# What to do today (Jan 31, 2023)?

## Part 2. Basic Statistical Inference (Chp 6-9)

*§2.1 Point Estimation (Chp 6)*

*§2.2 Interval Estimation (Chp 7)*

**§2.3 One-Sample Tests of Hypotheses (Chp 8)**

**§2.3.1 Introduction and Basic Concepts**

**§2.3.2 Tests about Population Mean**

*§2.3.3 Large Sample Tests*

*§2.3.4 Discussions*

*§2.4 Inference Based on Two-Samples*

**Some Logistics.**

- There is no homework this week.
- Midterm 1 is on Fri (Feb 3) 10:30-11:20. It will be open book, and cover Chp 6 and Chp 7.

# §2.3. One-Sample Tests of Hypotheses (Chp 8)

## §2.3.1 Introduction and Basic Concepts (Chp 8.1)

Practical problems are often formulated into $X \sim F(\cdot;\theta)$ and *to make inferences on* $\theta$

- *to estimate* $\theta$
  $\Rightarrow$ "point/interval estimation" (Chp 6,7)
- *to choose between two contradictory claims of* $\theta$
  $\Rightarrow$ "hypothesis testing" (Chp 8,9)

  e.g., to assess a new drug:
  - to evaluate a key quantity associated with it, or
  - to compare it with the standard drug – is it similar or better?

$\Longrightarrow$ **To Study Hypotheses Testing!**

**Statistical Hypotheses:**

- **null hypothesis.** ($H_0$)
  A claim is initially assumed to be true ("prior belief").
  eg, $H_0 : \theta = 0$
- **alternative hypothesis.** ($H_1$ or $H_a$)
  A contradictory statement to $H_0$ (big evidence needed to support it).
  eg, one-sided $H_1 : \theta > 0$; or two-sided $H_1 : \theta \neq 0$

**Test Procedure:**

To provide a rule, which is based on the sample data through a test statistic (say, $T$), to decide whether to reject $H_0$ or not.

*It can be presented by checking if $T_{obs}$ is in the "rejection region", a subset of all possible values of $T$.*

**Errors in Hypothesis Testing:**

- **type I error rate.** $\alpha = P\big(\text{the rule rejecting } H_0 \big| H_0\big)$
  - the significance level of the test, or the risk to take when rejecting $H_0$
  - usually to control it
- **type II error rate.** $\beta = P\big(\text{the rule accepting } H_0 \big| H_1\big)$
  - usually to make it as small as possible
  - $1 - \beta = P\big(\text{the rule rejecting } H_0 \big| H_1\big)$, the power (efficiency) of the test.

**Remarks:**

- $\alpha = .05$ *is often used*: to "protect" $H_0$ – the event of making type I error is a small probability event
- $1 - \beta = 80\%$ *is commonly used*: to achieve a satisfactory power

# 2.3.2 Tests about Population Mean

$H_0 : \mu = \mu_0$

**2.3.2A When Population is Normal:** $X \sim N(\mu, \sigma^2)$

**If $\sigma^2$ is known,**

- *test statistic.*

$$Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} \sim N(0, 1) \text{ under } H_0$$

- *rejection region.* with pre-determined $\alpha$

  (i) when $H_1 : \mu \neq \mu_0$: $\mathcal{R} = \left\{ z : |z| > z_{\alpha/2} \right\}$
  (ii) when $H_1 : \mu < \mu_0$: $\mathcal{R} = \left\{ z : z < -z_\alpha \right\}$
  (iii) when $H_1 : \mu > \mu_0$: $\mathcal{R} = \left\{ z : z > z_\alpha \right\}$

- *making decision.* Reject $H_0$ if $Z_{obs} \in \mathcal{R}$; otherwise, accept $H_0$.

**If $\sigma^2$ is unknown,**

- *test statistic.*

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t(n-1) \text{ under } H_0$$

  $S^2$ is the sample variance.

- *rejection region*, with pre-determined $\alpha$

  (i) when $H_1 : \mu \neq \mu_0$: $\mathcal{R} = \{t : |t| > t_{\alpha/2}(n-1)\}$
  (ii) when $H_1 : \mu < \mu_0$: $\mathcal{R} = \{t : t < -t_{\alpha}(n-1)\}$
  (iii) when $H_1 : \mu > \mu_0$: $\mathcal{R} = \{t : t > t_{\alpha}(n-1)\}$

- *making decision.* Reject $H_0$ if $T_{obs} \in \mathcal{R}$; otherwise, accept $H_0$.

