What to do today (Feb 7, 2023)?

Part 1. Introduction and Review (Chp 1-5)

Part 2. Basic Statistical Inference (Chp 6-9)

§2.1 Point Estimation (Chp 6)

§2.2 Interval Estimation (Chp 7)

§2.3 One-Sample Tests of Hypotheses (Chp 8)

§2.3.1 Introduction and Basic Concepts

§2.3.2 Tests about Population Mean

§2.3.3* Large Sample Tests

§2.3.4 Discussions

§2.4 Inference Based on Two-Samples

Part 3. Important Topics in Statistics (Chp 10-13)

Part 4. Further Topics (Selected from Chp 14-16)

Some Logistics.

- Homework 4 has been assigned. It's due on Monday next week.
- Midterm 1 has been marked, and exam papers were distributed after the tutorial yesterday.

§2.3.2 Tests about Population Mean (Chp 8.2 and 8.3)

$H_0 : \mu = \mu_0$

2.3.2A When Population is Normal (Chp 8.2): $X \sim N(\mu, \sigma^2)$

If σ^2 is known,

- *test statistic.*

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1) \text{ under } H_0$$

- *rejection region.* with pre-determined α

  (i) when $H_1 : \mu \neq \mu_0$: $\mathcal{R} = \left\{z : |z| > z_{\alpha/2}\right\}$
  (ii) when $H_1 : \mu < \mu_0$: $\mathcal{R} = \left\{z : z < -z_{\alpha}\right\}$
  (iii) when $H_1 : \mu > \mu_0$: $\mathcal{R} = \left\{z : z > z_{\alpha}\right\}$

- *making decision.* Reject H_0 if $Z_{obs} \in \mathcal{R}$; otherwise, accept H_0.

If σ^2 is unknown,

- *test statistic.*

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t(n-1) \text{ under } H_0$$

  S^2 is the sample variance.

- *rejection region*, with pre-determined α

  (i) when $H_1 : \mu \neq \mu_0$: $\mathcal{R} = \{t : |t| > t_{\alpha/2}(n-1)\}$
  (ii) when $H_1 : \mu < \mu_0$: $\mathcal{R} = \{t : t < -t_{\alpha}(n-1)\}$
  (iii) when $H_1 : \mu > \mu_0$: $\mathcal{R} = \{t : t > t_{\alpha}(n-1)\}$

- *making decision.* Reject H_0 if $T_{obs} \in \mathcal{R}$; otherwise, accept H_0.

2.3.2B When Population is Bernoulli (Chp 8.3)

Setup:

- $X \sim B(1, p)$
- $H_0 : \mu = \mu_0$ is then $H_0 : p = p_0$
- a random sample $\{X_1, \ldots, X_n\}$

Testing procedure:

- **Step 1. to construct test statistic.**
  $Y = \sum_{i=1}^{n} X_i \sim B(n, p_0)$ under H_0
- **Step 2. to choose rejection region.** with desired α

  (i) if $H_1 : p \neq p_0$: to choose c_1, c_2 such that
  $P(Y < c_1 | H_0) = \alpha/2$, $P(Y > c_2 | H_0) = \alpha/2$.
  $\Rightarrow \mathcal{R} = \{y : y < c_1 \text{ or } y > c_2\}$;
  (ii) if $H_1 : p < p_0$: to choose c such that $P(Y < c | H_0) = \alpha$,
  $\Rightarrow \mathcal{R} = \{y : y < c\}$
  (iii) if $H_1 : p > p_0$: to choose c such that $P(Y > c | H_0) = \alpha$,
  $\Rightarrow \mathcal{R} = \{y : y > c\}$;
- **Step 3. to make decision:**

  if $Y_{obs} \in \mathcal{R}$, reject H_0;
  if $Y_{obs} \notin \mathcal{R}$, do not reject H_0.

2.3.2C When n is large (Chp 8.2 and 8.3)

Recall, if $n \gg 1$,

- By CLT, $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ approximately. Equivalently,

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

  approximately.

