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SOME NOTES CONCERNING FUZZY LOGICS*

ABSTRACT. Fuzzy logics are systems of logic with infinitely many truth values. Such logics have been claimed to have an extremely wide range of applications in linguistics, computer technology, psychology, etc. In this note, we canvass the known results concerning infinitely many valued logics; make some suggestions for alterations of the known systems in order to accommodate what modern devotees of fuzzy logic claim to desire; and we prove some theorems to the effect that there can be no fuzzy logic which will do what its advocates want. Finally, we suggest ways to accommodate these desires in finitely many valued logics.

I. HISTORICAL NOTES

The concept of a *fuzzy set* and the related systems of *fuzzy set theory* and *fuzzy logic* have been claimed to have an extremely wide range of applications: reasoning involving inexact concepts (Goguen, 1969; Zadeh, 1972), psychological classification (Heider, 1971; Dreyfus *et al.*, 1975; Kochen, 1975; Kochen and Badre, 1975), threshold phenomena (Nakamura, 1962a,b; 1963a,b,c; 1964, 1965), pattern recognition (Bellman *et al.*, 1966), computer learning (Samuel, 1967), and the provision of an adequate description of certain linguistic phenomena (Lakoff, 1971, 1972). The purpose of this note is to point out certain shortcomings inherent in fuzzy logics which demonstrate their unsuitability for these purposes.

Presentations of fuzzy logics have thus far been semantic in nature and the syntax has been ignored.[1] It seems to be assumed by most of these writers that an adequate syntax, one in which exactly the (semantically) valid arguments are provable, can easily be constructed and is 'merely a task for logicians'. (For a dissenting view, see Goguen (1969; p. 326) who says that it is unlikely that we should find a purely syntactic logic for inexact concepts. We shall show that Goguen's intuition here is correct.)[2]

The basic semantic intuition behind a fuzzy propositional logic is that propositions cannot only take the two values 0 (false) and 1 (true), but any of

* We would like to express our thanks to Stan Peters for helpful conversations. The method in the Appendix is due to David Lewis.

[1] That is to say, the usual presentations give an account – informal and intuitive in Lakoff, more formal in Goguen (1969) and Zadeh (1965, 1971) – of the truth of certain kinds of sentences and validity of certain arguments. The syntax of fuzzy logic – a specification of the axioms and rules of inference – has been left unspecified. (Compare the discussion in Goguen, Sect. I.)

[2] Also Lakoff p. 216 seems to feel that no axiomatic treatment will be forthcoming. For some reason, however, he seems to think this is good and that it shows the superiority of fuzzy logics.

the non-denumerable real values in the closed interval 0 to 1. This range intuitively corresponds to 'degrees of truth' of a proposition. The values of complex expressions are computed as functions of the values of component expressions. This is alleged to give sense to the claim that

(1) Howard is tall or he is not tall

when said of 5′11″ Howard, is not tautologous in the sense of being valued 1. If 'Howard is tall' has the value 0.6, then 'Howard is not tall' has the value 0.4 and (1) has the value 0.6.

In this example we have followed the Lakoff/Zadeh presentation of the valuation function for 'or', which they give as the maximum value of the disjuncts, and for 'not' which is given as 1 minus the value of the argument. They also give the valuation function for 'and' as the minimum value of the conjuncts. We shall use 'v', '&', and '~' for these functions, and 'V' to stand for 'the value of'.

$$
\begin{aligned}
V(\phi \vee \psi) &= \max(V\phi, V\psi)\\
V(\phi \,\&\, \psi) &= \min(V\phi, V\psi)\\
V{\sim}\phi &= 1 - V\phi
\end{aligned}
$$

This is, of course, not the only possibility for regimenting the ordinary 'and', 'or', 'not' into infinite-valued logic; but they do satisfy certain intuitive beliefs about 'and', 'or', 'not'. In particular, 'not' satisfies the law of double negation, both 'and' and 'or' are commutative, associative, and idempotent, and the familiar distribution laws hold.

$$
\begin{aligned}
V{\sim}{\sim}\phi &= V\phi\\
V(\phi \vee \psi) &= V(\psi \vee \phi)\\
V(\phi \,\&\, \psi) &= V(\psi \,\&\, \phi)\\
V(\phi \vee (\psi \vee \theta)) &= V((\phi \vee \psi) \vee \theta)\\
V(\phi \,\&\, (\psi \,\&\, \theta)) &= V((\phi \,\&\, \psi) \,\&\, \theta)\\
V(\phi \vee \phi) &= V\phi\\
V(\phi \,\&\, \phi) &= V\phi\\
V{\sim}(\phi \vee \psi) &= V({\sim}\phi \,\&\, {\sim}\psi)\\
V{\sim}(\phi \,\&\, \psi) &= V({\sim}\phi \vee {\sim}\psi)\\
V(\phi \vee (\psi \,\&\, \theta)) &= V((\phi \vee \psi) \,\&\, (\phi \vee \theta))\\
V(\phi \,\&\, (\psi \vee \theta)) &= V((\phi \,\&\, \psi) \vee (\phi \,\&\, \theta)).
\end{aligned}
$$

Lakoff and Zadeh use this as the valuation function corresponding to 'if ... then ...': 1 if the value of ϕ is less than or equal to the value of ψ, 0 otherwise. We use '$\rightarrow$' for this relation.

$$
V(\phi \rightarrow \psi) = \begin{cases} 1 & \text{if } V\phi \leq V\psi \\ 0 & \text{otherwise} \end{cases}
$$

Most of the familiar relations of two-valued logic between 'if . . . then . . .' on the one hand and 'or', 'and', 'not' on the other do not hold in the infinite valued case using '$\rightarrow$'.

