STAT-285 Homework 7 Solutions

Section 10.2 Question 15 /4

Study Objective: Determine which of the six different concrete mixtures have differ with respect to their resistivity.

Formulation: Let

- X_{ij} denote the jth measurement for the ith concrete mixture, with $i = 1, \cdots, 6$, and $j = 1, \cdots, 26$ (ie $I = 6$ and $J = 26$).

We assume that $X_{ij} \sim N(\mu_i, \sigma^2)$, and we are given that

$$\bar{X}_{1.} = 14.18,\ \bar{X}_{2.} = 17.94,\ \bar{X}_{3.} = 18,$$
$$\bar{X}_{4.} = 18,\ \bar{X}_{5.} = 25.74,\ \bar{X}_{6.} = 27.67$$

Method: Apply Tukey's method to identify significant differences.

Using $\alpha = 0.05$, we find that $Q_{0.05,6,150} = 4.08$ (in R: `qtukey(0.95, nmeans = 6, df = 150)`). Since the question also gives $MSE = 13.929$, we have

$$W = Q_{0.05,6,150}\sqrt{\frac{MSE}{J}} = 4.08\sqrt{\frac{13.929}{26}} = 2.9863$$

Table 1 presents the sample mean differences between the six concrete mixtures. Note that if $\bar{X}_{j.} - \bar{X}_{i.} < W$, this implies that zero lies in the corresponding confidence interval. That is, using an underscoring pattern, we summarize our findings as

i	1	2	3	4	5	6
$\bar{X}_{i.}$	14.18	17.94	18	18	25.74	27.67

To interpret this, we see that there is no significant differences between

- concrete mixture 2 from concrete mixtures 3 or 4
- concrete mixture 5 from concrete mixture 6

Table 1: $\bar{X}_{j.} - \bar{X}_{i.}$ for $j > i$. The bold-faced elements correspond to the values less than W.

$i \setminus j$	1	2	3	4	5	6
1		3.76	3.82	3.82	11.56	13.49
2			**0.06**	**0.06**	7.8	9.73
3				**0**	7.74	9.67
4					7.74	9.67
5						**1.93**
6						

Section 11.1 Question 4 /14

Study Objective: Determine if there exists a difference between the mean coverage of light-bulb interior latex paint between brands of paint and rollers.

Formulation: Let

- X_{ij} denote the observation for the ith brand of paint and jth roller brand, with $i = 1, 2, 3, 4$, and $j = 1, 2, 3$ (ie $I = 4$ and $J = 3$).

We assume that $X_{ij} \sim N(\mu_{ij}, \sigma^2)$, where $\mu_{ij} = \mu + \alpha_i + \beta_j$, with $\sum_{i=1}^{4} \alpha_i = 0$ and $\sum_{j=1}^{3} \beta_j = 0$. We see that

- $\bar{X}_{..} = \sum_{i=1}^{4} \sum_{j=1}^{3} X_{ij}/12 = 44.75$

- $\bar{X}_{i.} = \sum_{j=1}^{3} X_{ij}/3$

  $\bar{X}_{1.} = 50.33$, $\bar{X}_{2.} = 45.67$, $\bar{X}_{3.} = 41.67$, $\bar{X}_{4.} = 41.33$

  $\bar{X}_{.j} = \sum_{i=1}^{4} X_{ij}/4$

  $\bar{X}_{.1} = 45.75$, $\bar{X}_{.2} = 42.25$, $\bar{X}_{.3} = 46.25$

Part A /6

We are to fill out the following ANOVA table:

- degrees of freedom for paint brand is $I - 1 = 3$
- degrees of freedom for roller brand is $J - 1 = 2$
- degrees of freedom for error is $(I - 1)(J - 1) = 6$

Source	df	Sum of Squares	Mean Square	F
Paint Brand (Factor A)	$I-1$	SSA	MSA	F_A
Roller Brand (Factor B)	$J-1$	SSB	MSB	F_B
Error	$(I-1)(J-1)$	SSE	MSE	
Total	$IJ-1$	SST		

- total degrees of freedom is $IJ-1=11$
- $SST=\sum_{i=1}^{4}\sum_{j=1}^{3}(X_{ij}-\bar{X}_{..})^2=238.25$
- $SSA=J\sum_{i=1}^{4}(\bar{X}_{i.}-\bar{X}_{..})^2=159.5833$
- $SSB=I\sum_{j=1}^{3}(\bar{X}_{.j}-\bar{X}_{..})^2=38$
- $SSE=SST-SSA-SSB=40.6667$
- $MSA=SSA/(I-1)=53.1944$
- $MSB=SSB/(J-1)=19$
- $MSE=SSE/((I-1)(J-1))=6.7778$
- $F_A=MSA/MSE=7.85$
- $F_B=MSB/MSE=2.80$

