STAT-285 Homework 2 Solutions

§6.1 Exercises, Question 7 /6

For this question, let X_i denote the gas usage (in therms) for the ith house in January in a particular area. We are told (in **Part B**) that there are N =10,000 houses in our population, but we only observe the gas usage for $n = 10$ houses. Table 1 presents the $n = 10$ observations in our random sample.

Table 1: Observations $X_1, \cdots, X_{10}$ for §6.1 Exercises, Question 7.

X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
103	156	118	89	125	147	122	109	138	99

Part A /2

Since we cannot observe all N observations, we cannot compute μ. Instead, we can estimate μ based on our random sample. Since μ is the population mean, it is sensible to estimate it with the *sample mean*

$$\hat{\mu} = \frac{1}{10}\sum_{i=1}^{10} X_i = 120.6$$

Other point estimates could be used however, but we use the sample mean because it is the *minimum variance unbiased estimator* of μ (see §6.1 of your textbook).

Part B /1

Here, $\mu = 1/N \sum_{i=1}^{N} X_i$ denote the average gas usage during January for all of the houses. Then with $N = 10,000$

$$\mu = \frac{1}{10,000} \underbrace{\sum_{i=1}^{10,000} X_i}_{\tau} = \frac{\tau}{10,000},$$

$$\Rightarrow \tau = 10,000 \times \mu.$$

Since we estimate μ with $\hat{\mu}$ in **Part A**, we can estimate τ with

$$\hat{\tau} = 10,000 \times \hat{\mu} = 1,206,000.$$

Part C /1

Let

$$Y_i = I(X_i \geq 100) = \begin{cases} 1 & \text{if } X_i \geq 100 \\ 0 & \text{if } X_i < 100 \end{cases}.$$

We can use our random sample to estimate the population proportion $p = E(Y)$ with the *sample proportion*

$$\hat{p} = \frac{1}{10}\sum_{i=1}^{10} Y_i = \frac{8}{10} = 0.8$$

Note that the *sample proportion* is simply the sample mean of $Y_1, \cdots, Y_n$, so it it is the *minimum variance unbiased estimator* of p.

Part D /2

Let $F(x)$ denote the cumulative distribution function of X, so that the (population) median is

$$M = F^{-1}(0.5),$$

that is, $F(M) = 0.5$. Since we do not know $F(x)$ we estimate it with the *empirical cumulative distribution function*

$$\hat{F}(x) = \frac{1}{10}\sum_{i=1}^{10} I(X_i \leq x).$$

Figure 1 illustrates $\hat{F}(x)$, in which we see $\hat{F}(x) = 0.5$ for $x \in [118, 122)$. Since we want a point estimate for M, we consider the midpoint of this interval

$$\hat{M} = \frac{118 + 122}{2} = 120.$$

§6.2 Exercises, Question 22 /14

See Table 2 for the observations $X_1, \cdots, X_{10}$ in our random sample.