There are other choices available for these functions that will change which of the familiar laws hold in the infinite-valued case. Following Łukasiewicz and Tarski (1930) for example, we could use this as the valuation function corresponding to 'if . . . then . . .': the minimum value of 1 or of 1 minus the value of ϕ plus the value of ψ. The function corresponding to '. . . or . . .' is: the minimum of 1 or of the value of ϕ plus the value of ψ. The function corresponding to '. . . and . . .' is: 1 minus the minimum of 1 and of the value ($\sim\phi$ plus $\sim\psi$). We shall use '$\supset$', '+', '$\cdot$' respectively for these functions.

$$V(\phi \supset \psi) = \min(1, 1 - V\phi + V\psi)$$
$$V(\phi + \psi) = \min(1, V\phi + V\psi)$$
$$V(\phi \cdot \psi) = 1 - \min(1, 2 - (V\phi + V\psi)).$$

Note that '+' and '$\cdot$' are commutative and associative but not idempotent – that is, it is not generally the case the $V(\phi + \phi) = V\phi$ and $V(\phi \cdot \phi) = V\phi$ (except when $V\phi = 0$ or 1). Note also that the following laws hold.

$$V(\phi \supset \psi) = V(\sim\phi \vee \psi)$$
$$V(\phi \supset \psi) = V\sim(\phi \cdot \sim\psi)$$
$$V\sim(\phi + \psi) = V(\sim\phi \cdot \sim\psi)$$
$$V\sim(\phi \cdot \psi) = V(\sim\phi + \sim\psi)$$

The other distribution laws do not generally hold, but note that

$$V(\phi + \sim\phi) = 1$$
$$V(\phi \cdot \sim\phi) = 0$$

while with the corresponding formulae using 'v' and '&' this is not so.

For the purposes of this paper, it is a matter of indifference to us which of these are chosen to represent the ordinary English 'and', 'or' and 'if . . . then . . .', so long as some formula having all these truth-functions can be described.[3]

The extension of fuzzy propositional logic to fuzzy predicate logic depends on the notion of 'degree of membership'. So Howard, the individual, belongs (to a certain degree) to the set of those (men?) who are tall – in this case 0.6. For predicate logic, the valuation of the quantifiers '$\forall$' and '$\exists$' are what one would expect: the former, when prefixed to a formula, gives the *least* truth value of all substitution instances of the free variable in the formula (or the greatest lower bound if there is no least), the latter gives the

[3] Lakoff says the system he follows is called $S_{\aleph}$ in Rescher (1969, pp. 47, 344). '$S_{\aleph}$' is not a happy name since there are more than $\aleph$ truth-values.

greatest value (or the least upper bound if there is no greatest). Thus the sentence

(2) Everything is green

will have the least value which is assigned to '*x* is green' for all the values of '*x*' (or the greatest lower bound of such values);

(3) Something is green

will have the greatest value assigned to '*x* is green' for all the values of '*x*' (or the least upper bound of such values). This reflects our intuitions that (2) will have whatever value is assigned to the least green object, and that (3) will have whatever value is assigned to the most green object.

We trust that the preceding discussion of the intuitions behind the semantics for fuzzy logics will indicate why so many writers in so many different fields have thought there to be promise in them. Further justification can be found in the works cited above, especially Lakoff (1971, 1972). We now turn to a discussion of the difficulties involved in giving a correct syntactic system (both the operators in the language and choice of axioms) to mirror these semantical intuitions. Without such a syntax, the semantical intuitions cannot be formulated in any orderly manner, and their value will be correspondingly diminished. Later we indicate more precisely in what areas the intuitions are not viable.

It is not possible to construct all truth functions by means of the connectives given above. This inability to construct all truth functions in terms of '$\rightarrow$', '$\sim$', 'v', '&', '$\supset$', '+', '$\cdot$' is described by saying that the fuzzy logic in terms of '$\rightarrow$', '$\sim$', '$\supset$', '+', '$\cdot$', 'v', and '&' is *functionally incomplete*. A simple cardinality argument would show that any fuzzy logic with any countable number of propositional connectives necessarily is functionally incomplete.

Can fuzzy logic be axiomatized? Some preliminaries are in order. Given a semantics, a *syntax* is said to be *correct* if and only if it is both *complete* and *sound* with respect to that semantics. That is, if and only if (a) every formula which is always (semantically) designated is (syntactically) provable, and (b) every provable formula is designated. A formula is (semantically) designated if and only if it always takes a 'favored' truth-value. Just what truth values shall be considered 'favored' is a decision to be made by the author of the system and such decisions will be heavily influenced by the purposes the author has in mind for the system. For example, Lakoff seems to have in mind (although he doesn't explicitly say) that the only designated value is 1. Thus a correct syntax would be one in which all and only those formulae whose truth value is always 1 are provable.[4] Other writers (Lee, 1972) seem to want the designated values to be those greater than $\frac{1}{2}$.

