STAT-285 Homework 1 Solutions

Question 1 /10

Part (i) /3

The sample space is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Part (ii) /3

With $A = \{\text{getting exactly one head}\}$ and $B = \{\text{the first toss lands on a head}\}$, we have

$$\begin{aligned} A &= \{HTT, THT, TTH\} \\ B &= \{HHH, HHT, HTH, HTT\} \\ A \cap B &= \{HTT\} \\ A \cup B &= \{HTT, THT, TTH, HHH, HHT, HTH\}. \end{aligned}$$

Then the probability of these events happening are

$$\begin{aligned} P(A) &= 3/8 \\ P(B) &= 1/2 \\ P(A \cap B) &= 1/8 \\ P(A \cup B) &= 3/4 \end{aligned}$$

Part (iii) /4

- Since $A \cap B \neq \emptyset$, events A and B are *not* mutually exclusive.
- Since $P(A)P(B) = 3/8 \times 1/2 = 3/16 \neq P(A \cap B)$, events A and B are *not* independent.

Question 2 /10

Part (i) /3

Let

- p denote the probability of suffering a side effect; we are given $p = 0.003$
- n denote the number of people in our random sample; we are given $n = 100$
- X denote the number of people having the side effect

$\Rightarrow X$ is a random variable, and $X \sim \text{Binomial}(100, 0.003)$.

$$P(X = 3) = \binom{100}{3} 0.003^3 (1 - 0.003)^{97} \approx 0.0033$$

Part (ii) /3

Let

- p denote the probability of suffering a side effect; we are given $p = 0.003$
- k denote the total number of people checked; we are given $k = 50$
- Y denote the number of people checked until the (first) side effect is found

$\Rightarrow Y$ is a random variable, and $Y \sim \text{Geometric}(50, 0.003)$.

$$P(Y = 50) = (1 - 0.003)^{50-1} 0.003 \approx 0.0026$$

Part (iii) /4

Let

- p denote the probability of suffering a side effect; we are given $p = 0.003$
- n denote the number of people in our random sample; we are given $n = 1000$
- X denote the number of people having the side effect

$\Rightarrow X$ is a random variable, $X \sim \text{Binomial}(1000, 0.003)$, and we want to compute

$$P(X < 5) = \sum_{x=0}^{4} \binom{1000}{x} 0.003^x (1 - 0.003)^{1000-x}$$

Solution 1: Directly compute this probability

$$P(X < 5) = \sum_{x=0}^{4} \binom{1000}{x} 0.003^x (1 - 0.003)^{1000-x} \approx 0.8155$$

Solution 2: Use the Poisson distribution as an approximation

Letting $\lambda = np = 3$,

$$P(X < 5) \approx \sum_{x=0}^{4} \frac{3^x}{x!} e^{-3} \approx 0.8153$$

Solution 3: Use the Normal distribution as an approximation

Letting $\mu = np = 3$ and $\sigma^2 = np(1-p)$,

$$\begin{aligned} P(X < 5) &\approx \sum_{x=0}^{4} P(X < x + 0.5) - P(X < x - 0.5) \\ &= P(X < 4.5) - P(X < -0.5) \\ &\approx P(X < 0.87) - P(Z < -2.02) \\ &\approx 0.7862 \end{aligned}$$

Question 3 /10

Part (i) /3

Let

- $N(y)$ denote the number of 911 calls within y minutes. For fixed y, $N(y)$ is a random variable, and $N(y) \sim \text{Poisson}(\lambda y)$, with $\lambda = 3/10$.
  Note: $N(10) = X$.

$\Rightarrow$ The trick is to recognize that if "$Y \leq y$" (the first 911 call occurred within the first y minutes), then "$N(y) \geq 1$" (there must be at least one 911 call within the first y minutes). That is, the events $\{Y \leq y\}$ and $\{N(y) \geq 1\}$ are equivalent. Then

$$\begin{aligned} F(y) &= P(Y \leq y) \\ &= P(N(y) \geq 1) \\ &= 1 - P(N(y) = 0) \\ &= 1 - e^{-\lambda y}. \end{aligned}$$

The probability density function of Y is therefore

$$f(y) = \frac{dF(y)}{dy} = \lambda e^{-\lambda y}.$$

That is, $Y \sim \text{Exponential}(\lambda)$, with $\lambda = 3/10$.

Part (ii) /1

Since $X \sim$ Poisson(3), then $E(X) = 3$ and $Var(X) = 3$.

Part (iii) /2

Since $Y \sim$ Exponential(10/3) from **Part (i)**,

$$E(Y) = \frac{1}{3/10} = \frac{10}{3}$$
$$Var(Y) = \frac{1}{(3/10)^2} = \frac{100}{9}.$$

Part (iv) /4

We have for $y > x$,

$$\begin{aligned} P(Y \geq y) &= 1 - P(Y > y) = e^{-\lambda y} \\ P(Y \geq y | Y \geq x) &= \frac{P(Y \geq y \cap Y \geq x)}{P(Y \geq x)} \\ &= \frac{P(Y \geq y)}{P(Y \geq x)}, \quad \text{since the event } \{Y \geq y\} \text{ implies } \{Y > x\} \\ &= e^{-\lambda(y-x)}. \end{aligned}$$

Then

$$P(Y \geq 2) = e^{-2\lambda} \approx 0.5488$$
$$P(Y \geq 6 | Y \geq 4) = e^{-\lambda(6-4)} = e^{-2\lambda} \approx 0.5488$$

$\Rightarrow$ Given that the first 911 call has not happened within the initial 4 minutes, the probability of the first call happening after 6 minutes (2 minutes later) is the same as the probability of the first call happening after the initial 2 minutes. This property is referred to as the *memoryless* property. The Exponential distribution is the only continuous probability distribution that has this property.