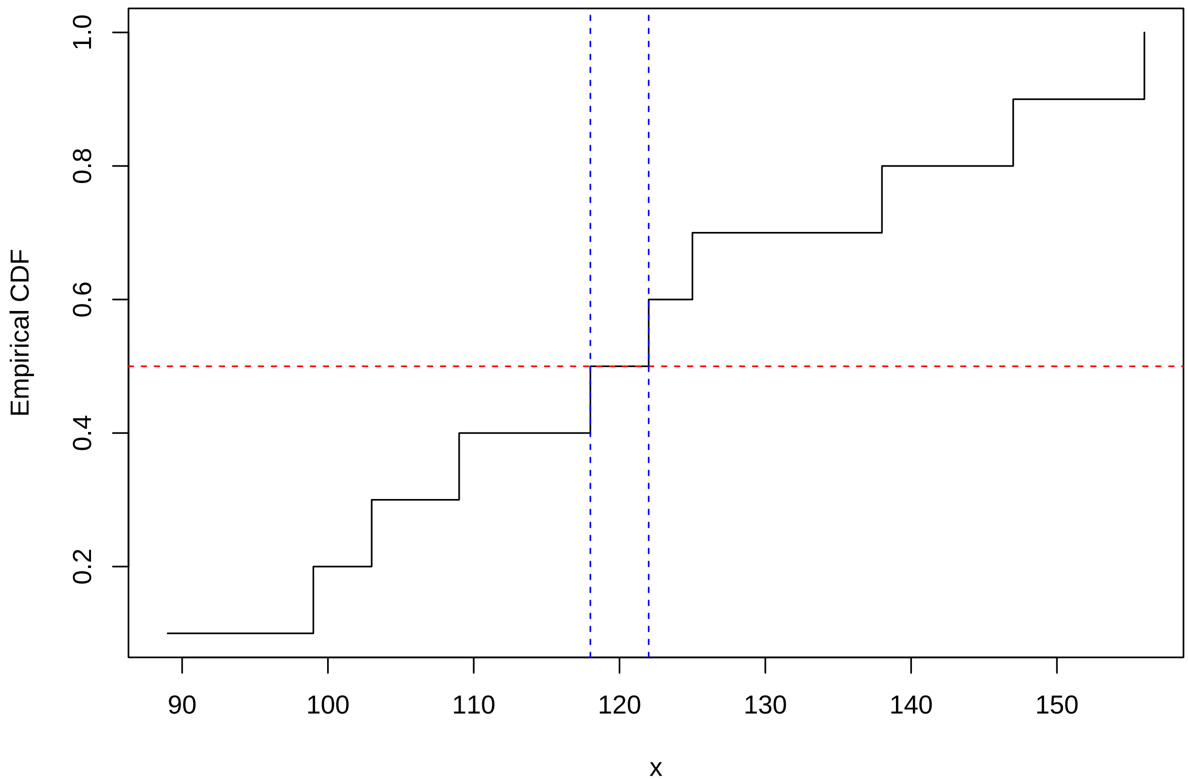

Figure 1: Illustration of x vs. $\hat{F}(x)$, for $x \in [89, 156]$. The red line corresponds to $\hat{F}(x) = 0.5$, and the blue lines illustrate the interval in which $\hat{F}(\cdot) = 0.5$

Table 2: Observations $X_1, \cdots, X_{10}$ for §6.2 Exercises, Question 22.

X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
0.92	0.79	0.90	0.65	0.86	0.47	0.73	0.97	0.94	0.77

Part A /6

We will obtain the method of moment estimator of θ by following the following steps:

Step 1: Obtain the population moment, $E(X)$:

$$\begin{aligned} E(X) &= \int_0^1 x f(x;\theta) dx \\ &= \int_0^1 (\theta+1) x^{\theta+1} \\ &= \left(\frac{\theta+1}{\theta+2}\right) x^{\theta+2} \Big|_{x=0}^{x=1} \\ &= \frac{\theta+1}{\theta+2}, \end{aligned}$$

since $x^{\theta+2} = 0$ for all $\theta > -1$.

Step 2: Equate $E(X)$ to the sample moment. That is,

$$\bar{X}_n = \frac{\theta+1}{\theta+2},$$

where $\bar{X}_n$ denotes the sample mean based on n observations.

Step 3: Solve for θ from **Step 2**

$$\begin{aligned} (\theta+2)\bar{X}_n &= \theta + 1 \\ &= \cdots \\ \hat{\theta}_{MoM} &= \frac{1 - 2\bar{X}_n}{\bar{X}_n - 1}. \end{aligned}$$

Step 4: Compute the method of moment estimate of θ

$$\begin{aligned} \bar{X}_{10} &= \frac{1}{10} \sum_{i=1}^{10} X_i = \frac{8}{10} = 0.8, \\ \Rightarrow \hat{\theta}_{MoM} &= \frac{1 - 2(0.8)}{(0.8) - 1} = 3 \end{aligned}$$

Part B /8

We will obtain the maximum likelihood estimator of θ by following the following steps:

Step 1: Write down the likelihood function

$$\begin{aligned} L(\theta|X_1, \cdots, X_n) &= f(X_1, \cdots, X_n; \theta) \\ &= \prod_{i=1}^{n} f(X_i; \theta) \quad \text{(by independence)} \\ &= (\theta+1)^n \prod_{i=1}^{n} X_i^{\theta}. \end{aligned}$$

Step 2: Write down the log-likelihood function

$$\begin{aligned} \ell(\theta|X_1, \cdots, X_n) &= \log L(\theta|X_1, \cdots, X_n) \\ &= n \log(\theta+1) + \theta \sum_{i=1}^{n} \log X_i. \end{aligned}$$

Figure 2 illustrates $\ell(\theta|X_1, \cdots, X_{10})$ vs. θ, in which we see the maximum of $\ell(\theta|X_1, \cdots, X_{10})$ corresponds to $\theta \approx 3.1161$. We will proceed to (analytically) find the value of θ that maximizes $\ell(\theta|X_1, \cdots, X_n)$.

