What to study today (Sept 16, 2020)?

2.1 Probability (Chp1.1-4)

2.1.1 Introduction

2.1.2 Set Theory

2.1.3 Preliminaries of Probability

2.1.4 Conditional Probability and Independence

2.1.3 Preliminaries of Probability: Definition

Consider an experiment.

- Its *sample space* S is the set of all its possible outcome.
- Its associated *events* are the subsets of S from a σ-field $\mathcal{F}$ definied on S.
- **Axiom Definition of Probability:** (Kolmogorov, 1933) A probability measure P satisfies three axioms (*basic probability rules*):
  1. For any event $A \in \mathcal{F}$, $P(A) \geq 0$.
  2. $P(S) = 1$ for the sample space S.
  3. If $A_1, A_2, \ldots$ are disjoint (mutually exclusive), $P\big(\bigcup_{j=1}^{\infty} A_j\big) = \sum_{j=1}^{\infty} P(A_j)$.

A probability measure P is a measure defined on $\mathcal{F}$ with $P(S) = 1$.

2.1.3 Preliminaries of Probability: Definition

Probability Space.

A probability space is the triplet $(\Omega, \mathcal{F}, P)$:

- a sample space Ω, the set of all possible outcomes.
- a σ-field $\mathcal{F}$, a set of events containing zero or more outcomes.
- a set function P (probability measure), the assignment of probabilities to the events

e.g. A fair coin is flipped twice. What is the probability of getting at least one head?

2.1.3 Preliminaries of Probability: Basic Properties

Useful derivations from the Kolmogorov defn (*induced probability rules*):

- $P(\bar{A}) = 1 - P(A) \implies P(\emptyset) = 0$
- If $A \subseteq B$, $P(A) \leq P(B)$
- Inclusion Exclusion Formula.
  - $P(A \bigcup B) = P(A) + P(B) - P(AB)$
  - $P(A \bigcup B \bigcup C) = \big[P(A)+P(B)+P(C)\big] - \big[P(AB)+P(BC)+P(AC)\big] + P(ABC)$
  - $P(\bigcup_{k=1}^{K} A_k) = \sum_{k=1}^{K} P(A_k) - \sum_{k<l} P(A_k A_l) + \sum_{k<l<m} P(A_k A_l A_m) ...$

2.1.3 Preliminaries of Probability: Counting Rules

For $0 \le r \le n$,

- **permutation.** The number of *permutations* of r objects chosen from n distinct objects

$$n^{(r)} = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

Special Case: $r = n$

- **combination.** The number of *combinations* of r objects chosen from n distinct objects

$$\binom{n}{r} = \frac{n^{(r)}}{r!} = \frac{n!}{r!(n-r)!}$$

- $\binom{n}{r} = \binom{n}{n-r}$.
- **binomial coefficients.** $\binom{n}{k}$:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

2.1.3 Preliminaries of Probability: Further Properties.

Let $\{C_n\}_{n=1}^{\infty}$ be a sequence of events from $(\Omega, \mathcal{F}, P)$.

- If $C_n \subseteq C_{n+1}$ ($\{C_n\}$ is nondecreasing), denote $\lim_{n\to\infty} C_n = \bigcup_{n=1}^{\infty} C_n$, and then

$$\lim_{n\to\infty} P(C_n) = P(\lim_{n\to\infty} C_n) = P(\bigcup_{n=1}^{\infty} C_n).$$

- If $C_n \supseteq C_{n+1}$ ($\{C_n\}$ is nonincreasing), denote $\lim_{n\to\infty} C_n = \bigcap_{n=1}^{\infty} C_n$, and then

$$\lim_{n\to\infty} P(C_n) = P(\lim_{n\to\infty} C_n) = P(\bigcap_{n=1}^{\infty} C_n).$$

- Boole's Inequality. $P(\bigcup_{n=1}^{\infty} C_n) \leq \sum_{n=1}^{\infty} P(C_n)$

2.1.4 Conditional Probability and Independence: Conditional probability

When only those outcomes that are elements of a subset of the sample space S are of interest ...

Conditional Probability: (*Very important concept!*)

The conditional probability of A given B is

$$P(A \mid B) = \frac{P(AB)}{P(B)},$$

provided that $P(B) \neq 0$.

Comments:

- What if $P(B) = 0$? What if $A = B$?
- Often use $P(AB) = P(A \mid B)P(B) = P(B \mid A)P(A)$.
- $P(A \mid B) \neq P(B \mid A)$.

Example 2.4. The birthday problem. Among 30 people, what is the probability that at least two of them share a common birthday?

$P(B) = P(\geq 2 \text{ people share a birthday}) = 1 - P(\text{all 30 birthdays are diff})$

A_K = K people all with different birthdays.

2.1.4 Conditional Probability and Independence: Conditional probability

Consider K mutually exclusive and exhaustive events $C_1, \ldots, C_K$: C_k's are disjoint and $\bigcup_{j=1}^{K} C_j = S$.

Bayes' Theorem. (law of total probability)

$$P(C_j \mid A) = \frac{P(C_j A)}{P(A)} = \frac{P(A|C_j)P(C_j)}{\sum_{i=1}^{K} P(C_i)P(A|C_i)}$$

2.1.4 Conditional Probability and Independence: Independence

The simplest relationship of two events ...
Independence: (*Very important concept!*) Events A and B are independent if and only if (iff)

$$P(AB) = P(A)P(B).$$

Two events are independent if the occurrence or nonoccurrence of one event does not affect the probability of the other event: If $A \perp\!\!\!\perp B$, $P(A \mid B) = P(A)$.

Definition. Events $A_1, A_2, \ldots, A_K$ are mutually independent iff the probability of th eintersection of any 2, 3, ..., K of these events equals the product of their respective probabilities.

2.1.4 Conditional Probability and Independence: Independence

Example of pairwise independence but not mutual independence: Roll two dice and define the following

- $A_1 =$ first die is odd.
- $A_2 =$ second die is odd.
- $A_3 =$ sum of both dice is odd.

What will we study in the next class?

I. Introduction

2. Probability and Distributions

2.1 Probability (Chp1.1-4)
2.2 Random Variables and Distributions (Chp1.5-10)
2.3 Multivariate Distributions (Chp2)
2.4 Some Important Distributions (Chp3)

3. Essential Topics in Mathematical Statistics (Chp 4-6)

4. Further Topics, Selected from Chp 7-11