),\end{aligned} \tag{22}$$

where

$$\delta = \delta(n) := \frac{n+1}{2n}. \tag{23}$$

We complete the proof by using Theorem 3 to evaluate the expansion of the right hand side of (22) as $n \longrightarrow \infty$. When $\frac{1}{n^2}\sum_{|u|<n}[C_{X_r,X_{r+\delta}}(u) - C_{X_{r+\delta},X_{r+2\delta}}(u)]^2$ is expanded into three sums, by Theorem 3 each of the sums is $O(1)$ as $n \longrightarrow \infty$. Therefore, by the Cauchy-Schwarz inequality, the additional contribution arising from the inclusion of the term $O(1)$ in the second square bracket of (22) is $O(n^{-1/2})$, which can be neglected. By applying Theorem 3 and (13) to the six resulting sums in the expansion of the right hand side of (22) (noting from (23) that $\delta(n) = \frac{1}{2} + O(n^{-1})$ as $n \longrightarrow \infty$), and then taking the limit, we obtain

$$\begin{aligned}\lim_{n\longrightarrow\infty}\frac{1}{n^2}\sum_{|u|<2n}&\big[C_{Y_{\widetilde{\rho}}}(u)\big]^2 \\ &= \psi(r,0,0) + \psi(r+\tfrac{1}{2},0,0) \\ &\qquad + 2\psi(r,0,\tfrac{1}{2}) + \psi(r,\tfrac{1}{2},0) + \psi(r+\tfrac{1}{2},\tfrac{1}{2},0) - 2\psi(r,\tfrac{1}{2},\tfrac{1}{2}) \\ &= \tfrac{2}{3} + 32\phi(r+\tfrac{1}{4},\tfrac{1}{4}) + 16\phi(0,\tfrac{1}{2})\end{aligned} \tag{24}$$

by the definition (12) of ψ and by (14). By the definition (11) of ϕ, in the given range $|r| \le \frac{1}{2}$ we have

$$\phi(0,\tfrac{1}{2}) = \tfrac{1}{4},$$

$$\phi(r+\tfrac{1}{4},\tfrac{1}{4}) = \begin{cases} r^2 & \text{for } |r| \le \frac{1}{4} \\ \big(|r| - \frac{1}{2}\big)^2 & \text{for } \frac{1}{4} \le |r| \le \frac{1}{2}. \end{cases} \tag{25}$$

Now, since $x_0 = 0$, the definition of Y gives $[E(Y_{\widetilde{\rho}})]^2 = (2n-2)^2$. Substitution of (25) into (24), together with the relation (3), then yields

$$\frac{1}{\lim_{n\longrightarrow\infty} F(Y_{\widetilde{\rho}})} = \begin{cases} \frac{1}{6} + 8r^2 & \text{for } |r| \le \frac{1}{4} \\ \frac{1}{6} + 8\left(|r| - \frac{1}{2}\right)^2 & \text{for } \frac{1}{4} \le |r| \le \frac{1}{2}, \end{cases}$$

as required. □

The maximum asymptotic merit factor obtained under the construction of Theorem 5 is 6, which occurs at $r = 0$ and at $r = \frac{1}{2}$. The special cases $r = 0$ and $r = \frac{1}{2}$ of Theorem 5 were proved by Xiong and Hall [11] in response to numerical evidence presented by Parker [8, Figs. 1,2].

Corollary 6. *Let $X = (x_0, x_1, \ldots, x_{n-1})$ be a Legendre sequence of prime length $n \equiv 1 \pmod 4$, and define Y to be the first $2n$ elements of the sequence $X \otimes (+,+,-,-)$. Then the sequence $Y;(-)$ is skew-symmetric of length $2n+1$ and has asymptotic merit factor 6.*

Proof. Take $r = 0$ in Theorem 5, and apply Proposition 1 to show that

$$\lim_{n\longrightarrow\infty} F(Y;(-)) = \lim_{n\longrightarrow\infty} F(Y) = 6.$$

It remains to show that $Y;(-)$ is skew-symmetric. Since $n \equiv 1 \pmod 4$, we know that $(-1\,|\,n) = 1$ (see [10, p. 184], for example), and it follows from (6) and (7) that

$$x_{n-j} = x_j \quad \text{for } 0 < j < n. \tag{26}$$

Now by writing $Y;(-) = (y_0, y_1, \ldots, y_{2n})$, from (15) we find that, for $0 < j < n$,

$$\begin{aligned} y_{n+j} &= (-1)^{\frac{(n+j)^2-(n+j)}{2}} x_j \\ &= (-1)^{j+\frac{(n-j)^2-(n-j)}{2}} x_{n-j} \end{aligned}$$

by (26). Then using (15) again we obtain

$$y_{n+j} = (-1)^j y_{n-j} \quad \text{for } 0 < j < n,$$

and by construction the same equation also holds for $j = n$ since $y_0 = x_0 = 1$. Therefore the sequence $Y;(-)$ is skew-symmetric, by the definition (1). □

To our knowledge, Corollary 6, together with Corollary 9 to be proved in Section 5, are the first known asymptotic results on the merit factor of a family of skew-symmetric sequences.

