STAT-285 Homework 6 Solutions

Section 10.1 Question 5 /7

Study Objective: Determine if there exists a difference between the mean modulus of elasticity for lumber of three different grades.

Formulation: Let

- X_{ij} denote the jth measurement for the ith grade, with $i = 1, 2, 3$, and $j = 1, \cdots, 10$ (ie $I = 3$ and $J = 10$).

We assume that $X_{ij} \sim N(\mu_i, \sigma^2)$. For this question, we are given

- $\bar{X}_{1.} = 1.63$, $\bar{X}_{2.} = 1.56$, and $\bar{X}_{3.} = 1.42$, so that

$$\bar{X}_{..} = \frac{1}{3}(\bar{X}_{1.} + \bar{X}_{2.} + \bar{X}_{3.}) = 1.54$$

- $S_1 = 0.27$, $S_2 = 0.24$, and $S_3 = 0.26$

Hypothesis Test: $H_0 : \mu_1 = \mu_2 = \mu_3$ vs. H_a: At least two of the μ_i's are different.

Test Statistic:

$$\frac{MSTr}{MSE} \sim F(2, 27).$$

Here,

$$MSTr = \frac{10}{2}[(1.63 - 1.54)^2 + (1.56 - 1.54)^2 + (1.42 - 1.54)^2] = 0.1145$$
$$MSE = \frac{1}{3}(0.27^2 + 0.24^2 + 0.26^2) = 0.06603.$$

Therefore,

$$F_{obs} = \frac{MSTr}{MSE} = \frac{0.1145}{0.06603} = 1.73$$

Method 1 - p-value: $P_{H_0}(F > F_{obs}) = 0.1963$, where $F \sim F(2, 27)$.
Since $0.01 < 0.1963$, we fail to reject H_0.

Method 2 - Rejection Region:

$$\begin{aligned}\mathcal{R}_{0.01} &= \{f : f > f_{0.01}(2, 27)\} \\ &= \{f : f > 5.49\}.\end{aligned}$$

Since $F_{obs} \notin \mathcal{R}_{0.01}$, we fail to reject H_0.

Section 10.1 Question 7 /7

Study Objective: Determine if there exists a difference between six different low-permeability concrete bridge deck mixtures.

Formulation: Let

- X_{ij} denote the jth measurement on concrete cylinders for the ith mixture, with $i = 1, \cdots, 6$, and $j = 1, \cdots, 26$ (ie $I = 6$ and $J = 26$).

We assume that $X_{ij} \sim N(\mu_i, \sigma^2)$.

Hypothesis Test: $H_0 : \mu_1 = \cdots = \mu_6$ vs. H_a: At least two of the μ_i's are different.

Test Statistic:

$$\frac{MSTr}{MSE} \sim F(I - 1, I(J - 1)).$$

We are asked to fill out the following ANOVA table:

Source	df	Sum of Squares	Mean Square	F
Mixture	$I - 1$	$SSTr$	$MSTr$	F_{obs}
Error	$I(J - 1)$	SSE	13.929	
Total	$IJ - 1$	5664.415		

Here, the

- degrees of freedom for treatment is $I - 1 = 5$
- total degrees of freedom is $IJ - 1 = 155$
- degrees of freedom for error is $I(J - 1) = 155 - 5 = 150$.
- $MSE = \frac{SSE}{I(J-1)} = 13.929$
  $\Rightarrow SSE = MSE \times I(J - 1) = 13.929 \times 150 = 2089.35$

- $SST = SSE + SSTr$
  $\Rightarrow STr = SST - SSE = 5664.415 - 2089.35 = 3575.065$
- $MSTr = \frac{SSTr}{I-1} = \frac{3575.065}{5} = 715.013$
- $F_{obs} = \frac{MSTr}{MSE} = \frac{715.013}{13.929} = 51.33$

Method 1 - p-value: $P_{H_0}(F > F_{obs}) \approx 8.3 \times 10^{-31}$, where $F \sim F(5, 150)$.
Taking $\alpha = 0.05$, we reject H_0 since $0.05 > 8.3 \times 10^{-31}$.

Method 2 - Rejection Region: Taking $\alpha = 0.05$:

$$\begin{aligned} \mathcal{R}_{0.05} &= \{f : f > f_{0.05}(5, 150)\} \\ &= \{f : f > 2.27\}. \end{aligned}$$

Since $F_{obs} \in \mathcal{R}_{0.05}$, we reject H_0.

