What to do today (Jan 27, 2023)?

Part 2. Basic Statistical Inference (Chp 6-9)

§2.1 Point Estimation (Chp 6)

§2.2 Interval Estimation (Chp 7)

§2.2.1 Some General Concepts

§2.2.2 Confidence Interval with Normal Population

§2.2.3 Approximate Confidence Intervals

§2.3 One-Sample Test (Chp 8)

§2.3.1 Introduction and Basic Concepts (Chp 8.1)

Some Logistics.

- Homework 3 is due on Monday 5:00pm.
- Midterm 1 is on Friday next week (Feb 3) 10:30-11:20. It will be open book, and cover Chp 6 and Chp 7.

§2.2.3. Approximate Confidence Intervals (Chp 7.2)

Framework.

- **Population**. $X \sim F(\cdot;\theta)$
- **Data**. a random sample $\{X_1, \ldots, X_n\}$ with $n \gg 1$ (*a large sample*)
- **Goal. to construct (approximate) CI of** θ

§2.2.3A When $\theta = \mu$:

- Recall $\bar{X}$, the "best" estimator of μ
- Recall the Central Limit Theorem (CLT),

$$\frac{\bar{X} - \mu}{\sqrt{\sigma^2/n}} \sim N(0,1) \text{ approximately as } n \gg 1$$

- Thus

$$P\Big(-Z_{1-\alpha/2} < \frac{\bar{X} - \mu}{\sqrt{\sigma^2/n}} < Z_{1-\alpha/2}\Big) \approx 1 - \alpha$$

An approximate $(1-\alpha)100\%$ CI of $\mu = \theta$:

$$\bar{X} \pm Z_{1-\alpha/2}\sqrt{\frac{\sigma^2}{n}}$$

Another approximate $(1-\alpha)100\%$ CI of μ:

$$\Big(\bar{X} - Z_{1-\alpha/2}\sqrt{\frac{S^2}{n}}, \quad \bar{X} + Z_{1-\alpha/2}\sqrt{\frac{S^2}{n}}\Big)$$

using S^2 to estimate σ^2 (since $S^2 \to \sigma^2$ as $n \to \infty$).

Example 3.5 (p290)

- **Study**. rate of ignition by a lighted cigarette
- **Data**. $n = 48$ trials in a laboratory: 16 resulted in ignition
- **Stats formulation**. r.v. $X \sim B(1, p)$; to estimate $p = P(X = 1)$ by a random sample with size of 48: $X_1, \ldots, X_{48}$.
  $E(X) = p$; $V(X) = p(1-p)$
- **Estimation of** p.
  - $\bar{X} = \sum X_i/48$ of $p = E(X)$: $\hat{p}_{obs} = 16/48$
  - Taking $48 \gg 1$, an approximate CI with level of $1 - \alpha$ can be derived from

$$P\Big(-z_{\alpha/2} < \frac{\bar{X} - p}{\sqrt{p(1-p)/n}} < z_{\alpha/2}\Big) \approx 1 - \alpha,$$

$$\Rightarrow \quad P\big((\hat{p}_L, \hat{p}_U) \ni p\big) \approx 1 - \alpha$$

with $\hat{p}_L, \hat{p}_U$ given by (7.10) in Chp7.2 of the textbook. (**the score type CI for** p). Using the data, the evaluation of the CI with $1 - \alpha = 95\%$ is $(.216, 474)$.

- **Estimation of p (cont'd).**
  - If estimating $V(X) = p(1-p)$ with $\bar{X}(1-\bar{X})$, a different approximate CI (**the Wald type CI for p**):

$$\bar{X} \pm Z_{1-\alpha/2}\sqrt{\frac{\bar{X}(1-\bar{X})}{n}}$$

    with the evaluation $(.200, .466)$
  - If estimating $V(X) = p(1-p)$ with $S^2 = n\bar{X}(1-\bar{X})/(n-1)$, another different approximate CI:

$$\bar{X} \pm Z_{1-\alpha/2}\sqrt{\frac{\bar{X}(1-\bar{X})}{n-1}}$$

    with the evaluation $(.198, .468)$.
- The CIs are wide. *Q: if to obtain a CI with length less than 10%, $n =$?*

Recall $p(1-p)$ is maximized at $p = .5$:
$n = 381$ by the score type; $n = 385$, by the Wald type.