In the case $n \equiv 3 \pmod 4$, the sequence constructed in Corollary 6 does not satisfy the skew-symmetry condition (1). However, in that case we can instead form the length $2n+1$ sequence $Y;(+)$ and then set the central element of this sequence to be 0. The resulting ternary sequence $(a_0, a_1, \ldots, a_{2n})$ also has asymptotic merit factor 6, and satisfies the following modification of the skew-symmetry condition:

$$a_{n+j} = (-1)^{j+1} a_{n-j} \quad \text{for } 0 < j \le n.$$

5. The Periodic Construction

In this section we present the second of our two constructions, involving a periodic rotation Y_r of a sequence Y of length $4n$ derived from a sequence X of odd length n. Lemma 7 analyses this construction for the case $r = \rho$, where $n\rho$ is integer (which forces the number $\lfloor 4nr \rfloor$ of rotated elements in Y_r to be a multiple of 4). We then show in Theorem 8 that, in the case that X is a Legendre sequence, Lemma 7 can be used to determine the asymptotic merit factor of Y_r for all real r.

Lemma 7. *Let X be a sequence of odd length n, each of whose elements is bounded in magnitude by a constant independent of n, and let Z be the 4-decimation of X. Let r be a real number, and write $\rho := \lfloor nr \rfloor / n$ and*

$$\delta := \begin{cases} \frac{3n+1}{4n} & \text{for } n \equiv 1 \pmod 4 \\ \frac{n+1}{4n} & \text{for } n \equiv 3 \pmod 4). \end{cases}$$

Let Y be the length $4n$ sequence $X \otimes (+,+,+,-)$. Then, as $n \longrightarrow \infty$,

$$\sum_{|u|<4n} \big[C_{Y_\rho}(u)\big]^2 = \sum_{k=0}^{3} \sum_{|u|<n} \left(\sum_{i=0}^{3} (-1)^{\frac{ik(i+k+2)}{2}} C_{Z_{r+i\delta}, Z_{r+(i+k)\delta}}(u) + O(1) \right)^2 .$$

Proof. This proof is closely modelled on that of Lemma 4, and so is presented in less detail.

Write $X = (x_0, x_1, \ldots, x_{n-1})$ and $Y = (y_0, y_1, \ldots, y_{4n-1})$. By definition of Y we can write

$$y_j = (-1)^{\frac{j^3-j^2}{2}} x_{j \bmod n} \quad \text{for } 0 \le j < 4n. \tag{27}$$

Since $n\rho = \lfloor nr \rfloor$, the definition (4) of Y_ρ then gives

$$[Y_\rho]_j = (-1)^{\frac{j^3-j^2}{2}} x_{(j+4n\rho) \bmod n} \quad \text{for } 0 \le j < 4n.$$

Therefore, for $0 \le u < 4n$,

$$\begin{aligned} C_{Y_\rho}(u) &= (-1)^{\frac{u^3-u^2}{2}} \sum_{j=0}^{4n-u-1} (-1)^{\frac{ju(j+u+2)}{2}} x_{(j+4n\rho) \bmod n}\, x_{(j+u+4n\rho) \bmod n} \\ &= (-1)^{\frac{u^3-u^2}{2}} \sum_{j=0}^{4n-u-1} (-1)^{\frac{ju(j+u+2)}{2}} w_j w_{j+u}, \end{aligned}$$

where $W = (w_0, w_1, \ldots, w_{4n-1}) := (X; X; X; X)_\rho$. Take the absolute value, and separate the summation index j according to its value mod 4 to obtain

$$|C_{Y_\rho}(u)| = \left| \sum_{i=0}^{3} (-1)^{\frac{iu(i+u+2)}{2}} \sum_{j=0}^{\lfloor \frac{4n-u-i-1}{4} \rfloor} w_{4j+i} w_{4j+i+u} \right| \quad \text{for } 0 \le u < 4n.$$

Since $n\delta$ is integer by definition of δ, a similar argument to that used in the proof of Lemma 4 shows that

$$w_{4j+i} = [Z_{r+i\delta}]_j \quad \text{for all integer } i, j \text{ satisfying } 0 \le 4j+i < 4n.$$

Therefore, for $0 \le u < n$ and $0 \le k \le 3$,

$$|C_{Y_\rho}(4u+k)| = \left|\sum_{i=0}^{3}(-1)^{\frac{ik(i+k+2)}{2}} \sum_{j=0}^{\lfloor\frac{4n-4u-k-i-1}{4}\rfloor} [Z_{r+i\delta}]_j [Z_{r+(i+k)\delta}]_{j+u}\right|$$

$$= \left|\sum_{i=0}^{3}(-1)^{\frac{ik(i+k+2)}{2}} C_{Z_{r+i\delta},Z_{r+(i+k)\delta}}(u) + O(1)\right|. \tag{28}$$

We deal with negative values of the argument of the left hand side of (28) as in the proof of Lemma 4, using (20). For $0 < u < n$ and $k = 0$ this gives

$$|C_{Y_\rho}(-4u)| = \left|\sum_{i=0}^{3} C_{Z_{r+i\delta}}(-u) + O(1)\right|, \tag{29}$$

while for $0 \le u < n$ and $k \in \{1,2,3\}$ we have

$$|C_{Y_\rho}(-4u-k)|$$
$$= \left|\sum_{i=0}^{3}(-1)^{\frac{ik(i+k+2)}{2}} \left(R_{Z_{r+i\delta},Z_{r+(i+k)\delta}}(u) - C_{Z_{r+i\delta},Z_{r+(i+k)\delta}}(u-n)\right) + O(1)\right|$$
$$= \left|\sum_{i=0}^{3}(-1)^{\frac{ik(i+k+2)}{2}} C_{Z_{r+i\delta},Z_{r+(i+k)\delta}}(u-n) + O(1)\right|, \tag{30}$$

where the terms involving the periodic crosscorrelation R cancel since, for each $k \in \{1,2,3\}$, $(-1)^{\frac{ik(i+k+2)}{2}}$ takes the value $+1$ twice and the value -1 twice as i ranges over $0 \le i \le 3$. By combining (29) and (30) with (28) we obtain the required result. □

We now take the sequence X of Lemma 7 to be a Legendre sequence and, using Theorem 3, derive the asymptotic merit factor of the resulting sequence Y at all periodic rotations.