A *syntactic theory of argumentation* tells us what conclusions we can draw from a given set of premisses. Such a theory consists of a recursively enumerable (perhaps finite) set of inferences rules and sometimes also a recursively enumerable (perhaps finite) set of axioms. An argument from the set of formulae Γ is just a finite sequence of expressions, each of which is an axiom (if there are any), is a member of Γ, or follows from previous formulae in the sequence by an inference rule; the sequence is said to be an argument from Γ to A, where A is the last member of the sequence. For an argument from Γ to A, we write $\Gamma \vdash A$. Note that since arguments are only finitely long, $\Gamma \vdash A$ just in case there is a finite subset of Γ, say Δ, such that $\Delta \vdash A$.

Good arguments should always lead us from true (designated) premisses to true (designated) conclusions. So to say that our theory of argumentation is *sound* is to say that it must be such that whenever there is an argument from Γ to A, if our semantics assigns every member of Γ a designated value, then A must be designated as well. We write '$\Gamma \Vdash A$' to mean that whenever every member of Γ is assigned a designated value, A is also assigned a designated value. So to say that our theory of argumentation is sound means that if $\Gamma \vdash A$ then $\Gamma \Vdash A$.

On the other hand, we want to be assured that our theory of argumentation is strong enough. Whenever the truth of all expressions in Γ guarantees the truth of A, we would like to know that our argument theory is strong enough to reflect this fact by allowing us to give an argument from Γ to A. We say that *our argument theory* is *complete* just in case if $\Gamma \Vdash A$ then $\Gamma \vdash A$.

The preceding characterization of soundness and completeness, viz. $\Gamma \vdash A$ iff $\Gamma \Vdash A$, is commonly called *argument correctness*, a concept to which we shall return presently. A special case of that is where there are no premisses[5] – what we above called a 'correct syntax'. That is, where all and only theorems are always designated: $\vdash A$ iff $\Vdash A$. If it can be shown that the set of formulae which are always designated does not form a recursively enumerable set, then the set of theorems is not specifiable, and hence the theory is not recursively axiomatizable (i.e., there is no correct syntax for the semantics).

[4] Goguen, p. 365 also seems to have this as a definition. But since he wants to be able to mirror 'fuzzy reasoning processes' involving 'decrease of truth-value' he concludes that there can be no such syntax.

[5] Suppose c is some connective which has standard conditional properties, namely that if $V(\phi \mathbf{c} \psi)$ and $V\phi$ are designated, then $V\psi$ must also be designated. (Note that '$\rightarrow$' has this property, but '$\supset$' does not: for the latter, suppose that $\frac{3}{4} \leq r \leq 1$ are designated, then if $V\phi = \frac{3}{4}$ and $V\psi = \frac{1}{2}$, $V(\phi \supset \psi) = \frac{3}{4}$.) Now suppose that the logic in question has an analogue of the deduction theorem with this connective: that $\Gamma \cup \{A\} \vdash B$ if and only if $\Gamma \vdash (A \mathbf{c} B)$. Then an equivalent special case is where Γ contains only a finite number of premisses.

The Łukasiewicz and Tarski (1930) axioms for the propositional logic together with Modus Ponens as its only rule of inference[6] was shown to be complete and consistent by Rose and Rosser (1958) and by Chang (1959) when 1 is taken as the only designated value. (Note that this is not a proof of *argument correctness*, but only of the special case of syntax completeness.) An interesting question to ask at this point would be: for what values of **r**, such that a formula is designated iff it always takes a value greater than or equal to **r**, is the set of corresponding formulae recursively enumerable? Since there are a non-denumerable number of such **r**'s (and since there are only a denumerable number of recursively enumerable sets built up from a finite number of connectives), there are an awful lot fewer **r**'s which can even stand a chance at being axiomatized than there are **r**'s which do not even stand a chance. This question is discussed in Belluce (1964).

The monadic predicate logic, using 1 as the only designated value, was correctly axiomatized by Rutledge (1959). A *weak completeness* theorem for the full predicate logic is given by Belluce (1960) using the axioms of Hay (1958). The weak completeness theorem states: If A always takes a value greater than $\frac{1}{2}$, then '$A+A$' is provable.[7] The crushing blow to infinite valued logics came with the result of Scarpellini (1962) showing that the set of formulae of the full predicate logic which always take the value 1 is not recursively enumerable. Hence there can be no axiomatization of them; there is no effective way to talk about the class of designated formulae of such systems.

II. METHODOLOGICAL NOTES

We think, however, that the Łukasiewicz/Tarski/Hays systems are not really what advocates of fuzzy logic are after. We mentioned the preceding

[6] One of the original five axioms was proved dependent by Meredith (1958) and by Chang (1958). It is not obvious that the Zadeh/Lakoff primitive symbols '∨', '&', '→' can define the Łukasiewicz/Tarski '+', '·', '⊃'. If not, the fact that the Łukasiewicz/Tarski system is correct does not imply that the Zadeh/Lakoff one is. A close-by axiom system is obtained by (a) deleting the value $\frac{1}{2}$ from the range of possible truth values, and (b) allowing the designated values to be those greater than $\frac{1}{2}$. Then the appropriate syntax is exactly ordinary two-valued logic. All and only theorems of two-valued logic will be theorems of this (modified) fuzzy logic. Such a move certainly takes away a lot of the appeal of fuzzy logics. If there were no '→' in the logic, and the conditional were equivalent to '$(\sim\phi \vee \psi)$' it turns out that no formula always takes the value 1. Hence axiomatization is trivial: the logic has no axioms. van Fraassen (1975) gives an axiomatic system for allowing values to be greater than or equal to $\frac{1}{2}$. He asserts without proof that this is, in a certain sense, equivalent to Zadeh/Lakoff and hence the latter is axiomatized by that set of axioms.