Question 4 /10

Part (i) /3

By independence of X_1 and X_2,

$$\begin{aligned} P(1.0 < X_1 < 2.5,\ 2.2 < X_2 < 3.5) &= P(1.0 < X_1 < 2.5)P(2.2 < X_2 < 3.5) \\ &= [F(2.5) - F(1.0)][F(3.5) - F(2.2)], \quad \text{with } F(x) = 1 - e^{-3x} \\ &= [e^{-7.5} - e^{-3}][e^{-6.6} - e^{-10.5}] \end{aligned}$$

$$\approx 0.0000656$$

By the mutual independence between X_1, X_2, and X_3:

$$\begin{aligned} P(1.0 < X_1 < 2.5,\ 2.2 < X_3 < 3.5 | X_2 < 4) &= P(1.0 < X_1 < 2.5,\ 2.2 < X_3 < 3.5) \\ &= P(1.0 < X_1 < 2.5) P(2.2 < X_3 < 3.5) \\ &= [F(2.5) - F(1.0)][F(3.5) - F(2.2)] \\ &\approx 0.0000656 \end{aligned}$$

Part (ii) /1

Since $X_1 \sim$ Exponential(3), $E(X_1) = 1/3$ and $Var(X_1) = 1/9$.

Part (iii) /3

$$\begin{aligned} E(2X_1 + X_2 - 4X_3) &= 2E(X_1) + E(X_2) - 4E(X_3) \\ &= 2\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right) - 4\left(\frac{1}{3}\right) \\ &= -1/3 \\ Var(2X_1 + X_2 - 4X_3) &= 4Var(X_1) + Var(X_2) + 16Var(X_3), \quad \text{since } X_1,\ X_2,\ X_3 \text{ are independent} \\ &= 4\left(\frac{1}{9}\right) + \left(\frac{1}{9}\right) + 16\left(\frac{1}{9}\right) \\ &= 21/9. \end{aligned}$$

Part (iv) /3

Note that for random variables X and Y, $E(X) = E\{\ E(X|Y)\ \}$. Also, $Var(X_2) = E((X_2 - 1/3)^2)$ and $Var(X_3) = E((X_3 - 1/3)^2)$. Then

$$\begin{aligned} E(X_1(X_2 - 1/3)^2) &= E\{\ E(X_1(X_2 - 1/3)^2 | X_1)\ \} \\ &= E\{\ X_1\ E((X_2 - 1/3)^2 | X_1)\ \} \\ &= E\{\ X_1 E((X_2 - 1/3)^2)\ \}, \quad \text{since } X_1 \text{ and } X_2 \text{ are independent} \\ &= E(X_1/9) = 1/27 \end{aligned}$$

$$\begin{aligned} E(X_1(X_2 - 1/3)^2(X_3 - 1/3)^2) &= E\{\ E(X_1(X_2 - 1/3)^2(X_3 - 1/3)^2 | X_1, X_2)\ \} \\ &= E\{\ X_1(X_2 - 1/3)^2\ E((X_3 - 1/3)^2 | X_1, X_2)\ \} \\ &= E\{\ X_1(X_2 - 1/3)^2\ E((X_3 - 1/3)^2)\ \}, \quad \text{by the mutual independence} \\ &= \underbrace{E\{X_1(X_2 - 1/3)^2\}}_{1/27}/9 \\ &= 1/243. \end{aligned}$$

Question 5 /10

Part (i) /2

$$P(X < 9.9) = P\left(\frac{X-\mu}{\sigma} < \frac{9.9-10}{1}\right) = P(Z < -0.1) \approx 0.4602$$
$$P(9.9 < X < 10.2) = P(X < 10.2) - P(X < 9.9) = P(Z < 0.2) - P(-0.1) \approx 0.1191$$

Part (ii) /4

Since $X_i \sim N(10,1) \Rightarrow \bar{X}_{100} \sim N(10, 1/100)$. Then

$$\begin{aligned} P(9.9 < \bar{X}_{100} < 10.2) &= P(\bar{X}_{100} < 10.2) - P(\bar{X}_{100} < 9.9) \\ &= P\left(Z < \frac{10.2-10}{\sqrt{1/100}}\right) - P\left(Z < \frac{9.9-10}{\sqrt{1/100}}\right) \\ &= P(Z < 2) - P(Z < 1) \\ &\approx 0.8186 \end{aligned}$$

Part (ii) /4

We have $Y = \sum_{i=1}^{100} I(X_i < 9.9)$, where

$$I(X_i < x) = \begin{cases} 1 & \text{if the } i\text{th mint is} < x \text{ grams} \\ 0 & \text{otherwise} \end{cases}.$$

Then $Y \sim \text{Binomial}(100, p)$, where $p \approx 0.4602$ from **Part (i)**, so that

$$P(Y \leq 20) = \sum_{y=0}^{20} P(Y = y)$$

Solution 1: Directly compute this probability

$$P(Y \leq 20) = \sum_{y=0}^{20} \binom{100}{y} 0.4602^y (1-0.4602)^{100-y} \approx 5.1417 \times 10^{-8} \approx 0$$

Solution 2: Use the Normal distribution as an approximation

Let $\mu = np = 46.02$, $\sigma^2 = np(1-p) = 24.8416$

$$\begin{aligned}
P(Y \leq 20) &\approx \sum_{y=0}^{20} P\left(\frac{(y-0.5)-46.02}{\sqrt{24.8416}} < Z < \frac{(y+0.5)-46.02}{\sqrt{24.8416}}\right) \\
&\approx P(Z < -5.12) - P(Z < -9.33) \\
&\approx 1.5277 \times 10^{-7} \\
&\approx 0
\end{aligned}$$