Theorem 8. *Let X be a Legendre sequence of prime length $n > 2$, and let r be a real number satisfying $|r| \le \frac{1}{2}$. Let Y be the length $4n$ sequence $X \otimes (+,+,+,-)$. Then*

$$\frac{1}{\lim\limits_{n\longrightarrow\infty} F(Y_r)} = \begin{cases} \frac{1}{6} + 8r^2 & \text{for } |r| \le \frac{1}{4} \\ \frac{1}{6} + 8(|r| - \frac{1}{2})^2 & \text{for } \frac{1}{4} \le |r| \le \frac{1}{2}. \end{cases}$$

Proof. Write $X = (x_0, x_1, \ldots, x_{n-1})$ and $\rho := \lfloor nr \rfloor / n$. By Proposition 1, we can take $x_0 = 0$. Furthermore, the length $4n$ sequences Y_r and Y_ρ share a common subsequence of length at least $4n-3$, and so by Proposition 1 it is sufficient to show that $1/F(Y_\rho)$ has the asymptotic behaviour claimed for $1/F(Y_r)$. Following the method of proof of Theorem 5, the 4-decimation Z of X is given by $Z = (4\,|\,n)\,X$, so by (6) we have $Z = (2\,|\,n)^2 X = X$. Application of Lemma 7 then gives

$$\frac{1}{n^2}\sum_{|u|<4n}\left[C_{Y_\rho}(u)\right]^2 = \frac{1}{n^2}\sum_{k=0}^{3}\sum_{|u|<n}\left(\sum_{i=0}^{3}(-1)^{\frac{ik(i+k+2)}{2}} C_{X_{r+i\delta},X_{r+(i+k)\delta}}(u) + O(1)\right)^2,$$

where

$$\delta = \delta(n) := \begin{cases} \frac{3n+1}{4n} & \text{for } n \equiv 1 \pmod 4 \\ \frac{n+1}{4n} & \text{for } n \equiv 3 \pmod 4). \end{cases} \tag{31}$$

As $n \longrightarrow \infty$ we can neglect the $O(1)$ term, as in the proof of Theorem 5, so that

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \frac{1}{n^2} \sum_{|u|<4n} \left[C_{Y_\rho}(u)\right]^2 \\
&= \lim_{n\longrightarrow\infty} \sum_{k=0}^{3}\sum_{i=0}^{3}\sum_{j=0}^{3} (-1)^{\frac{ik(i+k+2)+jk(j+k+2)}{2}} \\
&\qquad\cdot \frac{1}{n^2}\sum_{|u|<n} C_{X_{r+i\delta},X_{r+(i+k)\delta}}(u)C_{X_{r+j\delta},X_{r+(j+k)\delta}}(u) \\
&= \sum_{0\le i,j,k\le 3} (-1)^{\frac{ik(i+k+2)+jk(j+k+2)}{2}} \psi(r+i\Delta, k\Delta, (j-i)\Delta)
\end{aligned}$$

by Theorem 3 and (13), where from (31) we have $\delta(n) = \Delta + O(n^{-1})$ as $n \longrightarrow \infty$ and

$$\Delta := \begin{cases} \frac{3}{4} & \text{for } n \equiv 1 \pmod 4 \\ \frac{1}{4} & \text{for } n \equiv 3 \pmod 4. \end{cases} \tag{32}$$

By the definition (12) of ψ we then obtain

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \frac{1}{n^2} \sum_{|u|<4n} \left[C_{Y_\rho}(u)\right]^2 \\
&= \sum_{0\le i,j,k\le 3} (-1)^{\frac{ik(i+k+2)+jk(j+k+2)}{2}} \Bigg(\tfrac{1}{6} + 8\phi\left(r+\tfrac{i+j+k}{2}\Delta, \tfrac{1}{4}\right) \\
&\qquad\qquad + 2\phi\left(k\Delta, \tfrac{1}{2}\right) + 2\phi\left((j-i)\Delta, \tfrac{1}{2}\right)\Bigg) \\
&= \sum_{\ell=0}^{3} \sum_{\substack{0\le i,j,k\le 3\\ i+j+k\equiv \ell \pmod 4}} (-1)^{\frac{ik(i+k+2)+jk(j+k+2)}{2}} \left(\tfrac{1}{6} + 8\phi\left(r+\tfrac{\ell}{8}, \tfrac{1}{4}\right)\right) \\
&\quad + 2\sum_{0\le i,j,k\le 3} (-1)^{\frac{ik(i+k+2)+jk(j+k+2)}{2}} \left(\phi\left(k\Delta, \tfrac{1}{2}\right) + \phi\left((j-i)\Delta, \tfrac{1}{2}\right)\right) \qquad (33)
\end{aligned}$$

by (14), for both possible values of Δ given in (32). By direct calculation we find that

