STAT-285 Homework 5 Solutions

Section 9.2 Question 30 /12

Study Objective: Examine the difference in average ultimate load between *fibreglass grid* and *commercial carbon grid*.

Formulation: Let

- X_F denote the ultimate load of fibreglass grid.
- X_C denote the ultimate load of commercial carbon grid.

We assume that $X_F \sim N(\mu_F, \sigma_F^2)$, $X_C \sim N(\mu_C, \sigma_C^2)$, and X_F is independent from X_C.

Data: We observe

- $X_{F,1}, \cdots, X_{F,26}$: a random sample of 26 observations of ultimate load of fibreglass grid. We are given that $\bar{X}_F = 33.4$, and $S_F = 2.2$
- $X_{C,1}, \cdots, X_{C,26}$: a random sample of 26 observations of ultimate load of commercial carbon grid. We are given that $\bar{X}_C = 42.8$, (iii) $S_C = 4.3$

Part a /7

We aim to construct a 99% CI for $\mu_F - \mu_C$. Note that $X_F - X_C \sim N(\mu_F - \mu_C, \sigma_F^2 + \sigma_C^2)$, where σ_F^2 and σ_C^2 are unknown.

$\Rightarrow$ A point estimate for $\mu_F - \mu_C$ is $\bar{X}_F - \bar{X}_C$, with

$$Var(\bar{X}_F - \bar{X}_C) = \frac{\sigma_F^2}{26} + \frac{\sigma_C^2}{26}$$
$$\widehat{Var}(\bar{X}_F - \bar{X}_C) = \frac{S_F^2}{26} + \frac{S_C^2}{26}$$

Therefore

$$\frac{(\bar{X}_F - \bar{X}_C) - (\mu_F - \mu_C)}{\sqrt{\frac{S_F^2}{26} + \frac{S_C^2}{26}}} \sim t(\nu),$$

where

$$\begin{aligned}\nu &= \frac{(S_F^2/26 + S_C^2/26)^2}{\frac{(S_F^2/26)^2}{26-1} + \frac{(S_C^2/26)^2}{26-1}} \\ &= 37.24 \\ &\approx 37.\end{aligned}$$

Plugging in the observed values, a 99% CI for $\mu_F - \mu_C$ is

$$\begin{aligned}&(\bar{X}_F - \bar{X}_c) \pm t_{\alpha/2}(\nu)\sqrt{\frac{S_F^2}{26} + \frac{S_C^2}{26}} \\ &= (33.4 - 42.8) \pm 2.715\sqrt{\frac{2.2^2}{26} + \frac{4.3^2}{26}} \\ &= (-11.9718, -6.8282).\end{aligned}$$

Interpretation: We are 99% confident that the true mean difference between the ultimate load of fibreglass grid and commercial carbon grid lies in the interval $(-11.9718, -6.8282)$.

Part b /5

A 99% CI for $\mu_F - \mu_C$ is not the same as a 99% upper confidence bound for $\mu_F - \mu_C$.

- **Upper bound of CI**: $(\bar{X}_F - \bar{X}_C) + t_{\alpha/2}(\nu)\sqrt{\frac{S_F^2}{26} + \frac{S_C^2}{26}}$
- **Upper confidence bound**: $(\bar{X}_F - \bar{X}_C) + t_{\alpha}(\nu)\sqrt{\frac{S_F^2}{26} + \frac{S_C^2}{26}}$

A 99% upper confidence bound for $\mu_F - \mu_C$ is computed as

$$\begin{aligned}&= (33.4 - 42.8) + 2.431\sqrt{\frac{2.2^2}{26} + \frac{4.3^2}{26}} \\ &= -7.098\end{aligned}$$

Note that if the difference between the two types of grids are negligible, we would expect the upper bound to be approximately zero. Since not, it strongly suggests that the true average load for the carbon beams is larger than fibreglass beams.

Section 9.3 Question 36 /8

Study Objective: Check if there exists a difference in breaking load for fabrics in both unabraded condition and abraded condition.

Formulation: Let

- X_U denote the breaking load for fabrics in unabraded condition.

- X_A denote the breaking load for fabrics in abraded condition.

We assume that $X_U \sim N(\mu_U, \sigma_U^2)$ and $X_A \sim N(\mu_A, \sigma_A^2)$.

Data: $(X_{U,1}, X_{A,1}), \cdots, (X_{U,8}, X_{A,8})$.

Reformulation: Let $D = X_U - X_A \sim N(\mu_D, \sigma^2)$, where $\mu_D = \mu_U - \mu_A$, and σ_D^2 is unknown. We want to test if there is a difference between the breaking load for unabraded condition is larger than the breaking load for abraded condition.

Hypothesis Test: $H_0 : \mu_D = 0$ vs. $H_a : \mu_D > 0$ with $\alpha = 0.01$.