Section 10.1 Question 10 /16

Recall that we let X_{ij} denote the jth measurement for the ith group, with $i = 1, \cdots, I$, and $j = 1, \cdots, J$. We assume that $X_{ij} \sim N(\mu_i, \sigma^2)$.

Estimator for μ_i: We estimate μ_i with $\bar{X}_{i.} = \sum\limits_{j=1}^{J} X_{ij}/J$. Note that

$$\begin{aligned} E(\bar{X}_{i.}) &= E\left(\frac{1}{J}\sum_{j=1}^{J} X_{ij}\right) = \frac{1}{J}\sum_{j=1}^{J} E(X_{ij}) = \mu_i \\ Var(\bar{X}_{i.}) &= Var\left(\frac{1}{J}\sum_{j=1}^{J} X_{ij}\right) \underbrace{=}_{\text{by independence}} \frac{1}{J^2}\sum_{j=1}^{J} Var(X_{ij}) = \frac{\sigma^2}{J}. \end{aligned}$$

$\Rightarrow$ We will use this in **Part B**.

Part A /2

By definition, we have

$$\begin{aligned} \bar{X}_{..} &= \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J} X_{ij} \\ &= \frac{1}{I}\sum_{i=1}^{I}\left(\frac{1}{J}\sum_{j=1}^{J} X_{ij}\right) \\ &= \frac{1}{I}\sum_{i=1}^{I} \bar{X}_{i.}. \end{aligned}$$

Therefore,

$$E(\bar{X}_{..}) = E\left(\frac{1}{I}\sum_{i=1}^{I}\bar{X}_{i.}\right) = \frac{1}{I}\sum_{i=1}^{I}E(\bar{X}_{i.}) = \frac{1}{I}\sum_{i=1}^{I}\mu_i \equiv \mu.$$

That is, $\bar{X}_{..}$ is an unbiased estimator for μ.

Part B /4

Recall for a random variable X:

$$E(X^2) = Var(X) + E(X)^2$$

Therefore,

$$E(\bar{X}_{i.}^2) = Var(\bar{X}_{i.}) + E(\bar{X}_{i.})^2 = \frac{\sigma^2}{J} + \mu_i^2$$

Part C /3

We see that

$$\begin{aligned}
E(\bar{X}_{..}^2) &= Var(\bar{X}_{..}) + E(\bar{X}_{..})^2 \\
&= Var\left(\frac{1}{I}\sum_{i=1}^{I}\bar{X}_{i.}\right) + \mu^2 \\
&= \frac{1}{I^2}\sum_{i=1}^{I}Var(\bar{X}_{i.}) + \mu^2, \quad \text{by independence} \\
&= \frac{1}{I^2}\sum_{i=1}^{I}\frac{\sigma^2}{J} + \mu^2 \\
&= \frac{\sigma^2}{IJ} + \mu^2
\end{aligned}$$

Part D /4

Since $\bar{X}_{i.} = X_{i.}/J$ where $X_{i.} = \sum_{j=1}^{J} X_{ij}$, then $\bar{X}_{i.}^2 = X_{i.}^2/J^2$, and

$$J\sum_{i=1}^{I}\bar{X}_{i.}^2 = \frac{1}{J}\sum_{i=1}^{I}X_{i.}^2 \tag{1}$$

Also, since $\bar{X}_{..} = X_{..}/IJ$ where $X_{..} = \sum_{i=1}^{I}\sum_{j=1}^{J} X_{ij}$, then $X_{..} = IJ\bar{X}_{..}$, and

$$X_{..}^2 = (IJ)^2 \bar{X}_{..}^2 \tag{2}$$

Putting these two results together, we can express $SSTr$ as

$$\begin{aligned} SSTr &= \frac{1}{J}\sum_{i=1}^{I} X_{i.}^2 - \frac{1}{IJ}X_{..}^2 \quad \text{(see Page 416 in your textbook)} \\ &= \underbrace{J\sum_{i=1}^{I} \bar{X}_{i.}^2}_{\text{from (1)}} - \underbrace{IJ\ \bar{X}_{..}^2}_{\text{from (2)}}, \end{aligned}$$