$$\sum_{\substack{0\le i,j,k\le 3\\ i+j+k\equiv \ell \pmod 4}} (-1)^{\frac{ik(i+k+2)+jk(j+k+2)}{2}} = \begin{cases} 16 & \text{for } \ell = 2 \\ 0 & \text{for } \ell \in \{0,1,3\}, \end{cases}$$

and that

$$\begin{aligned}
\sum_{0\le i,j,k\le 3} (-1)^{\frac{ik(i+k+2)+jk(j+k+2)}{2}} \phi\left(k\Delta, \tfrac{1}{2}\right) \\
&= \sum_{0\le i,j,k\le 3} (-1)^{\frac{ik(i+k+2)+jk(j+k+2)}{2}} \phi\left((j-i)\Delta, \tfrac{1}{2}\right) = 4
\end{aligned}$$

for both possible values of Δ. Substitution in (33) then gives

$$\lim_{n\longrightarrow\infty} \frac{1}{n^2} \sum_{|u|<4n} \left[C_{Y_\rho}(u)\right]^2 = 16\left(\tfrac{7}{6} + 8\phi\left(r+\tfrac{1}{4}, \tfrac{1}{4}\right)\right),$$

and then, since $[E(Y_\rho)]^2 = (4n-4)^2$, by (3) and (25) we conclude that

$$\frac{1}{\lim\limits_{n\longrightarrow\infty} F(Y_\rho)} = \begin{cases} \frac{1}{6} + 8r^2 & \text{for } |r| \le \frac{1}{4} \\ \frac{1}{6} + 8\left(|r| - \frac{1}{2}\right)^2 & \text{for } \frac{1}{4} \le |r| \le \frac{1}{2}, \end{cases}$$

as required. □

The maximum asymptotic merit factor obtained under the construction of Theorem 8 is 6, which occurs at $r = 0$ and at $r = \frac{1}{2}$. Theorem 8 explains the numerical results obtained by Yu and Gong [12, Fig. 1], based on the sequence $Y = X \otimes (+,+,+,-)$, for all values of r. It can readily be modified to explain also their numerical results based on the related sequence $X \otimes (-,+,+,+)$.

Corollary 9. *Let $X = (x_0, x_1, \ldots, x_{n-1})$ be a Legendre sequence of prime length $n \equiv 1 \pmod 4$, and let Y be the length $4n$ sequence $X \otimes (+,+,+,-)$. Then the sequence $Y;(+)$ is skew-symmetric of length $4n+1$ and has asymptotic merit factor 6.*

Proof. The proof is similar to that of Corollary 6. Take $r = 0$ in Theorem 8, and apply Proposition 1 to show that

$$\lim_{n\longrightarrow\infty} F(Y;(+)) = 6.$$

To show that $Y;(+)$ is skew-symmetric, write $Y;(+) = (y_0, y_1, \ldots, y_{4n})$ and note that (26) holds since $n \equiv 1 \pmod 4$. Then from (27) we have, for $0 < j < 2n$,

$$\begin{aligned} y_{2n+j} &= (-1)^{\frac{(2n+j)^3-(2n+j)^2}{2}} x_{j \bmod n} \\ &= (-1)^{j+\frac{(2n-j)^3-(2n-j)^2}{2}} x_{(2n-j) \bmod n} \\ &= (-1)^j y_{2n-j}, \end{aligned}$$

and by construction the same equation also holds for $j = 2n$ since $y_0 = x_0 = 1$. □

6. Conclusion

Theorems 5 and 8 establish the asymptotic merit factor of a negaperiodic construction described in [8] and a periodic construction described in [12], based in both cases on Legendre sequences. Corollaries 6 and 9 provide families of skew-symmetric sequences having asymptotic merit factor 6, equal to the current best proven result for $\limsup_{n\to\infty} F_n$.

However there is considerable numerical evidence, though currently no proof, that an asymptotic merit factor greater than 6.34 can be achieved for a family of binary sequences constructed via periodic appending of a Legendre sequence [2]. Numerical results displayed by Yu and Gong [12, Fig. 3] for the periodic construction, and reported by Parker [9, p. 12] for the negaperiodic construction, indicate that both constructions also attain a merit factor greater than 6.34 under suitable appending. A possible direction for future research is to attempt to prove that the asymptotic merit factor of these appended sequences is equal to the (as yet undetermined) asymptotic merit factor of the periodically appended Legendre sequences described in [2].

Appendix: Proof of Theorem 3

This appendix contains a proof of Theorem 3. Our starting point is the method of Høholdt and Jensen [4], which calculates the reciprocal merit factor of an arbitrary sequence of odd length n as the sum of expressions involving complex nth roots of unity. We generalise this method in Lemma 10 in order to calculate the sum of products of crosscorrelations of periodic rotations of an arbitrary odd-length sequence. We then prove Theorem 3 by applying Lemma 10 to the specific case of a modified Legendre sequence.

