STAT-285 Homework 3 Solutions

§7.1 Exercises, Question 3 /12

Suppose that $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$, for $i = 1, \cdots, n$ with σ^2 known. We are given that a 95% confidence interval for μ is $(7.8, 9.4)$.

For **Part B - Part D**, there are multiple reasons why the statements are incorrect; see below for *a* correct reason why the statements are incorrect.

Part A /3

Note that a 0% CI for μ is

$$\left[\bar{X} - 0 \times \sqrt{\frac{\sigma^2}{n}},\ \bar{X} + 0 \times \sqrt{\frac{\sigma^2}{n}}\right] = \bar{X},$$

and a 100% CI for μ is

$$\left(\bar{X} - \infty \times \sqrt{\frac{\sigma^2}{n}},\ \bar{X} + \infty \times \sqrt{\frac{\sigma^2}{n}}\right) = (-\infty, \infty).$$

From these two extreme cases, we can see that the higher the confidence level, the wider the CI is, so a 90% confidence interval for μ is *narrower*.

We can also demonstrate that a 90% confidence interval for μ is *narrower*, since a (1-α)100% confidence interval for μ is

$$\left(\bar{X} - Z_{\alpha/2} \times \sqrt{\frac{\sigma^2}{n}},\ \bar{X} + Z_{\alpha/2} \times \sqrt{\frac{\sigma^2}{n}}\right).$$

Then

- $\alpha = 0.05 \Rightarrow Z_{0.025} = 1.96$
- $\alpha = 0.10 \Rightarrow Z_{0.05} = 1.64$

We can see that the interval will be *wider* with $\alpha = 0.05$

Part B   /3

This statement is *incorrect* because the observed confidence interval $(7.8, 9.4)$ was based on our current random sample. If we were to hypothetically obtain 99 new random samples (so that we have 100 random samples in total), it is very unlikely that 95 of the 100 confidence intervals will be $(7.8, 9.4)$; disproving the statement in question.

Part C   /3

This statement is *incorrect* because $(7.8, 9.4)$ is a 95% confidence interval for μ, not an interval pertaining to all observations.

Part D   /3

This statement is *incorrect*, because we may not observe 95 of the 100 intervals will contain μ, due to randomness. We can however *expect* that 95 of the 100 intervals will contain μ.

§7.1 Exercises, Question 10   /8

Part A   /4

A 95% confidence interval for λ is

$$P\left(\chi^2_{0.975}(30) < 2\lambda \sum_{i=1}^{n} X_i < \chi^2_{0.025}(30)\right) = 0.95,$$

where $df = 2n = 2(15) = 30$, and if $W \sim \chi^2(\nu)$, the critical value $\chi^2_\beta(\nu)$ satisfies $P(W > \chi^2_\beta(\nu)) = \beta$. Here,

- $\sum_{i=1}^{n} X_i = 63.2$
- $\chi^2_{0.975}(30) = 16.791$
- $\chi^2_{0.025}(30) = 46.979$

Then

$$P\left(\frac{16.791}{2 \times 63.2} < \lambda < \frac{46.979}{2 \times 63.2}\right) = 0.95,$$

so that a 95% CI for λ is $(0.133, 0.372)$.

However, we do not want a 95% confidence interval for λ Recall however that if $X \sim \text{Exponential}(\lambda)$, then $E(X) = 1/\lambda$. Therefore, a 95% CI for $E(X)$ is

$$(1/0.372, 1/0.133) = (2.691, 7.528).$$

Part B /2

We need to change the critical values

- $\chi^2_{0.995}(30) = 13.787$
- $\chi^2_{0.005}(30) = 53.672$

Then

$$P\left(\frac{13.787}{2 \times 63.2} < \lambda < \frac{53.672}{2 \times 63.2}\right) = 0.95,$$

so that a 95% CI for λ is $(0.109, 0.425)$.

Therefore, a 95% CI for $E(X) = 1/\lambda$ is

$$(1/0.425, 1/0.109) = (2.355, 9.171).$$

Part C /2

Note that $Var(X) = 1/\lambda^2$, so that $SD(X) = \sqrt{Var(X)} = 1/\lambda$, which is the same as $E(X)$. That is, a 95% CI for the standard deviation of the lifetime distribution is $(2.691, 7.528)$.