Part B /2

Hypothesis Test: $H_{0A}:\alpha_1=\alpha_2=\alpha_3=\alpha_4=0$ vs. H_{aA}: At least one $\alpha_i\neq 0$

Test Statistic:

$$F_A=\frac{MSA}{MSE}\sim F(3,6).$$

From **Part A**, we have $F_{A,obs}=7.85$

Method 1 - p-value: $P_{H_0}(F>F_{A,obs})=0.0169$, where $F\sim F(3,6)$.
Since $0.0169<0.05$, we reject H_{0A}.

Method 2 - Rejection Region:

$$\begin{aligned}\mathcal{R}_{A,0.05} &= \{f : f > f_{0.05}(3,6)\} \\ &= \{f : f > 4.76\}.\end{aligned}$$

Since $F_{A,obs} \in \mathcal{R}_{A,0.05}$, we reject H_{0A}.

Part C /2

Hypothesis Test: $H_{0B} : \beta_1 = \beta_2 = \beta_3 = 0$ vs. H_{aB}: At least one $\beta_j \neq 0$

Test Statistic:

$$F_B = \frac{MSB}{MSE} \sim F(2,6).$$

From **Part A**, we have $F_{B,obs} = 2.80$

Method 1 - p-value: $P_{H_0}(F > F_{B,obs}) = 0.1381$, where $F \sim F(2,6)$.
Since $0.05 < 0.1381$, we fail to reject H_{0B}.

Method 2 - Rejection Region:

$$\begin{aligned}\mathcal{R}_{B,0.05} &= \{f : f > f_{0.05}(2,6)\} \\ &= \{f : f > 5.14\}.\end{aligned}$$

Since $F_{B,obs} \notin \mathcal{R}_{B,0.05}$, we fail to reject H_{0B}.

Part D /4

Since we fail to reject H_{0B} in **Part C**, we only need to use Tukey's method to identify significant differences among the *paint* brands.

We start by computing

$$W = Q_{0.05,I,(I-1)(J-1)}\sqrt{\frac{MSE}{J}} = 4.90\sqrt{\frac{6.7778}{3}} = 7.3651$$

Table 2 presents the sample mean differences between the paint brands. We summarize our findings with the following underscoring pattern. (Note that I subtracted off 400 from each observation to simplify the computing!)

i	4	3	2	1
$\bar{X}_{i.}$	41.33	41.67	45.67	50.33

Although there is not a significant difference between paint brands 1 and 2, we can see *paint brand 1* appears to be the preferable paint brand.

Table 2: $\bar{X}_{k.} - \bar{X}_{i.}$ for $k > i$. The bold-faced elements correspond to the values less than W.

$i \setminus k$	1	2	3	4
1		**4.6667**	8.6667	9
2			**4**	**4.3333**
3				**0.3333**
4				

Section 11.2 Question 16 /14

Study Objective: Determine if there exists an effect of *curing time* and *mixture type* on the comprehensive strength of hardened cement cubes

Formulation: Let

- X_{ijk} denote the kth observation for the ith curing time and jth mixture type, with $i = 1, 2, 3$, $j = 1, 2, 3, 4$, and $k = 1, 2, 3$ (ie $I = 3$, $J = 4$, and $K = 3$).

We assume that $X_{ijk} \sim N(\mu_{ij}, \sigma^2)$, where $\mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij}$, with $\sum_{i=1}^{3} \alpha_i = 0$ $\sum_{j=1}^{4} \beta_j = 0$, $\sum_{i=1}^{3} \gamma_{ij} = 0$ for each j, and $\sum_{j=1}^{4} \gamma_{ij}$ for each i.