Write

$$\epsilon_j := e^{2\pi ij/n} \quad \text{for integer } j, \text{ where } i := \sqrt{-1}.$$

Given a sequence $A = (a_0, a_1, \ldots, a_{n-1})$ of length n, we define the *z-transform* of A to be the function $Q_A : \mathbb{C} \to \mathbb{C}$ given by

$$Q_A(z) := \sum_{k=0}^{n-1} a_k z^k,$$

and define

$$\Lambda_A(j,k,\ell) := \sum_{a=0}^{n-1} Q_A(\epsilon_a)\overline{Q_A(\epsilon_{a+j})}Q_A(\epsilon_{a+k})\overline{Q_A(\epsilon_{a+\ell})} \quad \text{for integer } j,\ k,\ \ell. \tag{34}$$

Lemma 10. *Let X be a sequence of odd length n. Let S and T be integers, and write $s := S/n$ and $t := T/n$. Then*

$$\frac{1}{n^2}\sum_{|u|<n} C_{X,X_s}(u)\, C_{X_t,X_{s+t}}(u) = \frac{2n^2+1}{3n^5}\,\Lambda_X(0,0,0) + B + C + D, \tag{35}$$

where

$$B = \frac{1}{n^5}\sum_{k=1}^{n-1}\Big[\big(\epsilon_k^T + \epsilon_k^S\big)\Lambda_X(0,0,k) + \big(\epsilon_k^{-(S+T-1)} + \epsilon_k\big)\overline{\Lambda_X(0,0,k)}\Big] \cdot \frac{1+\epsilon_k}{(1-\epsilon_k)^2},$$

$$\begin{aligned} C = -\frac{2}{n^5}\sum_{\substack{1\le k,\ell<n\\ k\ne\ell}} &\Big[\big(\epsilon_k^{-(S+T)}+1\big)\big(\epsilon_\ell^T + \epsilon_\ell^S\big)\Lambda_X(0,k,\ell) \\ &+ \epsilon_k^{S-1}\epsilon_\ell^T\,\Lambda_X(k,0,\ell) + \epsilon_k^{-(S+T)}\epsilon_\ell\,\overline{\Lambda_X(k,0,\ell)}\Big] \cdot \frac{\epsilon_k}{(1-\epsilon_k)(1-\epsilon_\ell)}, \end{aligned}$$

$$D = \frac{4}{n^5}\sum_{k=1}^{n-1}\big[\big(\epsilon_k^S + \epsilon_k^T\big)\Lambda_X(0,k,k) + \epsilon_k^{S+T-1}\,\Lambda_X(k,0,k)\big] \cdot \frac{1}{|1-\epsilon_k|^2}.$$

Proof. Let U, V, W, Y, Z be any sequences of length n. Straightforward manipulation shows that

$$Q_U(z)Q_V(z^{-1}) = \sum_{|u|<n} C_{U,V}(u) z^{-u} \quad \text{for all } z \ne 0.$$

Since $Q_U(\epsilon_j^{-1}) = \overline{Q_U(\epsilon_j)}$, we then have

$$\frac{1}{2n^3}\sum_{j=0}^{n-1} Q_U(\epsilon_j)\overline{Q_V(\epsilon_j)}\; Q_W(\epsilon_j)\overline{Q_Z(\epsilon_j)} + \frac{1}{2n^3}\sum_{j=0}^{n-1} Q_U(-\epsilon_j)\overline{Q_V(-\epsilon_j)}\; Q_W(-\epsilon_j)\overline{Q_Z(-\epsilon_j)}$$

$$= \frac{1}{2n^3}\sum_{j=0}^{n-1}\sum_{|u|<n}\sum_{|v|<n}\left(1+(-1)^{-(u+v)}\right) C_{U,V}(u)C_{W,Z}(v)\epsilon_j^{-(u+v)}$$

$$= \frac{1}{2n^2}\sum_{|u|<n}\sum_{u+v\in\{0,n,-n\}}\left(1+(-1)^{-(u+v)}\right) C_{U,V}(u)C_{W,Z}(v)$$

$$= \frac{1}{n^2}\sum_{|u|<n} C_{U,V}(u)C_{Z,W}(u), \tag{36}$$

because n is odd and by definition $C_{W,Z}(-u) = C_{Z,W}(u)$.

Let j be an integer and let r be real. By definition of Y_r,

$$Q_{Y_r}(\epsilon_j) = \epsilon_j^{-\lfloor nr\rfloor} Q_Y(\epsilon_j) \tag{37}$$

and then, by the interpolation formula

$$Q_Y(-\epsilon_j) = \frac{2}{n}\sum_{k=0}^{n-1} Q_Y(\epsilon_k)\frac{\epsilon_k}{\epsilon_k+\epsilon_j}$$

(see [4, p. 162], for example), we have

$$Q_{Y_r}(-\epsilon_j) = \frac{2}{n}\sum_{k=0}^{n-1} \epsilon_k^{-\lfloor nr\rfloor} Q_Y(\epsilon_k)\frac{\epsilon_k}{\epsilon_k+\epsilon_j}, \tag{38}$$

and therefore

$$\overline{Q_{Y_r}(-\epsilon_j)} = \frac{2}{n}\sum_{k=0}^{n-1} \epsilon_k^{\lfloor nr\rfloor} \overline{Q_Y(\epsilon_k)}\frac{\epsilon_j}{\epsilon_k+\epsilon_j}. \tag{39}$$