**Example 4.2** (p331)

- **Study.** tread life of a tire; average value is 30K or larger with $\alpha = 0.01$?
- **Data.** $n = 16$ observations with $\bar{x}$; $\sigma = 1.5$K
- **Formulation.** $X \sim N(\mu, \sigma^2)$; a random sample of size $n = 16$; $H_0 : \mu = 30K$ vs $H_1 : \mu > 30K$

- **Testing.**
  - *test statistic.* $Z = \frac{\bar{X}-30000}{1500/\sqrt{16}} \sim N(0,1)$ under $H_0$
  - *rejection region.* a critical value $c$ is chosen such that $P(Z > c|H_0) = 0.01$: $c = z_\alpha = 2.33$; $\mathcal{R} = \{z : z > 2.33\}$.
  - *making decision.* $Z_{obs} = \frac{\bar{x}-30000}{1500/\sqrt{16}}$: if $Z_{obs} > 2.33$, reject $H_0$; otherwise, do not reject $H_0$.

## Moreover … …

- What if $\sigma$ is unknown and $s = 1500$? $(t_{0.01}(15) = 2.60)$
- *Prob of making type II error when* $\mu = 31K$*.*

$\beta = P\big(Z \notin \mathcal{R} | \mu = 31K\big) = P\big(Z < 2.33 | \mu = 31K\big)$ if $\sigma = 1500$.
Because $Z \sim N(\frac{31K - 30K}{1500/\sqrt{16}}, 1)$ when $\mu = 31$K,

$$\begin{aligned}\beta &= P\big(Z - \frac{31K - 30K}{1500/\sqrt{16}} < 2.33 - \frac{31K - 30K}{1500/\sqrt{16}} | \mu = 31K\big) \\ &= \Phi(2.33 - \frac{31K - 30K}{1500/\sqrt{16}}) = 0.3669,\end{aligned}$$

- *How large is n to make the test power above* 90%*?*

$1 - \beta \geq .9$ is equivalent to

$$0.1 \geq \Phi(2.33 - \frac{31K - 30K}{1500/\sqrt{n}});$$

thus $2.33 - \frac{31K - 30K}{1500/\sqrt{n}} \leq 1.28. \Rightarrow n \geq 29.32$. So, $n \geq 30$.
(*Design for a Study with Hypothesis Testing!*)

# 2.3.2B When Population is Bernoulli ... ...

**Setup.**

- $X \sim B(1, p)$
- $H_0 : \mu = \mu_0$ is then $H_0 : p = p_0$
- a random sample $\{X_1, \dots, X_n\}$

**Testing procedure.**

- **Step 1. to construct test statistic**.
  $Y = \sum_{i=1}^{n} X_i \sim B(n, p_0)$ under $H_0$

- **Step 2. to choose rejection region**. with desired $\alpha$

  (i) if $H_1 : p \neq p_0$: to choose $c_1, c_2$ such that
  $P(Y < c_1|H_0) = \alpha/2$, $P(Y > c_2|H_0) = \alpha/2$.
  $\Rightarrow \mathcal{R} = \{y : y < c_1 \text{ or } y > c_2\}$;

  (ii) if $H_1 : p < p_0$: to choose $c$ such that $P(Y < c|H_0) = \alpha$,
  $\Rightarrow \mathcal{R} = \{y : y < c\}$

  (iii) if $H_1 : p > p_0$: to choose $c$ such that $P(Y > c|H_0) = \alpha$,
  $\Rightarrow \mathcal{R} = \{y : y > c\}$;

- **Step 3. to make decision.**

  if $Y_{obs} \in \mathcal{R}$, reject $H_0$;
  if $Y_{obs} \notin \mathcal{R}$, do not reject $H_0$.

**Example 4.3** (p350)

- **Study.** whether fewer than 90% trash cans survive over 6 yrs; desired $\alpha = 0.05$
- **Data.** $n = 20$ observations: 14 cans survived $> 6$ yrs
- **Formulation.** $X \sim B(1, p)$ with $X = 1, 0$ for a can survives over 6 yrs or not; $H_0 : p = .9$ vs $H_1 : p < .9$
- **Testing.**
  - *test statistic:* $Y = \sum_{i=1}^{20} X_i \sim B(20, .9)$ under $H_0$
  - *rejection region:* to choose $c$ such that $P(Y < c|H_0) = 0.05$; from Table A.1, $c \in (15, 16)$ $\Rightarrow \mathcal{R} = \{y : y \leq 15\}$.
- **Making decision.** $14 \in \mathcal{R}$: reject $H_0$.