[7] A somewhat different version of weak completeness is given in Hay (1958): If ϕ always takes a value greater than 0, then '$\phi+\phi+\ldots+\phi$' (k times) is provable for some value of k.

only to emphasize these things: (a) if advocates really are after the Łukasiewicz/Tarski/Hays systems, they can never get them in a usable form, and (b) advocates of fuzzy logic must more carefully state which results they are interested in and by what methods they hope to achieve them.

Most advocates of fuzzy logic want to 'use the semantics' of their systems, and they claim not to be all that interested in its syntax. Let us investigate what this amounts to. In a typical 'use of the semantics', certain sentences will be claimed to be true to a certain degree, compared with sentences true to some (possibly different) degree, and certain conclusions will be drawn from such comparisons. The original object language (i.e., the language of the syntax, the Łukasiewicz/Tarski/Hays systems) adopted for fuzzy logic is unable to express this sort of information. For example, suppose that

(4) Sam is tall

is true to degree 0.7, and that

(5) Al is tall

is true to degree 0.5. The original object language allows us no way of expressing the obvious conclusion that Sam is taller than Al (or: taller to a certain degree than Al). That is to say, we are provided with no method of mirroring the semantic information about Sam's and Al's heights in the syntax. Let us try to remedy this defect. Let us try to develop a formal language which is capable of mirroring the semantical remarks about different truth values of sentences of the original fuzzy logic. The standard remedy (see Rosser and Turquette, 1952) is to introduce special one-place sentential operators. These operators have the form 'J_k' where the subscript 'k' is one of the truth -values. Intuitively, prefixing such an operator before a sentence A has the effect of saying that A has exactly the truth-value k. So (4) and (5) would become

(4*) $J_{0.7}[p]$
(5*) $J_{0.5}[q]$

from which (with suitable other principles) we can infer that Sam is taller than Al. The J-operators are two-valued: if the formula ϕ has the value k then $J_k[\phi]$ has the value 1, otherwise the value 0. We should in this way be able to capture the Lakoff intention that the only designated value is 1, and yet prove things about formulae with values less than 1 – we prove the corresponding J-formula. The use of J-operators allows us to mirror the important novelties of the fuzzy semantics in the syntax; this autodescriptivity – the ability to mirror the semantics in the syntax – is regarded by some authors as necessary for the adequacy of any many-valued logic (Rescher, 1969; pp. 86–87).[8] Without it, the apparent many-valuedness is only illusory, since we cannot *say* anything in a many-valued way.

However, the intruduction of J-operators brings up a serious problem. The proofs of a number of important theorems about logical systems depend on the fact that there are only a denumerably infinite number of symbols in the system. The present system would have a non-denumerable infinity of J-operators and hence violate this restriction. So, for example, the technique of Gödel numbering could not be applied. Perhaps the most satisfactory change would be to allow as semantic values only the rational numbers between 0 and 1, rather than the reals. There would then be only a denumerable infinity of distinct J-operators.

As far as we can determine, the introduction of J-operators into an underlying fuzzy logic captures all of the formal properties claimed for fuzzy logic and also allows the use of the semantical apparatus desired by its advocates. We now turn to an accounting of three insuperable shortcomings any such system has.

III. CRITICAL NOTES

The first deficiency with fuzzy semantical systems (or rather, with this new logic intended to capture fuzzy semantical untuitions – the logic we shall from now on intend by the term 'fuzzy logic') is the lack of simple algorithmic methods which can be programmed for computers to provide a useful tool for logicians, linguists, or researchers in artificial intelligence and robotics. Fuzzy logic, as indicated in the first paragraph above, is claimed to be able to give us a mechanical aid in our investigations about classification, pattern recognition, language, computer learning, fuzzy reasoning, threshold phenomena, *inter alia*. By this we understand that there must be some kind of algorithm for detecting valid expressions which can be followed, leading

[8] We might also want to introduce in the object language a way of talking about upper and lower bounds; '$U_k(\phi)$' or '$L_k(\phi)$' would be interpreted as saying, in a two-valued way, 'the value of ϕ is less than k' or 'the value of ϕ is greater than k'. It would then be possible, for example, to express in the object language the relation between '$\sim J_k[\phi]$' and the value of ϕ: $\sim J_k[\phi] \equiv L_k(\phi) \vee U_k(\phi)$. That is, we can define the J-operators in terms of these new operators and '&' and '$\sim$':

$$J_k[\phi] =_{\mathrm{df}} \sim L_k(\phi) \;\&\; \sim U_k(\phi).$$

These operators are similar to the 'threshold operators' of Nakamura. One might also want to allow constant functions: names for each of the truth-values, say C_k, where $0 \leq k \leq 1$. Then the J-operators can be defined

$$J_k[\phi] =_{\mathrm{df}} (\phi \rightarrow C_k) \;\&\; (C_k \rightarrow \phi).$$

(This observation is due to David Lewis.)

to success if any method will. Of course, even if we have such an algorithm, it may not be of practical use for many applications simply because of computational complexity considerations. But without such an algorithm, certainly *no* practical use can be made of the theory. Our argument here will be that no simple generalization of the methods known for other logical systems will work for fuzzy predicate logic with J-operators. There is a stronger argument to the conclusion that *no* such procedure is available for any fuzzy predicate logic in which certain standard operators are definable. This conclusion easily follows from our discussion of the 'third deficiency' below. Here we consider only generalizations of old methods.