Take $U = X$, $V = X_s$, $W = X_{s+t}$, and $Z = X_t$ in (36) and substitute from (37), (38) and (39) to obtain

$$\frac{1}{n^2}\sum_{|u|<n} C_{X,X_s}(u)C_{X_t,X_{s+t}}(u)$$

$$= \frac{1}{2n^3}\sum_{j=0}^{n-1}|Q_X(\epsilon_j)|^4 + \frac{1}{2n^3}\left(\frac{2}{n}\right)^4\sum_{j=0}^{n-1}\sum_{0\le a,\,b,\,c,\,d<n}\epsilon_b^S\epsilon_c^{-(S+T)}\epsilon_d^T$$

$$\cdot\, Q_X(\epsilon_a)\overline{Q_X(\epsilon_b)}Q_X(\epsilon_c)\overline{Q_X(\epsilon_d)}\cdot\frac{\epsilon_a}{\epsilon_a+\epsilon_j}\;\frac{\epsilon_j}{\epsilon_b+\epsilon_j}\;\frac{\epsilon_c}{\epsilon_c+\epsilon_j}\;\frac{\epsilon_j}{\epsilon_d+\epsilon_j}$$

$$= \frac{1}{2n^3}\Lambda_X(0,0,0) + \frac{8}{n^7}\sum_{0\le a,\,b,\,c,\,d<n}\epsilon_b^S\epsilon_c^{-(S+T)}\epsilon_d^T\epsilon_a\epsilon_c$$

$$\cdot\, Q_X(\epsilon_a)\overline{Q_X(\epsilon_b)}Q_X(\epsilon_c)\overline{Q_X(\epsilon_d)}\cdot\sum_{j=0}^{n-1}\frac{\epsilon_j^2}{(\epsilon_a+\epsilon_j)(\epsilon_b+\epsilon_j)(\epsilon_c+\epsilon_j)(\epsilon_d+\epsilon_j)}, \tag{40}$$

by definition (34) of Λ_X. Høholdt and Jensen [4] calculated the value of the sum over j in (40) according to which of a, b, c, d (in the range $0 \le a,b,c,d < n$) are equal and which are distinct, distinguishing five cases:

$$\sum_{j=0}^{n-1} \frac{\epsilon_j^2}{(\epsilon_a+\epsilon_j)(\epsilon_b+\epsilon_j)(\epsilon_c+\epsilon_j)(\epsilon_d+\epsilon_j)}$$
$$= \begin{cases} 0 & \text{for } a,\ b,\ c,\ d \text{ distinct} \\ \dfrac{n^2(n^2+2)}{48}\cdot\dfrac{1}{\epsilon_a^2} & \text{for } a=b=c=d \\ \dfrac{n^2}{8}\cdot\dfrac{\epsilon_a+\epsilon_d}{\epsilon_a(\epsilon_a-\epsilon_d)^2} & \text{for } a=b=c\ne d \\ -\dfrac{n^2}{4}\cdot\dfrac{1}{(\epsilon_a-\epsilon_c)(\epsilon_a-\epsilon_d)} & \text{for } a=b, \text{ and } a,c,d \text{ distinct} \\ -\dfrac{n^2}{2}\cdot\dfrac{1}{(\epsilon_a-\epsilon_c)^2} & \text{for } a=b\ne c=d. \end{cases}$$

We therefore partition the sum over a, b, c, d in (40) into sums of the same four types as those giving a non-zero result above, written in the same order, noting that each type requires multiple contributions because of the asymmetry with respect to a, b, c, d of the ϵ terms. This gives

$$\frac{1}{n^2}\sum_{|u|<n} C_{X,X_s}(u)C_{X_t,X_{s+t}}(u) = \frac{1}{2n^3}\Lambda_X(0,0,0) + A + B + C + D, \tag{41}$$

where, after abbreviating Q_X to Q, the sums A, B, C, D are given by

$$A = \frac{n^2+2}{6n^5}\sum_{a=0}^{n-1}|Q(\epsilon_a)|^4,$$

$$B = \frac{1}{n^5}\sum_{\substack{0\le a,b<n\\ a\ne b}} \Big[\big(\epsilon_{b-a}^T+\epsilon_{b-a}^S\big)\big(Q(\epsilon_a)\big)^2\,\overline{Q(\epsilon_a)}\;\overline{Q(\epsilon_b)} + \big(\epsilon_{b-a}^{-(S+T-1)}+\epsilon_{b-a}\big)\big(\overline{Q(\epsilon_a)}\big)^2 Q(\epsilon_a)Q(\epsilon_b)\Big]\cdot\frac{\epsilon_a(\epsilon_a+\epsilon_b)}{(\epsilon_a-\epsilon_b)^2},$$

$$C = -\frac{2}{n^5}\sum_{\substack{0\le a,b,c<n\\ a,b,c \text{ distinct}}} \Big[\big(\epsilon_{b-a}^{-(S+T)}\epsilon_{c-a}^T+\epsilon_{b-a}^{-(S+T)}\epsilon_{c-a}^S+\epsilon_{c-a}^T+\epsilon_{c-a}^S\big)|Q(\epsilon_a)|^2 Q(\epsilon_b)\overline{Q(\epsilon_c)} + \epsilon_{b-a}^{S-1}\epsilon_{c-a}^T\big(Q(\epsilon_a)\big)^2\,\overline{Q(\epsilon_b)}\;\overline{Q(\epsilon_c)} + \epsilon_{b-a}^{-(S+T)}\epsilon_{c-a}\big(\overline{Q(\epsilon_a)}\big)^2 Q(\epsilon_b)Q(\epsilon_c)\Big]\cdot\frac{\epsilon_a\epsilon_b}{(\epsilon_a-\epsilon_b)(\epsilon_a-\epsilon_c)},$$

$$D = -\frac{4}{n^5}\sum_{\substack{0\le a,b<n\\ a\ne b}} \Big[\big(\epsilon_{b-a}^S+\epsilon_{b-a}^T\big)|Q(\epsilon_a)|^2|Q(\epsilon_b)|^2 + \epsilon_{b-a}^{S+T-1}\big(Q(\epsilon_a)\big)^2\big(\overline{Q(\epsilon_b)}\big)^2\Big]\cdot\frac{\epsilon_a\epsilon_b}{(\epsilon_a-\epsilon_b)^2}.$$

By the definition (34),

$$A = \frac{n^2+2}{6n^5}\,\Lambda_X(0,0,0),$$

and substitution in (41) gives the required term in $\Lambda_X(0,0,0)$. We complete the proof by showing that the remaining sums B, C, D in (41) have the required form.