Step 3: Differentiate $\ell(\theta|X_1, \cdots, X_n)$ with respect to θ:

$$\frac{d}{d\theta} \ell(\theta|X_1, \cdots, X_n) = \frac{n}{\theta+1} + \sum_{i=1}^{n} \log X_i.$$

Step 4: Equate $d\ell(\theta|X_1, \cdots, X_n)//d\theta$ to 0, and solve for θ:

$$\begin{aligned} \frac{n}{\theta+1} + \sum_{i=1}^{n} \log X_i = 0 \\ \cdots \\ \hat{\theta}_{MLE} = \frac{-\left(n + \sum_{i=1}^{n} \log X_i\right)}{\sum_{i=1}^{n} \log X_i} \end{aligned}$$

Step 5: Compute the maximum likelihood estimate of θ.

$$\begin{aligned} \sum_{i=1}^{10} \log X_i \approx -2.4295 \\ \Rightarrow \hat{\theta}_{MLE} \approx \frac{10 - 2.4295}{2.4295} = 3.1161 \end{aligned}$$

We can see in Figure 2 that $\hat{\theta}_{MLE}$ is indeed the maximum of $\ell(\theta|X_1, \cdots, X_{10})$.

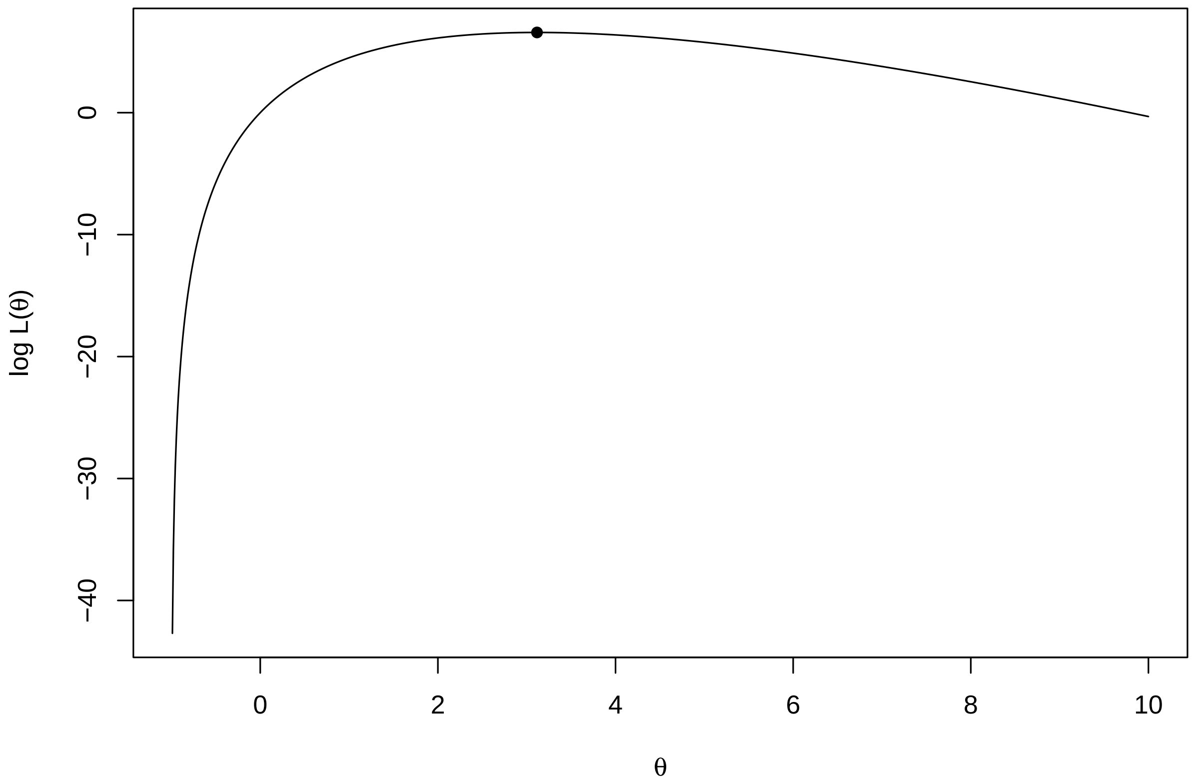

Figure 2: Illustration of $\ell(\theta|X_1, \cdots, X_{10})$ vs. θ. We can see that $\hat{\theta}_{MLE} = 3.1161$ is the maximum of $\ell(\theta|X_1, \cdots, X_{10})$.

- **Step 5**: Verify that $d^2\ell(\theta|X_1, \cdots, X_n)/d\theta^2 < 0$, evaluated at $\theta = \hat{\theta}_{MLE}$

$$\begin{aligned}\frac{d^2}{d\theta^2}\ell(\theta|X_1, \cdots, X_n) &= \frac{d}{d\theta}\left(\frac{n}{\theta+1}\sum_{i=1}^{n}\log X_i\right) \\ &= \frac{-n}{(\theta+1)^2} < 0,\end{aligned}$$

  for all $\theta > -1$.