§2.2.3B* A General Large Sample CI:

- **Approximate Normality of MLE $\hat{\theta}$.** If $n \gg 1$, $\hat{\theta} \sim N\big(\theta, [n\mathcal{I}(\theta)]^{-1}\big)$ approximately.

$$\text{Fisher Information: } \mathcal{I}(\theta) = E\Big[-\frac{\partial^2 \ln f(X;\theta)}{\partial \theta^2}\Big]$$

  if $f(\cdot;\theta)$ is the pdf/pmf.
- **Approximate** $(1-\alpha)100\%$ **CI:**

$$\hat{\theta} \pm Z_{1-\alpha/2}\sqrt{\frac{1}{n\mathcal{I}(\hat{\theta})}},$$

  because of

$$P\Big(-Z_{1-\alpha/2} < \frac{\hat{\theta}-\theta}{\sqrt{1/n\mathcal{I}(\theta)}} < Z_{1-\alpha/2}\Big) \approx 1-\alpha$$

  - In **Example 3.5**, $\hat{p} = \bar{X}$ is the MLE of p and $\mathcal{I}(p) = 1/p(1-p)$.
  - Applying the above to estimate $\theta = \mu$ when $F(\cdot;\theta) = N(\mu, \sigma^2)$, $\hat{\theta} = \bar{X}$ and $\mathcal{I}(\theta) = 1/\sigma^2$.

Q: consistent with before?

Q2: what if applying the above approach to estimate $\theta = \sigma^2$ when $F(\cdot;\theta) = N(0, \sigma^2)$?

Remarks:

- When $n \gg 1$ (large sample situations): to construct CI using MLE
- When $n \not\gg 1$: **Exact Confidence Interval**
  hard to compute!
  - In **Example 3.5**, $\bar{X} = \sum X_i / n$ and $Y = \sum X_i \sim B(n, p)$: $1 - \alpha$ CI is $(\hat{p}_L, \hat{p}_U)$ if

$$P\big(Y \le Y_{obs} | p = \hat{p}_U\big) = \alpha/2; P\big(Y \ge Y_{obs} | p = \hat{p}_L\big) = \alpha/2$$

- **Likelihood Interval**: with a constant c, such as 50%,

$$\big\{\theta : L(\theta)/\max L(\theta) > c\big\} = \big\{\theta : L(\theta)/L(\hat{\theta}) > c\big\}$$

  MLE $\hat{\theta}$ of θ
  – a potential good interval estimator

§2.3. One-Sample Tests of Hypotheses (Chp 8)

§2.3.1 Introduction and Basic Concepts (Chp 8.1)

Practical problems are often formulated into $X \sim F(\cdot;\theta)$ and *to make inferences on* θ

- *to estimate* θ
  $\Rightarrow$ "point/interval estimation"
- *to choose between two contradictory claims of* θ
  $\Rightarrow$ "hypothesis testing"
  e.g., to assess a new drug:
  - to evaluate a key quantity associated with it, or
  - to compare it with the standard drug – is it similar or better?

$\Longrightarrow$ **To Study Hypotheses Testing!**

What will we study next?

Part 1. Introduction and Review (Chp 1-5)

Part 2. Basic Statistical Inference (Chp 6-9)

2.1. Point Estimation (Chp 6)

2.2. Interval Estimation (Chp 7)

2.3. One-Sample Tests of Hypotheses (Chp 8)

2.3.1 Introduction and Basic Concepts (Chp 8.1)
2.3.2 Tests About Population Mean (Chp 8.2, 8.3)
2.3.3 Large Sample Tests (Chp 8.4)
2.3.4 Discussions (Chp 8.5)

2.4 Inference Based on Two-Samples

Part 3. Important Topics in Statistics (Chp 10-13)

Part 4. Further Topics (Selected from Chp 14-16)

Reminder: Midterm 1 (on Chp 6-7; open book; 10:30-11:20, Fri Feb 3)