In two-valued propositional logic there is the (automatic) method of truth tables for determining validity; for finitely many valued propositional logic there is an analogous many valued truth table. And even though ordinary first-order predicate logic is not decidable, there is the method of using Skolem functions to eliminate quantifiers and check the resulting formula for validity; this method is called a 'resolution procedure' (see, e.g., Chang and Lee, 1973). Its success depends on transforming formulae into logically equivalent ones wherein the quantifiers have 'wide scope' – that is, where all the quantifiers have the matrix of the formula in their scope. To find this equivalent form, one must be able to find equivalent ways of expressing a formula which does not have quantifiers with wide scope. For all the sentential connectives there is such an equivalent form, e.g.,

(6) $\sim(\forall x)Fx$

is equivalent to

(7) $(\exists x)\sim Fx$

This sort of process is repeated until all quantifiers have widest scope. If any of the connectives did not have the property that they could be distributed through quantifiers, there would be no resolution procedure. And if there be no resolution procedure, the syntax would be worthless as a computational aid. Above, in examples (2) and (3), it was mentioned that the interpretation of the universal quantifier was to be a greatest lower bound operator and that of the existential quantifier to be a least upper bound operator. In this, the universal quantifier is interpreted as being like a conjunction, the existential like a disjunction. A sentence like

(8) $J_k[(\forall x)Fx]$

would then say 'The greatest lower bound of the values of the Fx's is exactly k'. This is clearly different from

(9) $(\forall x)J_k[Fx]$

which says 'Every object is such that F of it has the value k'. Note that the truth of (8) is independent of the truth of even the following:

(10) $(\exists x)J_k[Fx]$

because (a) (10) can be true when (8) is false, as when some F does have the value k but other F's have values less than k, and (b) (8) can be true when (10) is false, as when the greatest lower bound of the F's is k, but there is no particular F which has the value k. In fact there is no formula equivalent to (8) where the J-operator has been 'moved inside' a quantifier (or the quantifier replaced by some non-quantificational formula). The closest that can be achieved is

(11) $\prod_{i=0}^{i<k} (\forall x) \sim J_i[Fx]$

where Π is the conjunction of the formulae of the form following, and the boundaries of the Π-operator indicate which subscripts should be attached to the J-operators. However, this will be an infinite conjunction, since there will be an infinite number of values between 0 and k, and hence (11) isn't a formula of fuzzy logic. We can also find a formula which is implied by (8) making use of the greatest lower bound operator (if we allow one in the syntax – see fn. 8), but it is not equivalent to (8) since it doesn't imply it:

(12) $(\forall x)L_k[Fx]$

Thus THEOREM 1: *There is no resolution procedure for fuzzy predicate logic.*

Without some computerized procedure, the application of fuzzy logic is essentially limited to *stating* intuitive principles. But even this application is vacuous, for as we shall see below, it is provable that there can be no adequate scheme for making inferences in fuzzy logic, even in fuzzy propositional logic.

The preceding observation was to the effect that even if there were a system which was correct for fuzzy logic, there does not seem to be any computational use to which it could be put. The second shortcoming is that there can be no correct account of argumentation for fuzzy logic with J-operators, whether the J-operators be primitive or defined.

We indicated above what a logical system of argumentation must do. It must be argument-correct – every time we are given a set of premisses all of which are designated, our rules of inference must always lead to a designated conclusion. But we can rigorously prove that there can be no such argumentation theory for fuzzy logics with J-operators. Our proof simply depends on noting that such fuzzy logics lack a property called semantic compactness (to be described below). We can show that any theory of argumentation with finite length proofs which is sound and complete must be semantically compact. Hence there can be no correct argument theory for fuzzy logics. That is, for independent reasons we want our language to be completely auto-descriptive (to allow us to *express* all of our supposed semantic insights) and we want the syntactic theory to be argument complete. We shall show that both desiderata cannot be simultaneously fulfilled: if we have argument completeness then we must also have semantic compactness, and if we have completely auto-descriptivity then we cannot have semantic compactness. It should also be noted that the result does not depend at all on what other syntactic machinery may or may not be around.

A formal language is said to be semantically compact just in case for any set of expressions Γ, Γ is satisfiable (there is a way to simultaneously assign every expression in Γ a designated value) if and only if every finite subset of Γ is satisfiable. Fuzzy logics, even fuzzy propositional logics, are generally not compact. Suppose our language has a negation operator '¬' such that '¬*Q*' takes an undesignated value when '*Q*' takes a designated value ('~' is such a negation). Now consider the following two infinite sets:

(13) $\Gamma = \{\neg J_k[Q]: 0 \leq k \leq 1\}$

(14) $\Gamma' = \{J_1[Fa_1], J_{1/2}[Fa_2], J_{1/4}[Fa_3], \ldots, \neg J_0[(\forall x)Fx]\}$.