For the sum B, replace the sum over b (where $0 \le b < n$ and $b \ne a$) by a sum over $k := (b-a) \bmod n$ (where $1 \le k < n$) to give

$$B = \frac{1}{n^5} \sum_{a=0}^{n-1} \sum_{k=1}^{n-1} \Bigg[\left(\epsilon_k^T + \epsilon_k^S\right) \left(Q(\epsilon_a)\right)^2 \overline{Q(\epsilon_a)}\ \overline{Q(\epsilon_{a+k})} + \left(\epsilon_k^{-(S+T-1)} + \epsilon_k\right) \overline{\left(Q(\epsilon_a)\right)^2 \overline{Q(\epsilon_a)}\ \overline{Q(\epsilon_{a+k})}} \Bigg] \cdot \frac{(1+\epsilon_k)}{(1-\epsilon_k)^2},$$

and then use (34). For the sum D, use the same procedure together with the identity

$$(\epsilon_a - \epsilon_b)^2 = -\epsilon_a \epsilon_b |1 - \epsilon_{b-a}|^2.$$

For the sum C, replace the sum over b and c by a sum over $k := (b-a) \bmod n$ and $\ell := (c-a) \bmod n$ to give

$$C = -\frac{2}{n^5} \sum_{a=0}^{n-1} \sum_{\substack{1 \le k,\ell < n \\ k \ne \ell}} \Bigg[\left(\epsilon_k^{-(S+T)} + 1\right) \left(\epsilon_\ell^T + \epsilon_\ell^S\right) |Q(\epsilon_a)|^2 Q(\epsilon_{a+k}) \overline{Q(\epsilon_{a+\ell})} + \epsilon_k^{S-1} \epsilon_\ell^T \left(Q(\epsilon_a)\right)^2 \overline{Q(\epsilon_{a+k})}\ \overline{Q(\epsilon_{a+\ell})} + \epsilon_k^{-(S+T)} \epsilon_\ell \overline{\left(Q(\epsilon_a)\right)^2 \overline{Q(\epsilon_{a+k})}\ \overline{Q(\epsilon_{a+\ell})}} \Bigg] \cdot \frac{\epsilon_k}{(1-\epsilon_k)(1-\epsilon_\ell)},$$

and then use (34). □

The method of Høholdt and Jensen [4] deals with the special case $S = T = 0$ of Lemma 10.

Proof of Theorem 3. We apply Lemma 10 to the sequence $X_{r(n)}$, setting $S := ns(n)$ and $T := nt(n)$. Since S and T are integer for each n by assumption, the sequences X_s, X_t and X_{s+t} appearing in (35) map to $X_{r(n)+s(n)}$, $X_{r(n)+t(n)}$, and $X_{r(n)+s(n)+t(n)}$, respectively, so that the left hand side of (35) becomes

$$\frac{1}{n^2} \sum_{|u|<n} C_{X_{r(n)},\, X_{r(n)+s(n)}}(u)\, C_{X_{r(n)+t(n)},\, X_{r(n)+s(n)+t(n)}}(u).$$

We will prove the desired result by finding the asymptotic form of the right hand side of (35) as $n \longrightarrow \infty$, evaluating the sum D and the term involving $\Lambda_{X_{r(n)}}(0,0,0)$, and bounding the sums B and C.

Write $R := \lfloor nr(n) \rfloor$ and $X = (x_0, x_1, \dots, x_{n-1})$. Since the modified Legendre sequence X satisfies

$$Q_X(\epsilon_j) = \begin{cases} x_j \sqrt{n} & \text{for } n \equiv 1 \pmod 4 \\ i x_j \sqrt{n} & \text{for } n \equiv 3 \pmod 4 \end{cases}$$

(see [10, p. 182], for example), where $i := \sqrt{-1}$, the definition (34) together with (37) gives

$$\Lambda_{X_{r(n)}}(j,k,\ell) = \epsilon_{j-k+\ell}^R \cdot n^2 \sum_{a=0}^{n-1} x_a\, x_{a+j}\, x_{a+k}\, x_{a+\ell}. \tag{42}$$

The term involving $\Lambda_{X_{r(n)}}(0,0,0)$. Since X is a modified Legendre sequence, we have

$$x_j^2 = \begin{cases} 0 & \text{for } j = 0 \\ 1 & \text{for } 0 < j < n. \end{cases} \tag{43}$$

Therefore (42) gives

$$\frac{2n^2+1}{3n^5}\,\Lambda_{X_{r(n)}}(0,0,0) = \frac{2n^2+1}{3n^5}\,n^2(n-1)$$

$$= \frac{2}{3} + O(n^{-1}) \quad \text{as } n \longrightarrow \infty. \tag{44}$$