- **Moreover ... ...**

  $$\beta = P(Y \notin \mathcal{R} | p = 0.8) = P(Y > 15 | p = 0.8) = .63$$

  – *high type II error rate!* (or, low test power 37%)

- *HWQ: how large is n such that $\beta = 0.2$ when $p = 0.8$?*

## 2.3.2C When $n$ is large … …

**Recall,** if $n \gg 1$,

- By CLT, $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ approximately. Equivalently,

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

  approximately.

- Further, $\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$ approximately: $t(n-1) \approx N(0,1)$ now. Thus, approximately

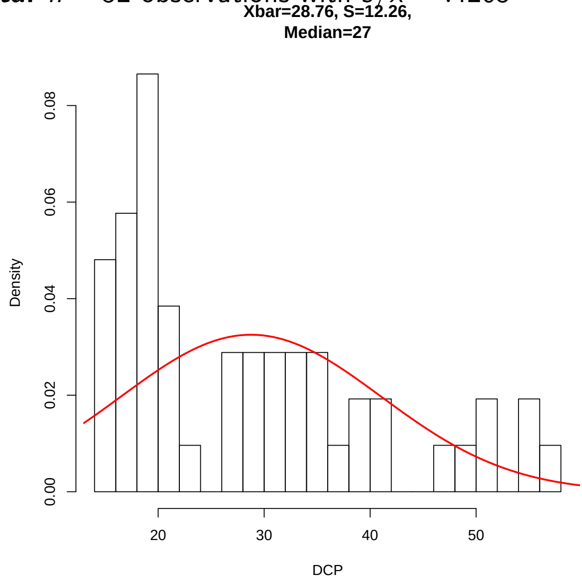
$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim N(0, 1).$$

$\Rightarrow$ **When** $n \gg 1$ **and for any population disn,**

- use $T = \frac{\bar{X}-\mu_0}{S/\sqrt{n}} \sim N(0,1)$ as the test statistic for $H_0 : \mu = \mu_0$
- use the procedure in **2.3.2A** to determine $\mathcal{R}$

**Example 4.4** (p332)

- **Study.** to check if the true average DCP of a type of pavement is below 30 as required.
- **Data.** $n = 52$ observations with $s/\bar{x} = .4265$



**Example 4.4** (p332)

- **Study.** to check if the true average DCP of a type of pavement is below 30 as required.
- **Data.** $n = 52$ observations with $s/\bar{x} = .4265$
- **Formulation.** $X \sim F(\cdot)$ with $E(X) = \mu$ but $F(\cdot)$ is not likely normal from the descriptive statistics
  $H_0 : \mu = 30$ vs $H_1 : \mu < 30$; taking $n = 52 \gg 1$
- **Testing.**
  *test statistic*: $Z = \frac{\bar{X} - 30}{S/\sqrt{52}} \sim N(0,1)$ under $H_0$
  *rejection region*: to choose $c$ such that $P(Z < c|H_0) = 0.05$;
  $c = -z_{0.05} = -1.645$
  $$\Rightarrow \mathcal{R} = \{z : z \leq -1.645\}.$$
- **Making decision.** $Z_{obs} = \frac{28.76 - 30}{12.26/\sqrt{52}} = -.73$:
  $Z_{obs} \notin \mathcal{R}$ and thus do not reject $H_0$.

# What will we study next?

*Part 1. Introduction and Review (Chp 1-5)*

**Part 2. Basic Statistical Inference (Chp 6-9)**

*§2.1. Point Estimation (Chp 6)*

*§2.2. Interval Estimation (Chp 7)*

**§2.3. One-Sample Tests of Hypotheses (Chp 8)**

*§2.3.1 Introduction and Basic Concepts (Chp 8.1)*
*§2.3.2 Tests About Population Mean (Chp 8.2, 8.3)*
**§2.3.3 Large Sample Tests (Chp 8.4)**
**§2.3.4 Discussions (Chp 8.5)**

Part 3. Important Topics in Statistics (Chp 10-13)

Part 4. Further Topics (Selected from Chp 14-16)

**Reminder: Midterm 1 (open book; 10:30-11:20 on Fri Feb 3, 2023)**