The set Γ is not satisfiable, since *Q* must take one of the values $0 \leq k \leq 1$, while the condition on membership amounts to saying that it cannot take any of them. But note that every finite subset of Γ is satisfiable by simply assigning *Q* some value not mentioned in the finite subset of *k*'s. And similarly for the predicate logic case, Γ′ is not satisfiable, since the greatest lower bound of the values of the Fa_i's is 0 and we are precluded from assigning $(\forall x)Fx$ the value 0 by the last expression listed in Γ′. But clearly every finite subset of Γ′ is satisfiable. Hence neither fuzzy propositional logic nor fuzzy predicate logic with *J*-operators is compact. Note that we can obtain the same result by using connectives other than '¬'. For example, suppose we have a conditional connective 'c' which has the semantic property that if '$\phi c \psi$' that designated value and 'ϕ' takes a designated value, then 'ψ' takes a designated value ('→' is such a connective). We can then replace '$\neg J_k[\phi]$' in (13) and (14) by '$J_k[\phi] c J_0[J_k[\phi]]$'. Therefore, any logic (or theory of 'semantical intuitions') which will allow one to define the *J*-operators – e.g., the 'threshold logics' of Nakamura – will not be semantically compact.

We will now use the fact that fuzzy logics are not semantically compact to prove that there can be no adequate theory of argumentation for them. Let Γ be a set of expressions which is not satisfiable but such that every finite subset is satisfiable. Since Γ is not satisfiable, it trivially follows that for any particular expression ϕ, $\Gamma \Vdash J_0[\phi]$ and also $\Gamma \Vdash J_1[\phi]$. Now, suppose (for *reductio*) that there were a sound and complete proof theory. Then from completeness it would follow that $\Gamma \vdash J_0[\phi]$ and also $\Gamma \vdash J_1[\phi]$. Since proofs are only finitely long, there must be finite subsets of Γ, say Δ and Δ', such that $\Delta \vdash J_0[\phi]$ and also $\Delta' \vdash J_1[\phi]$. But then we would have $\Delta \cup \Delta' \vdash J_0[\phi]$ and also $\Delta \cup \Delta' \vdash J_1[\phi]$. From soundness it follows that $\Delta \cup \Delta' \Vdash J_0[\phi]$ and also $\Delta \cup \Delta' \Vdash J_1[\phi]$. But since $J_0[\phi]$ and $J_1[\phi]$ cannot be jointly satisfied, it follows that $\Delta \cup \Delta'$ cannot be satisfied. Now note that $\Delta \cup \Delta'$ is a finite subset of Γ, and thus contradicts the assumption that every finite subset of Γ is satisfiable.

So, THEOREM 2: *There can be no sound and complete theory of argumentation for fuzzy logics, even fuzzy propositional logics.*[9]

Advocates of fuzzy logic are therefore faced with an unavoidable dilemma. Either one does not allow the introduction of J-operators (or equivalent ways of naming truth values) and so the language is not expressively adequate for the motivating semantical intuitions,[10] or else there can be no adequate theory of argumentation. Adopting either alternative makes fuzzy logic inadequate for those tasks for which it initially seemed to be promising.

The third failing attaches to fuzzy predicate logic. We shall show that the set of valid formulae of fuzzy predicate logic (with J-operators) is not recursively enumerable, and hence not only is there no correct theory of argumentation but also no correct syntax for fuzzy predicate logic. (In the Appendix we shall show that fuzzy propositional logic is decidable.) The proof proceeds by showing that the introduction of J-operators does not alter the proof of Scarpellini (1962) that fuzzy logic without J-operators does not have a recursively enumerable class of valid formulae.

We define a J-free formula to be one which has no J-operators in it. We first prove that the class of valid J-free formulae of the system with J-operators is identical to the class of valid formulae of the system without J-operators. (That is, that the addition of J-operators to the logic does not

[9] This includes the attempts to give a natural deduction system for fuzzy logic, as for example James McCawley 'Quantifiers and Fuzzy Logic' (read at a Semantics Workshop in Vancouver, Oct. 1975) or his chapter on fuzzy logic in *Everything Linguists ever Wanted to Know About Logic* (unpublished).

[10] And even if one tries to swallow his desire for autodescriptivity by neither allowing J-operators nor allowing them to be defined, this problem would still remain (albeit in a different form), since Scarpellini's (1962) result shows that such a logic does not permit a recursively enumerable statement of the set of formulae which always take the value 1.

alter the class of valid J-free formulae – the only new valid formulae added have J's in them.) The proof of this would proceed by induction on the depth of embedding of atomic formulae within connectives. (We define this in the usual way.) Note that any atomic formula which is satisfied in the original system by a valuation **V** is satisfied in the system after adding J-operators by the same valuation **V**. Now suppose, for induction, that every formula of the old logic with embedding less n is satisfied by a valuation **V** if and only if it is a J-free formula of the new logic and satisfied by **V**. It is trivial to verify that the addition of any sentential connective (except J's) or quantifier retains this property. Thus every J-free formula of the new J-logic is satisfied by a valuation **V** if and only if it is a formula of the old logic which was satisfied by **V**. And so the class of valid J-free formulae is identical to the class of valid formulae of the language without J-operators.

Now, the class of all formulae of the old language without J-operators is recursively enumerable (call this class A). But the class of its valid formulae is not recursively enumerable (call this class B). To prove that the class of valid formulae of the new language (call it C) is not recursively enumerable, we merely note that, in light of the result of the preceding paragraph, $B = A \cap C$. But if A and C were recursively enumerable, then B, their intersection, would be also. But B isn't. And since A is trivially recursively enumerable, C cannot be. So the class of valid formulae of fuzzy logic with J-operators is not recursively enumerable.