so that

$$\begin{aligned} E(SSTr) &= J\sum_{i=1}^{I} E(\bar{X}_{i.}^2) - IJ\ E(\bar{X}_{..}^2) \\ &= J\sum_{i=1}^{I}\left(\underbrace{\frac{\sigma^2}{J} + \mu_i^2}_{\text{from part (b)}}\right) - IJ\ \left(\underbrace{\frac{\sigma^2}{IJ} + \mu^2}_{\text{from part (c)}}\right) \\ &= I\sigma^2 + J\sum_{i=1}^{I}\mu_i^2 - \sigma^2 - IJ\ \mu^2 \\ &= (I-1)\sigma^2 + J\left(\sum_{i=1}^{I}\mu_i^2 - I\mu^2\right) \\ &= (I-1)\sigma^2 + J\left(\sum_{i=1}^{I}\mu_i^2 - 2I\mu^2 + I\mu^2\right) \quad \textbf{Note}\text{: } I\mu = \sum_{i=1}^{I}\mu_i \\ &= (I-1)\sigma^2 + J\sum_{i=1}^{I}(\mu_i - \mu)^2. \end{aligned}$$

Therefore,

$$\begin{aligned} E(MSTr) &= E\left(\frac{SSTr}{I-1}\right) \\ &= \frac{1}{I-1}\left((I-1)\sigma^2 + J\sum_{i=1}^{I}(\mu_i-\mu)^2\right) \\ &= \sigma^2 + \frac{J}{I-1}\sum_{i=1}^{I}(\mu_i-\mu)^2, \end{aligned}$$

as desired.

Part E /3

Recall that we are conducting the following hypothesis test:

$$H_0 : \mu_1 = \cdots = \mu_I \quad \text{vs.} \quad H_a : \text{At least two of the } \mu_i\text{'s are distinct.}$$

If H_0 is true, then $\mu_1 = \cdots = \mu_I \equiv \mu^*$, and

$$\mu = \frac{1}{I}\sum_{i=1}^{I} \mu_i = \frac{1}{I}\sum_{i=1}^{I} \mu^* = \mu^*.$$

We see that

$$\sum_{i=1}^{I}(\mu_i - \mu)^2 = \sum_{i=1}^{I}(\mu^* - \mu^*)^2 = 0,$$

and based on our work from **Part D**:

$$E(MSTr) = \sigma^2 + \frac{J}{I-1}\sum_{i=1}^{I}(\mu_i - \mu)^2, = \sigma^2.$$

That is, $MSTr$ is an unbiased estimator for σ^2. However, if H_0 is false (i.e. H_a is true), we see that

$$E(MSTr) > \sigma^2,$$

since

$$\frac{J}{I-1}\sum_{i=1}^{I}(\mu_i - \mu)^2 > 0.$$

BONUS Section 10.2 Question 15 /10

Study Objective: Determine which of the six different concrete mixtures differ with respect to their resistivity.

Formulation: Let

- X_{ij} denote the jth measurement for the ith concrete mixture, with $i = 1, \cdots, 6$, and $j = 1, \cdots, 26$ (ie $I = 6$ and $J = 26$).

We assume that $X_{ij} \sim N(\mu_i, \sigma^2)$, and we are given that

$$\bar{X}_{1.} = 14.18,\ \bar{X}_{2.} = 17.94,\ \bar{X}_{3.} = 18,$$
$$\bar{X}_{4.} = 18,\ \bar{X}_{5.} = 25.74,\ \bar{X}_{6.} = 27.67$$

Method: Apply Tukey's method to identify significant differences.

Using $\alpha = 0.05$, we find that $Q_{0.05,6,150} = 4.08$
(in R: `qtukey(0.95, nmeans = 6, df = 150)`). Since the question also gives $MSE = 13.929$, we have

$$W = Q_{0.05,6,150}\sqrt{\frac{MSE}{J}} = 4.08\sqrt{\frac{13.929}{26}} = 2.9863$$

Table 1 presents the sample mean differences between the six concrete mixtures. Note that if $\bar{X}_{j.} - \bar{X}_{i.} < W$, this implies that zero lies in the corresponding confidence interval. That is, using an underscoring pattern, we summarize our findings as

i	1	2	3	4	5	6
$\bar{X}_{i.}$	14.18	17.94	18	18	25.74	27.67

To interpret this, we see that there is no significant differences between

- concrete mixture 2 from concrete mixtures 3 or 4
- concrete mixture 5 from concrete mixture 6

Table 1: $\bar{X}_{j.} - \bar{X}_{i.}$ for $j > i$. The bold-faced elements correspond to the values less than W.

$i \setminus j$	1	2	3	4	5	6
1		3.76	3.82	3.82	11.56	13.49
2			**0.06**	**0.06**	7.8	9.73
3				**0**	7.74	9.67
4					7.74	9.67
5						**1.93**
6						