STAT-285 Homework 4 Solutions

Section 7.4 Question 44 /6

With $n = 9$, $\chi^2_{0.025}(n-1) \approx 17.5346$ and $\chi^2_{0.975}(n-1) \approx 2.1797$, and we are given $S = 2.81$

A 95% confidence interval for σ^2 is (see lecture notes `Week3Bb.pdf`):

$$\begin{aligned}\left(\frac{(n-1)S^2}{\chi^2_{0.025}(n-1)}, \frac{(n-1)S^2}{\chi^2_{0.975}(n-1)}\right) &\approx \left(\frac{8 \times 2.81^2}{17.5346}, \frac{8 \times 2.81^2}{2.1797}\right) \\ &\approx (3.6025, 28.9805).\end{aligned}$$

Similarly, a 95% CI for σ is

$$\begin{aligned}\left(\sqrt{\frac{(n-1)S^2}{\chi^2_{0.025}(n-1)}}, \sqrt{\frac{(n-1)S^2}{\chi^2_{0.975}(n-1)}}\right) &\approx \left(\sqrt{\frac{8 \times 2.81^2}{17.5346}}, \sqrt{\frac{8 \times 2.81^2}{2.1797}}\right) \\ &\approx (1.8980, 5.3834).\end{aligned}$$

Section 8.1 Question 12 /14

Part A /2

The null hypothesis (H_0) and alternative hypothesis (H_a) are

$$H_0 : \mu = 1300 \ \text{ vs. } \ H_a : \mu > 1300$$

Part B /5

Under the null hypothesis $H_0 : \mu = \mu_0$,

$$Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \sim N(0, 1).$$

Given that $\bar{x} = 1340$, $\sigma = 60$, and $n = 10$, our test statistic is

$$\begin{aligned} Z_{obs} &= \frac{1340 - 1300}{60/\sqrt{10}} \\ &\approx 2.11 \end{aligned}$$

There are two ways we can conduct the hypothesis test of interest:

Method 1: Assess if Z_{obs} is inside the rejection region

The *rejection region* for a pre-determined α is

$$\mathcal{R}_\alpha = \{z : z > z_\alpha\}.$$

By specifying $\alpha = 0.01$

$$\mathcal{R}_{0.01} = \{z : z > z_{0.01}\} \approx \{z : z > 2.33\}.$$

Since $Z_{obs} \notin \mathcal{R}_{0.01}$, we fail to reject H_0 with $\alpha = 0.01$

Method 2: Compute the p-value

The p-value is

$$P(Z \geq Z_{obs}) = P(Z \geq 2.11) = 0.0174$$

Since $0.0174 > 0.01$, we fail to reject H_0 with $\alpha = 0.01$

Part C /7

Note that when $\mu = 1350$,

$$Z_0 = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

The probability of making a type II error when $\mu = 1350$ with $\alpha = 0.01$ is

$$\begin{aligned} \beta &= P(Z \notin \mathcal{R}_{0.01} | \mu = 1350) \\ &= P\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} < 2.33 \middle| \mu = 1350\right) \\ &= P\left(\bar{X} < \mu_0 + 2.33 \times \frac{\sigma}{\sqrt{n}} \middle| \mu = 1350\right) \\ &= P\left(\bar{X} < 1300 + 2.33 \times \frac{60}{\sqrt{10}} \middle| \mu = 1350\right) \\ &= P(\bar{X} < 1344.209 | \mu = 1350) \end{aligned}$$

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\begin{aligned}
&= P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \frac{1344.209 - \mu}{\sigma/\sqrt{n}} \middle| \mu = 1350 \right) \\
&= \Phi(-0.3052) \\
&= 0.3801
\end{aligned}
$$