Hence THEOREM 3: *There is no correct syntax for fuzzy predicate logic with J-operators.*

Of course, a non-recursively-enumerable logic *might* be axiomatizable by an infinitary rule of inference, but we have here shown various theoretical reasons for eschewing infinite-valued logics: we must give up either autodescriptivity or argument completeness, there are no resolution procedures, and they are not recursively axiomatized. Certainly, finitary rules and procedures are preferable to infinitary ones for *applications*. So unless we are otherwise forced to infinite-valued logic, we should use finite-valued ones.

IV. SOME CONCLUDING NOTES

There is, of course, nothing wrong with many-valued logics in general. The problem here is with letting the range of values be infinite. Lakoff (1972, p. 187) says

> ... human beings cannot perceive that many distinctions. Perhaps it would be psychologically more real not to have an infinity of degrees of set-membership, but rather some relatively small

number of degrees, say the usual 7 ± 2. On the other hand, one might consider the interesting possibility that the finiteness of human perceptual distinctions is what might be called a surface phenomenon. It might be the case that the perception of degrees of tallness is based on an underlying continuous assignment of values. ... The finite number of perceived distinctions would then result from 'low level' perceptual factors. ... I think that the latter proposal has a high degree of plausibility, and I think that some of the facts discussed below will make it even more plausible. For this reason, I will stick to continuous assignments of values.

So, following Lakoff we have entertained the 'interesting possibility' and found that it is not really a possibility; and on the assumption that what is psychologically plausible must not be logically impossible, we suggest that a finite course of values – 7 ± 2 or 3 000 000 – is to be preferred. Such logics always have an adequate theory of argumentation, as well as being expressively adequate for the semantics (see Rosser and Turquette, 1952). In addition, resolution procedures have been developed for such logics, making them computationally useful tools (see Morgan, 1973).

Another suggestion for what the best choice of a course of semantic values is (given that we wish to have recourse to many-valued logics) is that what happens in human perception or human reasoning is that we 'focus' or 'change frames' in processing apparently fuzzy data. For example, in grading term papers one might be tempted to read through them and sort into three piles: good, indifferent, bad. One might then take each pile separately and sort into good, indifferent, bad (given that it has already been classified into a gross group). One might continue this process for a long time (if one has sufficient grades which can be assigned). It is clear that such judgments as this can be accounted for by a 3-valued logic with a 'focus'. No infinite-valued logic is required. We suggest that the same sort of apparatus could be made to work in all the fields of interest to advocates of fuzzy logic. Another possibility is that different domains might have different sizes of semantic values: There might be only 2 or 3 'degrees' of being a member of homo sapiens or of being the metal gold; but there might be 20 or 30 degrees of being a bird (cf. Heider, 1971, and Dreyfus *et al.*, 1975). In sum, we conclude that all the intuitions which are truly motivating fuzzy logic can be accommodated in a finite-valued logic without loss. So it's not that the intuitions are wrong, merely their expression.

APPENDIX: THE METHOD OF HYPOTHESES

In this appendix we give a method for determining whether any formula of the fuzzy propositional logic with J-operators is valid. (The method is, in outline, due to David Lewis, personal communication.) We shall first present the method for a specific fuzzy propositional logic, one which includes the Lakoff/Zadeh fuzzy propositional logic. We then consider the class of languages to which the method can be altered.

In fn. 8 it was observed that the J-operators could be defined in terms of '~', '&', and a set of constants of the form 'C_k', $0 \leq k \leq 1$, where these constants name (i.e., always take as semantic value) the truth value indicated by the subscript. Let us first consider the language with these constants, an infinite set of propositional variables, and the connectives '~', '&', '·', 'v', '+', '→', and '⊃' (as semantically defined in the body of the paper and with the usual formation rules).

The method of hypotheses is essentially a way of using finite valued truth tables, but where there is no fixed number of values and the formula itself determines 'how many' values are required. For the language under consideration, the method of hypotheses works like this, where 1 is the only designated value. Given a formula ϕ, consider the set of all sub-formulae of ϕ, call this set $\langle\phi\rangle$. Note that $\langle\phi\rangle$ is finite. An effective method for constructing hypotheses is the following. If $\langle\phi\rangle$ contains any of the C_k's, list them in order. Call each of the numbers **k** of the C_k's a *point of reference*: also allow 0 and 1 to be *points of reference*. Define a *reference interval* to be either a point of reference or the open interval between two adjacent points of reference. Note that for each formula there are finitely many reference intervals. We are now ready to construct individual hypotheses. (1) For each hypothesis, assign every propositional variable in $\langle\phi\rangle$ a reference interval. (2) Consider members of $\langle\phi\rangle$ with one connective; for each hypothesis generated by (1), arbitrarily assign each of these members of $\langle\phi\rangle$ a reference interval.... (n) Consider members of $\langle\phi\rangle$ with $n-1$ connectives; for each hypothesis generated by step $(n-1)$, arbitrarily assign each of these members of $\langle\phi\rangle$ a reference interval.

