Introduction to Intensional Logic

Ling 406/802

Read *Meaning and Grammar*, Ch. 5.1-5.2

Towards Intensional Semantics

- Extensional semantics that models the meaning of sentences based on the extensions of linguistic expressions is limited and cannot handle these intensional constructions.

- What is common in these intensional constructions is that they call for a consideration of extensions that an expression may have in circumstances other than the one in which it is evaluated. This is called an INTENSION of a linguistic expression.

- In order to get at the intensions, we need to consider alternative ways in which the world might have been, alternative sets of circumstances, or POSSIBLE WORLDS.

- The framework that models the meaning of sentences based on the intensions of linguistic expressions is called INTENSIONAL SEMANTICS or POSSIBLE-WORLDS SEMANTICS.

Possible Worlds and Intensions

- Possible worlds are possible circumstances in which some (or all) events or states are different from what they in fact are in the actual circumstance.

w_1	w_2	w_3
Mat and Sue are funny Sue is tall Stevenson is the pres. of SFU	Sue, Pete and John are funny Sue is tall Anderson is the pres. of SFU	Mat and Pete are funny Sue is not tall Anderson is the pres. of SFU

- W is a set of all possible worlds.
  $W = \{w_1, w_2, w_3, ...\}$

- The intension of a sentence S: proposition (e.g., *Mat is funny*)

  The set of possible worlds in which it is true.
  Function from possible worlds to truth values.

$$p = \begin{bmatrix} w_1 \rightarrow 1 \\ w_2 \rightarrow 0 \\ w_3 \rightarrow 1 \\ w_4 \rightarrow 0 \\ . \\ . \\ . \end{bmatrix}$$

Possible Worlds and Intensions (cont.)

- The intension of a VP: property (e.g., *is funny*)

  Function from possible worlds to sets of individuals

$$[\![\text{is funny}]\!] = \begin{bmatrix} w_1 \rightarrow \{Mat, Sue\} \\ w_2 \rightarrow \{Sue, Pete, John\} \\ w_3 \rightarrow \{Mat, Pete\} \\ w_4 \rightarrow \{Sue, John, Pete\} \\ . \\ . \\ . \end{bmatrix}$$

- The intension of an NP: individual concept (e.g., *the president of SFU*)

  Function from possible worlds to individuals

$$[\![\text{the president of SFU}]\!] = \begin{bmatrix} w_1 \rightarrow Stevenson \\ w_2 \rightarrow Anderson \\ w_3 \rightarrow Anderson \\ w_4 \rightarrow Pete \\ . \\ . \\ . \end{bmatrix}$$

Possible Worlds, Tense and Intensions

- To incorporate tense, we need to consider possible worlds at different times. That is, we will want to consider not just possible worlds, but possible world-time pairs (circumstances).

  W = $\{< w_1, i_1 >$, $< w_1, i_2 >$, ..., $< w_2, i_5 >$, ..., $< w_3, i_4 >$, ...$\}$

- Proposition: a set of possible world-time pairs, or a function from possible world-time pairs to truth values.

$$[\![\text{Mat is funny}]\!] = \begin{bmatrix} < w_1, i_1 > \rightarrow 1 \\ < w_1, i_2 > \rightarrow 0 \\ < w_2, i_1 > \rightarrow 1 \\ < w_3, i_3 > \rightarrow 0 \\ . \\ . \\ . \end{bmatrix}$$

Possible Worlds, Tense and Intensions (cont.)

- Property: a function from possible world-time pairs to sets of individuals.

$$[\![\text{is funny}]\!] = \begin{bmatrix} < w_1, i_1 > \rightarrow \{Mat, Sue\} \\ < w_1, i_2 > \rightarrow \{Sue, Pete, John\} \\ < w_2, i_1 > \rightarrow \{Mat, Pete\} \\ < w_3, i_3 > \rightarrow \{Sue, John, Pete\} \\ . \\ . \\ . \end{bmatrix}$$

- Individual concept: a function from possible world-time pairs to individuals.

$$[\![\text{the president of SFU}]\!] = \begin{bmatrix} < w_1, i_1 > \rightarrow Stevenson \\ < w_1, i_2 > \rightarrow Anderson \\ < w_2, i_1 > \rightarrow Anderson \\ < w_3, i_3 > \rightarrow Pete \\ . \\ . \\ . \end{bmatrix}$$

Possible Worlds, Tense and Intensions (cont.)