Part A /6

We are to fill out the following ANOVA table:

Source	df	Sum of Squares	Mean Square	F
Curing Time (Factor A)	$I - 1$	SSA	MSA	F_A
Mixture Type (Factor B)	$J - 1$	SSB	MSB	F_B
Interaction	$(I - 1)(J - 1)$	$SSAB$	$MSAB$	F_{AB}
Error	$IJ(K - 1)$	SSE	MSE	
Total	$IJK - 1$	SST		

- degrees of freedom for curing time is $I - 1 = 2$
- degrees of freedom for mixture type is $J - 1 = 3$
- degrees of freedom for the curing time and mixture type interaction is $(I-1)(J-1) = 6$
- degrees of freedom for error is $IJ(K - 1) = 24$
- total degrees of freedom is $IJK - 1 = 35$
- $SSA = 30763.0$ (given to us)
- $SSB = 34185.6$ (given to us)
- $SSE = 97436.8$ (given to us)
- $SST = 205966.6$ (given to us)
- $SSAB = SST - SSA - SSB - SSE = 43581.2$
- $MSA = SSA/(I - 1) = 15381.5$
- $MSB = SSB/(J - 1) = 11395.2$
- $MSAB = SSAB/((I - 1)(J - 1)) = 7263.533$
- $MSE = SSE/(IJ(K - 1)) = 4059.867$
- $F_A = MSA/MSE \approx 3.79$
- $F_B = MSB/MSE \approx 2.81$
- $F_{AB} = MSAB/MSE \approx 1.79$

Part B /2

Hypothesis Test: $H_{0AB} : \gamma_{ij} = 0$ for all i, j vs. H_{aA}: At least one $\gamma_{ij} \neq 0$

Test Statistic:

$$F_{AB} = \frac{MSAB}{MSE} \sim F(6, 24).$$

From **Part A**, we have $F_{AB,obs} \approx 1.79$

Method 1 - p-value: $P_{H_0}(F > F_{AB,obs}) = 0.1440$, where $F \sim F(6, 24)$.
Since $0.1440 > 0.05$, we fail to reject H_{0AB}.

Method 2 - Rejection Region:

$$\begin{aligned}\mathcal{R}_{AB,0.05} &= \{f : f > f_{0.05}(6, 24)\} \\ &= \{f : f > 2.51\}.\end{aligned}$$

Since $F_{AB,obs} \notin \mathcal{R}_{AB,0.05}$, we fail to reject H_{0AB}.

Part C /2

Hypothesis Test: $H_{0A} : \alpha_1 = \alpha_2 = \alpha_3 = 0$ vs. H_{aA}: At least one $\alpha_i \neq 0$

Test Statistic:

$$F_A = \frac{MSA}{MSE} \sim F(2, 24).$$

From **Part A**, we have $F_{A,obs} \approx 3.79$

Method 1 - p-value: $P_{H_0}(F > F_{A,obs}) = 0.0372$, where $F \sim F(2, 24)$.
Since $0.0372 < 0.05$, we reject H_{0A}.

Method 2 - Rejection Region:

$$\begin{aligned}\mathcal{R}_{A,0.05} &= \{f : f > f_{0.05}(2, 24)\} \\ &= \{f : f > 3.40\}.\end{aligned}$$

Since $F_{A,obs} \in \mathcal{R}_{A,0.05}$, we reject H_{0A}.

Part D /2

Hypothesis Test: $H_{0B} : \beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$ vs. H_{aB}: At least one $\beta_j \neq 0$

Test Statistic:

$$F_B = \frac{MSB}{MSE} \sim F(3, 24).$$

From **Part A**, we have $F_{B,obs} \approx 2.81$

Method 1 - p-value: $P_{H_0}(F > F_{B,obs}) = 0.0612$, where $F \sim F(3, 24)$.
Since $0.0612 > 0.05$, we fail to reject H_{0B}.

Method 2 - Rejection Region:

$$\begin{aligned}\mathcal{R}_{B,0.05} &= \{f : f > f_{0.05}(3,24)\} \\ &= \{f : f > 3.01\}.\end{aligned}$$

Since $F_{B,obs} \notin \mathcal{R}_{B,0.05}$, we fail to reject H_{0B}.

Part E /2

Note that we can apply Tukey's method, since we failed to reject H_{0AB} in **Part B**, and rejected H_{0A} in **Part C**.

We start by computing

$$W = Q_{0.05,I,IJ(K-1)}\sqrt{\frac{MSE}{JK}} = 3.53\sqrt{\frac{4059.867}{12}} = 64.9292$$

Table 3 presents the sample mean differences between the curing times. We summarize our findings with the following underscoring pattern

$$\begin{array}{cccc} k & 3 & 1 & 2 \\ \bar{X}_{k..} & \underline{3960.02 \quad 4010.88} & & 4029.10 \\ & & \underline{4010.88 \quad 4029.10} & \end{array}$$

Although there is not a significant difference between curing times 1 and 3, we can see a significant difference between curing times 3 and 2.