The sums B and C. We have

$$|B| \le \frac{1}{n^5}\sum_{k=1}^{n-1}\frac{8\,|\Lambda_{X_{r(n)}}(0,0,k)|}{|1-\epsilon_k|^2}$$

$$= \frac{8}{n^5}\sum_{k=1}^{n-1}\frac{n^2\,|R_X(k)|}{|1-\epsilon_k|^2}$$

by (42), (43) and the definition of R_X in Section 2. Then by (8),

$$|B| \le \frac{8}{n^3}\sum_{k=1}^{n-1}\frac{1}{|1-\epsilon_k|^2}. \tag{45}$$

Since $|\Lambda_{X_{r(n)}}(0,k,\ell)| = |\Lambda_{X_{r(n)}}(k,0,\ell)|$ by (42), we also have

$$|C| \le \frac{2}{n^5}\sum_{\substack{1\le k,\ell<n\\ k\ne\ell}}\frac{6\,|\Lambda_{X_{r(n)}}(0,k,\ell)|}{|1-\epsilon_k|\cdot|1-\epsilon_\ell|}$$

$$= \frac{12}{n^5}\sum_{\substack{1\le k,\ell<n\\ k\ne\ell}}\frac{n^2\,|R_X((k-\ell)\bmod n)-x_kx_\ell|}{|1-\epsilon_k|\cdot|1-\epsilon_\ell|}$$

by (42) and (43). Then using (8) and the inequality $|x_kx_\ell|\le 1$,

$$|C| \le \frac{24}{n^3}\sum_{\substack{1\le k,\ell<n\\ k\ne\ell}}\frac{1}{|1-\epsilon_k|\cdot|1-\epsilon_\ell|}$$

Combining with (45), we obtain

$$3|B|+|C| \le \frac{24}{n^3}\left(\sum_{k=1}^{n-1}\frac{1}{|1-\epsilon_k|}\right)^2$$

$$= O(n^{-1}(\log n)^2) \quad \text{as } n \longrightarrow \infty, \tag{46}$$

since $\sum_{k=1}^{n-1}\frac{1}{|1-\epsilon_k|}\le n\log n$ (see [4, p. 162], for example).

The sum D. By (42), for $1\le k<n$ we have $\Lambda_{X_{r(n)}}(0,k,k)=n^2(n-2)$ and $\Lambda_{X_{r(n)}}(k,0,k)=\epsilon_k^{2R}\cdot n^2(n-2)$, so that

$$D = \frac{4(n-2)}{n^3}\sum_{k=1}^{n-1}\frac{\epsilon_k^S+\epsilon_k^T+\epsilon_k^{2R+S+T-1}}{|1-\epsilon_k|^2}.$$

We wish to apply the identity

$$\sum_{k=1}^{n-1}\frac{\epsilon_k^j}{|1-\epsilon_k|^2} = \frac{n^2}{2}\left(\frac{|j|}{n}-\frac{1}{2}\right)^2 - \frac{n^2+2}{24} \quad \text{for integer } j \text{ satisfying } |j|\le n \tag{47}$$

(see, [7, p. 621], for example). By the definition of R, S, and T and the assumptions $r(n) = r + O(n^{-1})$, $s(n) = s + O(n^{-1})$, and $t(n) = t + O(n^{-1})$, as $n \longrightarrow \infty$ we have

$$R = nr + O(1),\ S = ns + O(1), \text{ and } T = nt + O(1). \tag{48}$$

Then, since $|s| \le \frac{1}{2}$ and $|t| \le \frac{1}{2}$ by assumption, we know that $|S| \le n$ and $|T| \le n$ for all sufficiently large n.

Case 1*: $|2R+S+T-1| \le n$ for all but finitely many n.* In this case, we apply (47) to show that, for all sufficiently large n,

$$D = \frac{4(n-2)}{n^3}\left(\frac{n^2}{2}\left[\left(\frac{|S|}{n}-\frac{1}{2}\right)^2 + \left(\frac{|T|}{n}-\frac{1}{2}\right)^2 + \left(\frac{|2R+S+T-1|}{n}-\frac{1}{2}\right)^2\right] - \frac{n^2+2}{8}\right). \tag{49}$$

By (48), as $n \longrightarrow \infty$ we find that

$$D = 2\left(|s|-\tfrac{1}{2}\right)^2 + 2\left(|t|-\tfrac{1}{2}\right)^2 + 8\left(|r+\tfrac{s+t}{2}|-\tfrac{1}{4}\right)^2 - \tfrac{1}{2} + O(n^{-1}). \tag{50}$$

Case 2*: $|2R+S+T-1| > n$ for infinitely many n.* In this case (48), together with the assumption $|r + \frac{s+t}{2}| \le \frac{1}{2}$, implies that

$$|2r+s+t| = 1. \tag{51}$$

For each sufficiently large n, either $|2R+S+T-1| \le n$, so that (49) holds; or else $n < |2R+S+T-1| \le 2n$, in which case (by applying (47) with $j = 2R+S+T \pm n - 1$, choosing the appropriate sign for $\pm$ so that $|j| \le n$) equation (49) again holds but with the term $\left(\frac{1}{n}|2R+S+T-1| - \frac{1}{2}\right)^2$ replaced by $\left(\frac{1}{n}|2R+S+T-1| - \frac{3}{2}\right)^2$. By (48) and (51), the contribution of this term as $n \longrightarrow \infty$ is $(1-\frac{3}{2})^2 = (1-\frac{1}{2})^2$, so the same conclusion (50) applies as in Case 1.

The result now follows by substituting the asymptotic forms (44), (46) and (50) in (35). □

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