Test Statistic:

$$T = \frac{\bar{D} - 0}{S_D/\sqrt{n}} \sim t(n-1) \text{ under } H_0$$

Here,

$$\bar{D} = \sum_{i=1}^{8} X_{U,i} - X_{A,i} = 7.25$$

$$S_D = \frac{1}{7}\sqrt{\sum_{i=1}^{8}(X_{U,i} - X_{A,i} - \bar{D})^2} = 11.86$$

$\Rightarrow$ Plugging in the observed data:

$$T_{obs} = \frac{\bar{D}}{S_D/\sqrt{n}} = \frac{7.25}{11.86/\sqrt{8}} = 1.73$$

Method 1 - p-value: $P_{H_0}(T > T_{obs}) \approx 0.064$, where $T \sim t(7)$.
Since $0.01 < 0.064$, we fail to reject H_0.

Method 2 - Rejection Region:

$$\begin{aligned}\mathcal{R}_{0.01} &= \{t : t > t_{0.01}(7)\} \\ &= \{t : t > 3.0\}.\end{aligned}$$

Since $T_{obs} \notin \mathcal{R}_{0.01}$, we fail to reject H_0.

Section 9.4 Question 53 /10

Study Objective: Assess if there exists a difference in the incidence rate between the control and treatment groups.

Formulation: Let

- X_C denote the number of individuals that experience an adverse GI event in the TG control group.
- X_T denote the number of individuals that experience an adverse GI event in the olestra treatment group.

We assume that $X_C \sim \text{Binomial}(n_C, p_C)$ and $X_T \sim \text{Binomial}(n_T, p_T)$.

Part a /6

Hypothesis Test: $H_0 : p_C = p_T$ vs. $H_a : p_C \neq p_T$ with $\alpha = 0.05$.

Data: We have

- $\hat{p}_C = 0.176$, $n_C = 529$
- $\hat{p}_T = 0.158$, $n_T = 563$

Test Statistic Taking $n_C >> 1$ and $n_T >> 1$,

$$Z = \frac{(\hat{p}_C - \hat{p}_T) - 0}{\sqrt{\hat{p}(1-\hat{p})(1/n_C + 1/n_T)}} \dot{\sim} N(0,1) \text{ under } H_0.$$

Here,

$$\begin{aligned}\hat{p} &= \frac{n_C}{n_C + n_T}\hat{p}_C + \frac{n_T}{n_C + n_T}\hat{p}_T \\ &= \frac{529}{529 + 563} \times 0.176 + \frac{563}{529 + 563} \times 0.158 \\ &= 0.166\end{aligned}$$

Plugging in our values:

$$Z_{obs} = \frac{0.176 - 0.158}{\sqrt{0.166(1 - 0.166)(1/529 + 1/563)}} = 0.8$$

Method 1 - p-value: $P_{H_0}(|Z| > |Z_{obs}|) = 2(1 - \Phi(0.8)) = 0.4237$.
Since $0.4237 > 0.05$, we fail to reject H_0.

Method 2 - Rejection Region:

$$\begin{aligned}\mathcal{R}_{0.05} &= \{z : |z| > z_{\alpha/2}\} \\ &= \{z : |z| > z_{0.025}\} \\ &= \{z : z < -1.96 \text{ or } z > 1.96\}.\end{aligned}$$

Since $Z_{obs} \notin \mathcal{R}_{0.05}$, we fail to reject H_0.

Method 3 - Confidence Interval: An approximate 95% CI for $p_C - p_T$ is

$$
\begin{aligned}
&(\hat{p}_C - \hat{p}_T) \pm Z_{0.025}\sqrt{\frac{\hat{p}_C(1-\hat{p}_C)}{n_C} + \frac{\hat{p}_T(1-\hat{p}_T)}{n_T}} \\
&= (0.176 - 0.158) \pm 1.96\sqrt{\frac{0.176(1-0.176)}{529} + \frac{0.158(1-0.158)}{563}} \\
&= (-0.0046, 0.0406).
\end{aligned}
$$

Since $0 \in (-0.0046, 0.0406)$, we fail to reject H_0.

Part b /4

With $\alpha = 0.05$, we want to find samples sizes for which $\beta = 0.10$ when $p_C = 0.15$ and $p_T = 0.20$. Assuming equal sample sizes ($n_T = n_C \equiv n$), the required sample size is obtained from Equation (9.7) from the textbook:

$$
n = \frac{\left(Z_{\alpha/2}\sqrt{(p_C + p_T)(1 - p_C + 1 - p_T)/2} + Z_\beta\sqrt{p_C(1-p_C) + p_T(1-p_T)}\right)^2}{(p_C - p_T)^2}.
$$

Plugging in our values yield

$$
\begin{aligned}
n &= \frac{\left(1.96\sqrt{(0.15 + 0.20)(0.85 + 0.80)/2} + 1.28\sqrt{0.15 \times 0.85 + 0.20 \times 0.80}\right)^2}{(0.15 - 0.20)^2} \\
&= 1210.4 \\
&\approx 1211.
\end{aligned}
$$