- Structural properties of propositions in terms of set-theoretic operations

1. p entails q $=_{df}$ $p \subseteq q$
2. p is equivalent to q $=_{df}$ $p = q$
3. p and q are contradictory $=_{df}$ $p \cap q = \emptyset$ (there is no world-time pair in which p and q are both true)
4. $\neg p$ $=_{df}$ $\{< w, i > \in W \; : \; < w, i > \notin p\}$ (the world-time pairs in which p is not true)
5. $p \wedge q$ $=_{df}$ $p \cap q$ = $\{< w, i > \in W \; : \; < w, i > \in p$ and $< w, i > \in q\}$
6. $p \vee q$ $=_{df}$ $p \cup q$ = $\{< w, i > \in W \; : \; < w, i > \in p$ or $< w, i > \in q\}$
7. p is possible $=_{df}$ $p \neq \emptyset$ (there is at least one world-time pair in which p is true
8. p is necessary $=_{df}$ $p = W$ (there is no world-time pair in which p is false)

Syntax of Intensional Predicate Calculus (IPC)

1. Primitive vocabulary

   (a) A set of terms:

   A set of individual constants: a, b, c, d, ..., john, mary, pete, stevenson

   A set of individual variables: x, y, z, x_0, x_1, x_2, ..., he, she, it

   (b) A set of predicates: P, Q, R, ...

   One-place predicates: is tall, is funny, is the president of SFU
   Two-place predicates: like, hate, love, hit
   Three-place predicates: introduce, give

   (c) A binary identity predicate: =

   (d) The connectives of IPC: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

   (e) Quantifiers: $\forall, \exists$

   (f) Modal, temporal operators: $\Box$, $\Diamond$, **P**, **F**

   (g) brackets: (,), [,]

Syntax of IPC (cont.)

2. Syntactic rules

   (a) If P is an n-place predicate and $t_1, \ldots, t_n$ are all terms, then $P(t_1, \ldots, t_n)$ is an atomic formula.

   tall(john), loves(john,mat), introduce(john,mat,sue)

   (b) If t_1 and t_2 are individual constants or variables, then $t_1 = t_2$ is a formula.

   john=bill: “John is Bill”

   (c) If ϕ is a formula, then $\neg\phi$ is a formula.

   (d) If ϕ and ψ are formulas, so are $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$.

   (e) If ϕ is a formula and x is a variable, then $\forall x\phi$, and $\exists x\phi$ are formulas too.

   $\forall x$ happy(x), $\exists x$ loves(x,j), $\forall x\ \exists y$ loves(x, y), $\exists y\ \forall x$ loves(x, y)

   (f) If ϕ is a formula, then the following are formulas too.

   $\Box\phi$: “It is necessarily the case that ϕ.
   $\Diamond\phi$: “It is possibly the case that ϕ.
   Pϕ: “It was the case that ϕ.
   Fϕ: “It will be the case that ϕ.

   (g) Nothing else is a formula in IPC.

A Model for IPC

A model for IPC is a 5-tuple $< W, I, <, U, V >$, where the following constraints hold:

1. W is a set of worlds.

2. I is a set of instants ordered by the relation $<$.

   For i, i' in I, $i < i'$ is to be read as "i precedes i'," or " i is earlier than $'$."

3. U is a domain of individuals.

4. V is a function that assigns an intension to the constants of IPC.

   (a) If β is an individual constant of IPC, $V(\beta)$ is a function from $W \times I$ (=$\{< w, i >$: $w \in W$ and $i \in I\}$) to U.