§7.3 Exercises, Question 36 /10

We are given that $\bar{X} = 370.69$, $S = 24.36$, and $n = 26$.

Part A /2

The upper confidence bound for μ is

$$\bar{X} + t_{0.05}(n-1)\frac{S}{\sqrt{n}} = 370.69 + 1.708 \times \frac{24.36}{\sqrt{26}} = 378.85.$$

Part B /4

Let X_{27} denote the predicted value for the escape time of a single additional worker. We know that

- $X_{27} \sim N(\mu, \sigma^2)$
- $\bar{X} \sim N(\mu, \sigma^2/n)$

Then

$$\bar{X} - X_{27} \sim N\left(0, \sigma^2\left(1 + \frac{1}{n}\right)\right).$$

Since σ^2 is unknown, we estimate it with S^2. Therefore,

$$T = \frac{\bar{X} - X_{27}}{\sqrt{S^2(1+1/n)}} \sim t(n-1),$$

and

$$P\left(\mu < \bar{X} + t_{0.05}(n-1) \times S\sqrt{1+1/n}\right) = 0.95.$$

That is, the upper prediction bound is

$$\bar{X} + t_{0.05}(n-1) \times S\sqrt{1+1/n} = 370.69 + 1.708 \times 24.36\sqrt{1+1/26} = 413.09.$$

Note that we anticipate the width of PIs to be wider than CIs due to more uncertainty.

Part C /4

We know that

- $X_{27}, X_{28} \sim N(\mu, \sigma^2)$
- $\bar{X} \sim N(\mu, \sigma^2/n)$

Therefore,

$$E(\bar{X}_{new}) = E\left(\frac{X_{27} + X_{28}}{2}\right) = \frac{\mu + \mu}{2} = \mu$$
$$Var(\bar{X}_{new}) = Var\left(\frac{X_{27} + X_{28}}{2}\right) = \frac{\sigma^2 + \sigma^2}{4} = \frac{\sigma^2}{2}.$$

That is, $\bar{X}_{new} \sim N(\mu, \sigma^2/2)$. Then we see that

$$\bar{X} - \bar{X}_{new} \sim N\left(0, \sigma^2\left(\frac{1}{n} + \frac{1}{2}\right)\right)$$

By estimating σ^2 with S^2, we see that

$$T = \frac{\bar{X} - \bar{X}_{new}}{S\sqrt{1/n + 1/2}} \sim t(n-1)$$

Then a 95% (two-sided) PI for $\bar{X}_{new}$ is

$$\begin{aligned} &\bar{X} \pm t_{0.025}(n-1) \times S\sqrt{\frac{1}{n} + \frac{1}{2}} \\ &= 370.69 \pm 2.060 \times 24.36\sqrt{\frac{1}{26} + \frac{1}{2}} \\ &\approx (333.87, 407.51). \end{aligned}$$

§7.3 Exercises, Question 37 /10

We are given that $\bar{X} = 0.9255$, $S = 0.0809$, and $n = 20$.

Part A /4

A 95% CI for μ is

$$\begin{aligned}
&\bar{X} \pm t_{0.025}(n-1) \times \frac{S}{\sqrt{n}} \\
&= 0.9255 \pm 2.093 \times \frac{0.0809}{20} \\
&\approx (0.8876, 0.9634)
\end{aligned}$$

The probability that $\mu \in (0.8876, 0.9634)$ is 95%.

Part B /4

Letting X_{21} to denote the new observation, a 95% PI for X_{21} is

$$\begin{aligned}
&\bar{X} \pm t_{0.025}(n-1) \times S\sqrt{1 + \frac{1}{n}} \\
&= 0.9255 \pm 2.093 \times 0.0809\sqrt{1 + \frac{1}{20}} \\
&= (0.7520, 1.0990)
\end{aligned}$$

The probability that $X_{21} \in (0.7520, 1.0990)$ is 95%.

Part C /2

Tolerance Interval

Let $k \in (0, 100)$. A *tolerance interval* for capturing at least k% of the values from a $N(\mu, \sigma^2)$ distribution has the form

$$\bar{X} \pm (\text{tolerance critical value}) \times S.$$

With $k = 99$ and $n = 20$, the tolerance critical value is 3.615. Therefore, the corresponding interval is

$$0.9255 \pm 3.615 \times 0.0809 \approx (0.6330, 1.218).$$