- Further, $\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$ approximately: $t(n-1) \approx N(0,1)$ now. Thus, approximately

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim N(0,1).$$

$\Rightarrow$ **When** $n \gg 1$ **and for any population disn,**

- use $T = \frac{\bar{X}-\mu_0}{S/\sqrt{n}} \sim N(0,1)$ as the test statistic for $H_0 : \mu = \mu_0$
- use the procedure in **2.3.2A** to determine $\mathcal{R}$

§2.3.3* Large Sample Tests

Setup:

- *Population.* r.v. $X \sim F(\cdot;\theta)$
- *Data.* a random sample of size $n \gg 1$: $\{X_1, \ldots, X_n\}$
- *Goal.* to test on $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$ with desired α

2.3.3A Using MLE of θ: Z-Test (Wald-Test)

Recall that if $\hat{\theta}$ is the MLE, $\hat{\theta} \sim N(\theta, 1/n\mathcal{I}(\theta))$ approximately if $n \gg 1$.

test statistic: $Z = \frac{\hat{\theta}-\theta_0}{\sqrt{1/n\mathcal{I}(\theta_0)}} \sim N(0,1)$ approximately under H_0.

rejection region: ... ...

decision making: ... ...

2.3.3B Likelihood Ratio Testing (LRT)

Test statistic:
Consider

$$R(\theta_0) = \frac{L(\theta_0|X_1, \ldots, X_n)}{\max_\theta L(\theta|X_1, \ldots, X_n)} = \frac{L(\theta_0)}{L(\hat{\theta})}$$

- $0 \leq R(\theta_0) \leq 1$: the value of $R(\theta_0)$ tells how likely the population where the data are from has $\theta = \theta_0$, compared to the situation with the highest possibility
- Under $H_0 : \theta = \theta_0$, if $n \gg 1$, approximately

$$W = -2 \ln R(\theta_0) \sim \chi^2(1)$$

$\Longrightarrow R(\theta_0)$ a test statistic!

Rejection region: $\mathcal{R} = \{w : w > c\}$ with $P_{H_0}(W > c) = \alpha$

Example 4.5

- **Study.** wine bottles's contamination.
- **Data.** 16 of 91 bottles were contaminated
- **Formulation.** r.v. $X \sim B(1, p)$ with p the prob of a bottle is contaminated; to test $H_0 : p = .15$ vs $H_1 : p \neq .15$ using $\alpha = 0.1$
- **Testing.**

  (i) *Using $\hat{p}$'s approximate distribution* (Z-test)

$$L(p|\text{data}) = \prod_{i=1}^{91} p^{X_i}(1-p)^{1-X_i} = p^{\sum X_i}(1-p)^{91-\sum X_i}$$

$$\ln L(p) = \left(\sum X_i\right)\ln p + \left(91 - \sum X_i\right)\ln(1-p)$$

$\Rightarrow$ MLE of p: $\hat{p} = \sum X_i/91 \sim N(p, p(1-p)/91)$ approximately, taking $91 \gg 1$.

Test statistic: approximately under H_0,

$$Z = \frac{\hat{p} - .15}{\sqrt{.15(1-.15)/91}} \sim N(0,1)$$

Rejection region: with $\alpha = .1$, to choose c such that $P_{H_0}(|Z| > c) = .1$; $\mathcal{R} = \{z : |z| > z_{0.05} = 1.645\}$.
Making decision:

$$Z_{obs} = (16/91 - .15)/\sqrt{.15(1-.15)/91} = .69 \notin \mathcal{R}$$

and thus do not reject the null hypothesis.

(ii) *Likelihood ratio test:* $R(p_0) = L(0.15)/L(\hat{p})$
Test statistic:

$$W = -2\ln R(p_0) \sim \chi^2(1)$$

approximately under H_0.
Rejection region: with $\alpha = .1$, to choose c such that $P_{H_0}(W > c) = 0.1$; $c = 2.706$; $\mathcal{R} = \{w : w > c\}$.
Making decision: $W_{obs} = 0.455 \notin \mathcal{R}$ and do not reject H_0.

Remarks:

- Both Z-test (Wald-test) and likelihood ratio test (LRT) require a large sample in general:
  they are based on the approximate distributions of the test statistics under H_0 when $n \gg 1$.

- The score-test, a different likelihood-based testing procedure, requires a large sample, too.

- Relatively to the Z-test, the LRT requires smaller sample, and is more efficient.

- How to select a testing procedure?

§2.3.4 Discussion

2.3.4A. A duality between CI and hypothesis test

- Suppose $(\hat{\theta}_L, \hat{\theta}_U)$ is a CI of θ with level of $1 - \alpha$.

  $\Longrightarrow$ Reject $H_0 : \theta = \theta_0$ if $\theta_0 \notin (\hat{\theta}_L, \hat{\theta}_U)$.

  This testing has a type I error rate of α at most.