   For any $w \in W$ and $i \in I$, $V(\beta)(< w, i >) \in U$.

   NOTE: We will assume that names like *Mat, Stevenson* are RIGID DESIGNATORS: a name maps onto the same value in every possible circumstances. But definite descriptions like *the president of SFU* are not rigid designators.

(b) If β is a one-place predicate of IPC, $V(\beta)$ is a function from $W \times I$ to sets of elements of U.

For any $w \in W$ and $i \in I$, $V(\beta)(< w, i >) \subseteq U$.

For any $w \in W$ and $i \in I$, $V(\beta)(< w, i >) \in \wp(U)$.

(c) If β is a two-place predicate of IPC, $V(\beta)$ is a function from $W \times I$ to sets of ordered pairs of elements of U.

For any $w \in W$ and $i \in I$, $V(\beta)(< w, i >) \subseteq U \times U$.

For any $w \in W$ and $i \in I$, $V(\beta)(< w, i >) \in \wp(U \times U)$.

(d) QUESTION: If β is a three-place predicate of IPC?

NOTE: Assignment function g from variables to individuals assigns values to variables.

Semantics of IPC

In IPC, interpretation function $[\![\ ]\!]$ is relativized to a model M, an assignment function g, a world w and a time i.

1. If β is a constant, $[\![\beta]\!]^{M,w,i,g} = V(\beta)(< w, i >)$.

2. If β is a variable, $[\![\beta]\!]^{M,w,i,g} = g(\beta)$.

   $$g_1 = \begin{bmatrix} x_1 \rightarrow John \\ x_2 \rightarrow Mary \\ x_n \rightarrow Pete \end{bmatrix} \text{ where } n \geq 3$$

   $[\![\text{He}_3 \text{ is happy}]\!]^{M,w,i,g_1} = [\![\text{happy}(x_3)]\!]^{M,w,i,g_1}$ = “Pete is happy.”

3. If β is an n-ary predicate and $t_1,...,t_n$ are all terms, then $[\![\beta(t_1,...,t_n)]\!]^{M,w,i,g}$ = 1 iff $< [\![t_1]\!]^{M,w,i,g}, ..., [\![t_n]\!]^{M,w,i,g} > \in [\![\beta]\!]^{M,w,i,g}$

4. If ϕ and ψ are formulas, then,

$[\![\neg\phi]\!]^{M,w,i,g}$ = 1 iff $[\![\phi]\!]^{M,w,i,g}$ = 0
$[\![\phi \wedge \psi]\!]^{M,w,i,g}$ = 1 iff $[\![\phi]\!]^{M,w,i,g}$ = 1 and $[\![\psi]\!]^{M,w,i,g}$ = 1
$[\![\phi \vee \psi]\!]^{M,w,i,g}$ = 1 iff $[\![\phi]\!]^{M,w,i,g}$ = 1 or $[\![\psi]\!]^{M,w,i,g}$ = 1
$[\![\phi \longrightarrow \psi]\!]^{M,w,i,g}$ = 1 iff $[\![\phi]\!]^{M,w,i,g}$ = 0 or $[\![\psi]\!]^{M,w,i,g}$ = 1
$[\![\phi \leftrightarrow \psi]\!]^{M,w,i,g}$ = 1 iff $[\![\phi]\!]^{M,w,i,g}$ = $[\![\psi]\!]^{M,w,i,g}$

5. If ϕ is a formula, and v is a variable, then,

$[\![\forall v\phi]\!]^{M,w,i,g}$ = 1 iff for all individuals $d \in U$, $[\![\phi]\!]^{M,w,i,g[d/v]}$ = 1.
$[\![\exists v\phi]\!]^{M,w,i,g}$ = 1 iff for some individual $d \in U$, $[\![\phi]\!]^{M,w,i,g[d/v]}$ = 1.

