What to do today (Jan 20, 2023)?

Part 2. Basic Statistical Inference (Chp 6-9)

§2.1 Point Estimation
§2.1.1 Some General Concepts
§2.1.2 Methods of Point Estimation

§2.2 Interval Estimation

§2.3 One-Sample Test

§2.4 Inference Based on Two-Samples

Reminder: Homework 2 is due on Monday 5:00pm.

Example 2.6 (textbook p271)

- **Study.** To find about my waiting time for a bus and 5 waiting times (in minutes) were recorded: 2.3, 3.7, 1.5, .4, 3.2
- **Statistical formulation.** The waiting time $X \sim U(0, \theta)$.

  To estm θ with a random sample $X_1, \ldots, X_5$?
- **MME.** Recall $E(X) = \mu_1(\theta) = \theta/2$, the population mean. The MME is the solution to $\bar{X} = \mu_1(\theta)$.

  Thus MME $\hat{\theta} = 2\bar{X}$, and $\hat{\theta}_{obs} = 4.44$.
- **MLE.** Recall $f(x;\theta) = 1/\theta$, for $0 \leq x \leq \theta$. The likelihood function of θ is

$$L(\theta) = \prod_{i=1}^{5} f(x_i;\theta) = \begin{cases} 1/\theta^5, & \text{if } \min(x_i) \geq 0, \max(x_i) \leq \theta \\ 0, & \text{otherwise} \end{cases}$$

  The MLE $\hat{\theta} = \max(X_i)$, and $\hat{\theta}_{obs} = 3.7$

Why to use MLE?

Invariance Property of MLE. If $\hat{\theta}$ is the MLE of θ, $\hat{\phi} = h(\hat{\theta})$ is the MLE of $\phi = h(\theta)$.

Large Sample Properties of MLE. With a random sample of size n, the MLE is $\hat{\theta}_n$.

- As $n \to \infty$, $E(\hat{\theta}_n) \to \theta$: approximately unbiased
- As $n \to \infty$, $\sigma_{\hat{\theta}_n} \to \sigma^* = \min \sigma_{\tilde{\theta}}$ with unbiased $\tilde{\theta}$
- The distribution of $\hat{\theta}_n$ is approximately $N(\theta, (\sigma^*)^2)$

Remarks. MLE is widely used, especially when $n >> 1$.

- Given the underlying population distribution, MLE is mechanically derived by calculus-based techniques.
- MLE is almost the best estimator that can be obtained.
- It is convenient to use to make statistical inference.

What have we studied on point estimation?

- *Some general concepts*: statistical literacy!

- *MME and MLE*: the two general procedures to obtain "good" estimators

  – *Study more examples from the textbook and* **practice**.

A point estimate of θ is sufficient? *Any alternative estimates?*

§2.2 Interval Estimation (Chp 7)

§2.2.1 Some General Concepts (Chp 7.1)

Goal: Suppose $X \sim F(\cdot;\theta)$ and $X_1, \ldots, X_n$ iid observations from the population. To obtain a 'good' interval estimator of θ.

Example 3.1 (p277-279)

- **Study.** To determine the true average preferred height for a keyboard. $n = 31$ trained typists were selected and their preferred height were recorded with $\bar{x} = 80.0$ cm.
- **Formulation.** The preferred height of a typist is r.v. $X \sim N(\mu, \sigma^2)$ with $\sigma = 2.0$ cm. What is μ?
- **Point estimation.**
  Both MME and MLE are $\bar{X} = \frac{1}{31}(X_1 + \ldots + X_{31})$, and $\bar{X}_{obs} = \bar{x} = 80.0$.

Suppose $X \sim F(\cdot;\theta)$ and $X_1, \ldots, X_n$ iid observations from the population. What is a 'good' interval estimator?

$100(1-\alpha)\%$ **confidence interval of** θ:

- **Definition**. $\hat{\theta}_L$ and $\hat{\theta}_U$ are two statistics. The random interval $(\hat{\theta}_L, \hat{\theta}_U)$ is a $100(1-\alpha)\%$ CI of θ is

  $$P\big(\theta \in (\hat{\theta}_L, \hat{\theta}_U)\big) = 100(1-\alpha)\%.$$

  Here, $(1-\alpha)$ is called the confidence level of the CI.
  - eg, $\alpha = 0.05$, a $100(1-\alpha)\%$ CI of θ is a CI with confidence level of 95%.
  - eg, in **Example 3.1**, $\mu = \theta$ and $\alpha = 5\%$:

    $$\begin{array}{cc} \text{lower limit } \hat{\theta}_L & \text{upper limit } \hat{\theta}_U \\ \Big(\bar{X} - Z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, & \bar{X} + Z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\Big) \end{array}$$

- **Interpretation**. (frequentist)
  With 100 experiments' outcomes, there're at least $100(1-\alpha)$ out of the 100 CI realizations containing the true value of θ.

  Bayesian interpretation: different!

- **Confidence Level, Precision, and Sample Size**.
  - $100(1-\alpha)\%$ CI $(\hat{\theta}_L, \hat{\theta}_U)$: the confidence level is $1-\alpha$.

    $$P\big(\theta \in (\hat{\theta}_L, \hat{\theta}_U)\big) = 1 - \alpha$$

  - Length (Width) of CI: $\hat{\theta}_U - \hat{\theta}_L$, about CI's **precision/accuracy**.

  - Often to determine the sample size n such that a $1-\alpha$ CI has a desired precision $\Rightarrow$ **Study Design** eg, recall **Example 3.1**, $(1-\alpha)$ CI of μ with $\sigma = 2$ in $N(\mu, \sigma^2)$:

$$\left(\bar{X} - Z_{1-\alpha/2}\frac{2}{\sqrt{n}}, \quad \bar{X} + Z_{1-\alpha/2}\frac{2}{\sqrt{n}}\right)$$

- If $1-\alpha = 95\%$, the CI is $\left(\bar{X} - 1.96\frac{2}{\sqrt{n}}, \bar{X} + 1.96\frac{2}{\sqrt{n}}\right)$ with length $2(1.96)(2/\sqrt{n})$.
- If n increases, the length decreases.
- If the desired length is A, the required sample size should be $\{(7.84)A\}^2$

What will we do next?

Part 1. Introduction and Review (Chp 1-5)

Part 2. Basic Statistical Inference (Chp 6-9)

2.1 Point Estimation

2.2 Confidence Interval

2.3 One-Sample Test

2.4 Inference Based on Two-Samples

Part 3. Important Topics in Statistics (Chp 10-13)

Part 4. Further Topics (Selected from Chp 14-16)

Homework 2 is due on Monday 5:00pm.