Table 3: $\bar{X}_{l..} - \bar{X}_{i..}$ for $l > i$. The bold-faced elements correspond to the values less than W.

$i \setminus l$	3	1	2
3		**50.86**	69.08
1			**18.22**
2			

Section 11.2 Question 22 /8

Study Objective: Determine if there exists a difference between the writing lifetimes of four premium brands of pens. However, it is believed that the writing surface might affect the writing lifetime.

Formulation: Let

- X_{ijk} denote the kth observation for the ith brand type and jth writing surface, with $i = 1, 2, 3, 4$, $j = 1, 2, 3$, and $k = 1, 2$ (ie $I = 4$, $J = 3$, and $K = 2$).

We assume that $X_{ijk} \sim N(\mu_{ij}, \sigma^2 + \sigma_B^2 + \sigma_G^2)$, where $\mu_{ij} = \mu + \alpha_i + B_j + G_{ij}$, with $\sum_{i=1}^{3} \alpha_i = 0$, $B_j \overset{iid}{\sim} N(0, \sigma_B^2)$, and $G_{ij} \overset{iid}{\sim} N(0, \sigma_G^2)$.

Hypothesis Test:

$$H_{0A} : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0 \text{ vs. } H_{aA} : \text{At least one } \alpha_i \neq 0$$
$$H_{0B} : \sigma_B^2 = 0 \text{ vs. } H_{aB} : \sigma_B^2 > 0$$
$$H_{0G} : \sigma_G^2 = 0 \text{ vs. } H_{aG} : \sigma_G^2 > 0$$

Note: It is customary to test H_{0A} and H_{0B} if we fail to reject H_{0G}. To conduct the hyptohesis tests, let's fill out the following ANOVA table:

Source	df	Sum of Squares	Mean Square	F
Pen Brand (Factor A)	$I - 1$	SSA	MSA	F_A
Writing Surface (Factor B)	$J - 1$	SSB	MSB	F_B
Interaction	$(I-1)(J-1)$	$SSAB$	$MSAB$	F_{AB}
Error	$IJ(K-1)$	SSE	MSE	
Total	$IJK - 1$	SST		

- degrees of freedom for pen brand is $I - 1 = 3$
- degrees of freedom for writing surface is $J - 1 = 2$
- degrees of freedom for the pen brand and writing surface interaction is $(I-1)(J-1) = 6$
- degrees of freedom for error is $IJ(K-1) = 12$
- total degrees of freedom is $IJK - 1 = 23$
- $SST = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(X_{ijk} - \bar{X}_{...})^2 = 20591.83$
- $SSE = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(X_{ijk} - \bar{X}_{ij.})^2 = 8216.0$
- $SSA = JK\sum_{i=1}^{I}(\bar{X}_{i..} - \bar{X}_{...})^2 = 1387.5$
- $SSB = IK\sum_{j=1}^{J}(\bar{X}_{.j.} - \bar{X}_{...})^2 = 2888.083$
- $SSAB = SST - SSA - SSB - SSE = 8100.25$

- $MSA = SSA/(I-1) = 462.5$
- $MSB = SSB/(J-1) = 1444.042$
- $MSAB = SSAB/((I-1)(J-1)) = 1350.042$
- $MSE = SSE/(IJ(K-1)) = 684.6667$
- $F_A = MSA/MSAB \approx 0.34$
- $F_B = MSB/MSAB \approx 1.07$
- $F_{AB} = MSAB/MSE \approx 1.97$

Test Statistics:

$$F_A = \frac{MSA}{MSAB} \sim F(I-1, (I-1)(J-1))$$
$$F_B = \frac{MSB}{MSAB} \sim F(J-1, (I-1)(J-1))$$
$$F_{AB} = \frac{MSAB}{MSE} \sim F((I-1)(J-1), IJ(K-1))$$

Decision:

$P_{H_{0G}}(F_{AB} > F_{AB,obs}) = 0.1492$, where $F_{AB} \sim F(6, 12)$. Since $0.05 < 0.1492$, we fail to reject H_{0G}.

$P_{H_{0A}}(F_A > F_{A,obs}) = 0.7960$, where $F_A \sim F(3, 6)$ since $0.05 < 0.7960$, we fail to reject H_{0A}.

$P_{H_{0B}}(F_B > F_{B,obs}) = 0.4006$, where $F_A \sim F(2, 6)$ since $0.05 < 0.4006$, we fail to reject H_{0B}.