$g[d/v]$: the variable assignment g' that is exactly like g except (maybe) for $g(v)$, which equals the individual d.

g_1 = $\begin{bmatrix} x_1 \rightarrow John \\ x_2 \rightarrow Mary \\ x_3 \rightarrow Pete \\ x_n \rightarrow Pete \end{bmatrix}$ where $n \geq 4$

$g_1[John/x_3]$ = $\begin{bmatrix} x_1 \rightarrow John \\ x_2 \rightarrow Mary \\ x_3 \rightarrow John \\ x_n \rightarrow Pete \end{bmatrix}$ where $n \geq 4$

$g_1[[John/x_3]Pete/x_1]$ = $\begin{bmatrix} x_1 \rightarrow Pete \\ x_2 \rightarrow Mary \\ x_3 \rightarrow John \\ x_n \rightarrow Pete \end{bmatrix}$ where $n \geq 4$

6. If ϕ is a formula, then, $[\![\Box\phi]\!]^{M,w,i,g}$ = 1 iff for all $w' \in W$ and all $i' \in I$, $[\![\phi]\!]^{M,w',i',g}$ = 1.

   A sentence is necessarily true in a given circumstance iff it is true in every possible circumstance.

7. If ϕ is a formula, then, $[\![\Diamond\phi]\!]^{M,w,i,g}$ = 1 iff there exists at least one $w' \in W$ and one $i' \in I$ such that $[\![\phi]\!]^{M,w',i',g}$ = 1.

   A sentence is possibly true iff there exists at least one possible circumstance in which it is true.

8. If ϕ is a formula, then, $[\![\mathbf{P}\phi]\!]^{M,w,i,g}$ = 1 iff there exists an $i' \in I$ such that $i' < i$ and $[\![\phi]\!]^{M,w,i',g}$ = 1.

   "It was the case that ϕ" is true iff there is a moment that precedes the time of evaluation (time of utterance) at which ϕ is true.

9. If ϕ is a formula, then, $[\![\mathbf{F}\phi]\!]^{M,w,i,g}$ = 1 iff there exists an $i' \in I$ such that $i < i'$ and $[\![\phi]\!]^{M,w,i',g}$ = 1.

   "It will be the case that ϕ" is true iff there is a moment that follows the time of evaluation (time of utterance) at which ϕ is true.

Compositional Interpretation

Assume $M_4 = < W_4, I_4, <_4, U_4, V_4 >$, where

a. $W_4 = \{w', w''\}$, $I_4 = \{i', i'', i'''\}$, $<_4 = \{< i', i'' >, < i'', i''' >, < i', i''' >\}$
$U_4 = \{frodo, sam, aragorn\}$

b. V_4(frodo)($< w, i >$) = *Frodo*, for any $w \in W_4$ and $i \in I_4$.

V_4(the ring bearer) =

$$\begin{bmatrix} < w', i' > \rightarrow Frodo \\ < w', i'' > \rightarrow Frodo \\ < w', i''' > \rightarrow Aragorn \\ < w'', i' > \rightarrow Sam \\ < w'', i'' > \rightarrow Sam \\ < w'', i''' > \rightarrow Sam \end{bmatrix}$$

V_4(think about the ring) =

$$\begin{bmatrix} < w', i' > \rightarrow \{Frodo, Sam, Aragorn\} \\ < w', i'' > \rightarrow \{Frodo, Sam\} \\ < w', i''' > \rightarrow \{Aragorn\} \\ < w'', i' > \rightarrow \{Sam, Aragorn\} \\ < w'', i'' > \rightarrow \{Frodo, Sam\} \\ < w'', i''' > \rightarrow \{Frodo\} \end{bmatrix}$$

Also, assume the following assignment function:

$$g_4 = \begin{bmatrix} x \rightarrow Frodo \\ y \rightarrow Sam \\ z \rightarrow Sam \\ . \\ . \\ . \end{bmatrix}$$

Compositional Interpretation (cont.)

Compute the truth conditions and truth values for the following formulas compositionally.