- Suppose a hypothesis test has the test statistic $T = T(\theta_0; \text{data})$ and the rejection region $\mathcal{R}$ with type I error rate of α.

  $\Longrightarrow$ The set of θ values, $\{\theta : T(\theta; \text{data}) \notin \mathcal{R}\}$, is a CI of θ with level at least $1 - \alpha$.

Example 4.6 $X \sim N(\mu, 1)$ with $X_1, \ldots, X_n$.

- Recall that a 95% CI of μ is $\bar{X} \pm \frac{1.96}{\sqrt{n}}$.
  $\Rightarrow$ to test $H_0 : \mu = 0$ vs $H_1 : \mu \neq 0$ is to check if $0 \in (\bar{X} - \frac{1.96}{\sqrt{n}}, \bar{X} + \frac{1.96}{\sqrt{n}})$.
  This testing's type I error rate is

$$P_{H_0}(0 \notin CI) = P_{H_0}(\bar{X} > \frac{1.96}{\sqrt{n}}) + P_{H_0}(\bar{X} < -\frac{1.96}{\sqrt{n}}) = 0.05$$

- Recall that the test on $H_0 : \mu = 0$ vs $H_1 : \mu \neq 0$ with test statistic $Z = \frac{\bar{X}-0}{1/\sqrt{n}}$ has the rejection region $\mathcal{R} = \{z : |z| > 1.96\}$ when $\alpha = 0.05$.
  $\Rightarrow$ to construct a 95% CI is to obtain $\{\mu : |\frac{\bar{X}-\mu}{1/\sqrt{n}}| \notin \mathcal{R}\}$, which is

$$\{\mu : |\frac{\bar{X} - \mu}{1/\sqrt{n}}| \leq 1.96\} = (\bar{X} - \frac{1.96}{\sqrt{n}}, \bar{X} + \frac{1.96}{\sqrt{n}}).$$

- If $n = 100$ and $\bar{x} = -0.2$: 95% CI of μ is $(-0.396, -0.004)$; $Z_{obs} = -2 \in \mathcal{R} = \{z : |z| > 1.96\}$.

2.3.4B Significance test

- *making inference vs making decision*

  practical consideration?

- *significance test:*

  (a) to calculate the p-value – the probability of obtaining a test statistic at least as extreme as the one that was actually observed under H_0

  (b) if the p-value is small (usually using ≤ 0.05), conclude that there is strong evidence against H_0; otherwise, conclude that there is no strong evidence against H_0

Comparison with decision making: difference and connection

- Given a test on H_0 vs H_1 with T, $\mathcal{R}$ with rate of type I error (significance level) α: reject or fail to reject H_0 based on $T_{obs} \in \mathcal{R}$ or $T_{obs} \notin \mathcal{R}$, respectively.
  $\Rightarrow$ The p-value of the significance test using the same test statistic and data must be not larger than α or larger than α, respectively.

- Given a significance test on H_0 vs H_1 using T with p-value calculated: conclude there is strong evidence against H_0 if p-value $< \alpha$
  $\Rightarrow$ Then $T_{obs} \in \mathcal{R}$: $\mathcal{R}$ is constructed using T and α in the corresponding hypothesis test.

Example 4.6 (cont'd)

$X \sim N(\mu, 1)$ with $X_1, \ldots, X_{100}$.
To test on $H_0 : \mu = 0$ vs $H_1 : \mu < 0$ with $\alpha = 0.05$ (rate of type I error or significance level), when $\bar{x} = -0.2$.

- Recall that $Z = (\bar{X} - 0)/\frac{1}{\sqrt{n}} \sim N(0, 1)$ under H_0.
- hypothesis testing:
  $\mathcal{R} = \{z : z < -z_{0.05}\} = \{z : z < -1.65\}$;
  $Z_{obs} = -0.2/(1/\sqrt{100}) = -2 \in \mathcal{R}$; reject H_0
- significance test:
  p-value$= P_{H_0}(Z < Z_{obs}) = 0.023 < 0.05$; the data show strong evidence against H_0.

What will we study next?

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§2.2. Interval Estimation (Chp 7)

§2.3. One-Sample Tests of Hypotheses (Chp 8)

§2.4. Two-Sample Tests of Hypotheses (Chp 9)

- **§2.4.1 Population Means with Normal Populations**
- **§2.4.2 Population Means Based on Large Sample**
- §2.4.3 Inferences on Two Population Variances

Part 3. Important Topics in Statistics (Chp 10-13)

Part 4. Further Topics (Selected from Chp 14-16)