Following (1) – (n) is a finite task, since $\langle\phi\rangle$ is finite. The (finite) set of hypotheses generated now needs to be pared down, since not all of them are possible hypotheses. Consider each hypothesis in the following way. $(n+1)$ For each formula with embedding of depth 1, determine whether it could occupy the reference interval it does, given the reference interval occupied by the atomic formulae composing it (according to the valuation rules for the connective involved). If it is not possible, discard this hypothesis. (For example: suppose '$(P \& Q)$' were assigned the (open) interval $(\frac{1}{2}, 1)$, we check 'P' and find it to be assigned the interval $(\frac{1}{4}, \frac{1}{2})$, then given the valuation rule for '&' this is not a possible hypothesis.) It may be that, although possible, the hypothesis contains insufficient information and new reference intervals need be added. Thus for example, '$\sim P$' may be hypothesized to be in the (open) interval $(\frac{1}{2}, 1)$ while 'P' was hypothesized to be in the interval $(\frac{1}{4}, \frac{1}{2})$. In this case we want to say that '$\sim P$' is really in the interval $(\frac{1}{2}, \frac{3}{4})$; so we add the reference point $\frac{3}{4}$; and the original hypothesis breaks into three sub-hypotheses with respect to '$\sim P$' – one where '$\sim P$' is assigned $(\frac{1}{2}, \frac{3}{4})$, one for $(\frac{3}{4}, 1)$ and one for $\frac{3}{4}$ (of course the last two are not possible hypotheses and will subsequently be ruled out). And for any other member of $\langle\phi\rangle$ which had been assigned $(\frac{1}{2}, 1)$, each of the three sub-hypotheses breaks down into three sub-sub-hypotheses in which that other member of $\langle\phi\rangle$ is assigned one of $(\frac{1}{2}, \frac{3}{4})$, $(\frac{3}{4}, 1)$, $\frac{3}{4}$. If **n** other members of $\langle\phi\rangle$ had been assigned to $(\frac{1}{2}, 1)$ in this example, the original hypothesis would give rise to 3^n sub-hypotheses; for each of these we must check the formulae with embedding depth 1 to see if they are possible hypotheses. This number might be large, but it is finite. $(n+2)$ For each formula with depth of embedding 2, check whether it is possible on the basis of its component formulae: discard it if not, add new reference intervals and new sub-hypotheses when necessary.... Note that since $\langle\phi\rangle$ is finite, this is a finite task.

Now, consider the set of possible hypotheses, the ones that remain after this process. Since ϕ is a member of $\langle\phi\rangle$, each hypothesis will have assigned it a reference interval. If all possible hypotheses assign it the reference interval 1, it is valid. If all possible hypotheses assign it the reference interval 0, it is contradictory. Otherwise it is contingent. The method can be extended to have other values designated than just 1. Either $(r, 1]$ or $[r, 1]$ can be had by deleting 0 and 1 as reference points, but adding $[0, 1-r)$ and $(r, 1]$ or $[0, 1-r]$ and $[r, 1]$ respectively as reference intervals.

Hence the method will effectively decide, for each formula, whether it is valid: a decision procedure. Since our logic has a connective **c** obeying standard conditional conditions (that if $V(\phi \mathbf{c} \psi)$ and $V\phi$ are both designated so is $V\psi$), namely '$\rightarrow$', it follows that the method will also work for arguments with finitely many premisses by applying it to the conditionalization of the argument. (Of course there is still no general theory of argumentation possible, as the discussion in the main text of this paper shows.) If we allow the value 1 to be the only designated value, then '$\supset$' also has this standard conditional property and so it can also be used to conditionalize arguments.

This method resembles many-valued truth-tables in that propositional variables are assigned a range of values and the appropriate range of compound expressions is determined on the basis of the ranges of the simpler ones. And like truth tables, there are certain shortcuts available – for example the method of falsifying conclusions and attempting to make a consistent assignment to the premisses. For the logic with J-operators, a shortcut can be made by noting that any member of $\langle\phi\rangle$ which has a J-operator as its main connective must take value 0 or 1. So all possible hypotheses must assign them one of these values. In any hypothesis in which it is assigned 1, the constituent expression must be assigned the value indicated by the subscript (and this may involve constructing a new point of reference at that number).

It is clear that the *standard* propositional logics introduced in the fuzzy logic literature will be decidable: these logics all have some subset of '·', '&', '~', 'v', '+', '→', and '⊃' as their connectives, and use something equivalent to our J-operators (which can be seen as an application of the present C-constants). Obviously, many other connectives could be added, so long as a reference *interval* value can be computed from the reference *interval* values of the components. (We give an example below to show the importance of the italicized 'interval'.)

It also would seem to follow from the decidability of the +-fragment of arithmetic that any fuzzy propositional logic with finitely many connectives, the semantic value of which can be expressed as a function in the +-fragment of arithmetic of its components, will be decidable. (Any of the following functions can be so expressed: min, max, +, −, =, ⩽, ⩾, and absolute value.) It would therefore seem that there are decidable fuzzy propositional logics for which the method of hypotheses fails. For example, suppose there is a 4-place connective '∘', whose semantic value is

$$V[\circ(p,q,r,s)] = \begin{matrix} 1 \text{ if } |V(p)-V(q)| \geqslant |V(r)-V(s)| \\ 0 \text{ otherwise} \end{matrix}$$

Here the method of hypotheses cannot be applied because we cannot effectively indicate the reference *intervals* corresponding the absolute values indicated. Yet since this function can be expressed in the +-fragment of arithmetic, it would seem that a fuzzy propositional logic with '∘' as its only connective is decidable.

In the literature (see especially Lakoff and Zadeh), there is use of many exotic truth functions – weighted averages, exponential functions, etc. Since the fragment of arithmetic embodying resources sufficient to define such functions is not decidable, it would seem that the fuzzy propositional logics invoking them are not either. So in Lakoff and Zadeh, it would seem that there is not a way to decide validity even in the simple case of propositional logic.

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