1. $[\![\Box \text{think about the ring(frodo)}]\!]^{M_4,w'',i',g_4}$

2. $[\![\Diamond \text{think about the ring(sam)}]\!]^{M_4,w'',i',g_4}$

3. $[\![\Box\exists x[\text{think about the ring}(x)]]\!]^{M_4,w'',i',g_4}$

4. $[\![\exists x[\Box\text{think about the ring}(x)]]\!]^{M_4,w'',i',g_4}$

5. $[\![\mathbf{P}\forall x[\text{think about the ring}(x)]]\!]^{M_4,w',i'',g_4}$

6. $[\![\exists x[\mathbf{FP}\text{think about the ring}(x)]]\!]^{M_4,w'',i',g_4}$

7. $[\![\mathbf{P}\text{think about the ring}(x) \wedge \Diamond\mathbf{F}\forall x[\text{think about the ring}(x)]]\!]^{M_4,w',i'',g_4}$

Definitions of Truth, Validity, Entailment and Equivalence

1. A (closed) formula ϕ is TRUE in a model M with respect to a world w and a time i iff for any assignment g, $[\![\phi]\!]^{M,w,i,g}$ = 1.

2. A formula ϕ is VALID iff for any model M, any world w and any time i, ϕ is true in M with respect to w and i.

   $\Box\phi \leftrightarrow \neg\Diamond\neg\phi$ $\quad$ $\forall x\text{P}(x) \leftrightarrow \neg\exists x\neg\text{P}(x)$

   $\Diamond\phi \leftrightarrow \neg\Box\neg\phi$ $\quad$ $\exists x\text{P}(x) \leftrightarrow \neg\forall x\neg\text{P}(x)$

3. A formula ϕ ENTAILS a formula ψ iff for any model M, any world w, any time i, any assignment g, if $[\![\phi]\!]^{M,w,i,g}$ = 1 then $[\![\psi]\!]^{M,w,i,g}$ = 1

   $\Box[\phi \rightarrow \psi]$ entails $[\Box\phi \rightarrow \Box\psi]$, but $[\Box\phi \rightarrow \Box\psi]$ does not entail $\Box[\phi \rightarrow \psi]$

   $\forall x[\text{P}(x) \rightarrow \text{Q}(x)]$ entails $[\forall x\text{P}(x) \rightarrow \forall x\text{Q}(x)]$, but
   $[\forall x\text{P}(x) \rightarrow \forall x\text{Q}(x)]$ does not entail $\forall x[\text{P}(x) \rightarrow \text{Q}(x)]$

   $\exists x\Box\text{P}(x)$ entails $\Box\exists x\text{P}(x)$, but $\Box\exists x\text{P}(x)$ does not entail $\exists x\Box\text{P}(x)$

   $\Diamond\forall x\text{P}(x)$ entails $\forall x\Diamond\text{P}(x)$, but $\forall x\Diamond\text{P}(x)$ does not entail $\Diamond\forall x\text{P}(x)$

   $\Box$P(m) does not entail $\exists x\Box\text{P}(x)$ (where m is a definite description like *the president of SFU*)

4. A set of formulas $\Omega = \{\phi_1, ..., \phi_n\}$ ENTAILS a formula ψ iff for any model M, any world w, any time i, and any assignment g, if $[\![\phi_i]\!]^{M,w,i,g}$ = 1 for all ϕ_i in Ω, then $[\![\psi_i]\!]^{M,w,i,g}$ = 1.

5. Two formulas ϕ and ψ are EQUIVALENT iff they entail each other.

$\Box\phi \Leftrightarrow \neg\Diamond\neg\phi$

$\Diamond\phi \Leftrightarrow \neg\Box\neg\phi$

$\exists x \Diamond \mathsf{P}(x) \Leftrightarrow \Diamond\exists x \mathsf{P}(x)$

$\forall x \Box \mathsf{P}(x) \Leftrightarrow \Box\forall x \mathsf{P}(x)$

$\Box\Diamond\phi \Leftrightarrow \Diamond\phi$

$\exists x \mathbf{F} \mathsf{P}(x) \Leftrightarrow \mathbf{F}\exists x \mathsf{P}(x)$