What to study today (Sept 21, 2020)?

2.1 Probability (Chp1.1-4)

- *2.1.1 Introduction*
- *2.1.2 Set Theory*
- *2.1.3 Preliminaries of Probability*
- **2.1.4 Conditional Probability and Independence**

2.2 Random Variable and Distribution (Chp1.5-10)

- **2.2.1 Basic Concepts**
- **2.2.2 Discrete Random Variable**
- **2.2.3 Continuous Random Variable**
- *2.2.4 Expectation and Related*

Probability Definition and Related

Consider an experiment with its *sample space* S and its associated *events*, subsets of S from a σ-field $\mathcal{F}$ definied on S.

Axiom Definition of Probability: (Kolmogorov, 1933)

A probability measure P satisfies three axioms (*basic probability rules*):

1. For any event $A \in \mathcal{F}$, $P(A) \geq 0$.
2. $P(S) = 1$ for the sample space S.
3. If $A_1, A_2, \ldots$ are disjoint (mutually exclusive), $P\big(\bigcup_{j=1}^{\infty} A_j\big) = \sum_{j=1}^{\infty} P(A_j)$.

Useful derivations from the Kolmogorov defn:

- $P(\bar{A}) = 1 - P(A) \Longrightarrow P(\emptyset) = 0$
- If $A \subseteq B$, $P(A) \leq P(B)$
- Inclusion Exclusion Formula.
  - $P(A \bigcup B) = P(A) + P(B) - P(AB)$
  - $P(\bigcup_{k=1}^{K} A_k) = \sum_{k=1}^{K} (-1)^k \big[\sum_{i_1 < \ldots < i_k} P(A_{i_1} \ldots A_{i_k})\big]$

2.1.4 Conditional Probability and Independence

Conditional Probability: (*Very important concept!*)
The conditional probability of A given B is

$$P(A \mid B) = \frac{P(AB)}{P(B)},$$

provided that $P(B) \neq 0$.

The simplest relationship of two events ...
Independence: (*Very important concept!*) Events A and B are independent if and only if (iff)

$$P(AB) = P(A)P(B).$$

Consider K mutually exclusive and exhaustive events $C_1, \ldots, C_K$: C_k's are disjoint and $\bigcup_{j=1}^{K} C_j = \mathcal{S}$.

Bayes' Theorem. (law of total probability)

$$P(C_j \mid A) = \frac{P(C_j A)}{P(A)} = \frac{P(A|C_j)P(C_j)}{\sum_{i=1}^{K} P(C_i)P(A|C_i)}$$

2.2 Random Variable and Distribution (Chp1.5-10): 2.2.1 Basic Concepts

Consider an experiment with its sample space S and the collection of its associated events $\mathcal{F}$.

Definition of Random Variable:

A random variable (rv) X is a real-valued function defined on S and with $\{c \in S : X(c) \in (a, b)\}$ being an event for any real interval $(a, b) \subseteq \mathcal{R}$.

For example ... ...

- A coin is flipped 10 times: let X be the number of heads.
- A student waits for a bus to school: let Y be the waiting time.
- The sum of two dice.
- The S&P/TSX closing price tomorrow.

2.2.1 Basic Concepts: Cumulative Distribution Function (cdf)

Consider a rv X: $S \longrightarrow \mathcal{R}$.
Definition. The function

$$F(x) = P\big(\{c \in S : X(c) \leq x\}\big), \quad \forall x \in \mathcal{R}$$

is called X's **cumulative distribution function** (cdf).

Often the presentation is shorten as $F(x) = P(X \leq x)$.

Remarks

- While a rv X is a *measurable* function defined on S, its distribution is associated with a particular probability space $(S, \mathcal{F}, P)$.

2.2.1 Basic Concepts: Cumulative Distribution Function (cdf)

Basic Properties. If $F(x)$ is the cdf of a rv X,

- $F(x)$ is nondecreasing.
- $F(x)$ is right-continuous. $(F(x_0+) = \lim_{x \downarrow x_0} F(x) = F(x_0))$
- $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$.

Examples of Application. If $F(x)$ is the cdf of X, then

- $P(a < X \leq b) = F(b) - F(a)$.
  - A special case: $P(X = x) = F(x) - F(x-)$.
- If $Y = \alpha + \beta X$ with $\beta > 0$, $P(Y \leq y) = F\big((y - \alpha)/\beta\big)$.
- If $Z = \log(X)$, $P(Z \leq z) = F\big(\exp(z)\big)$.

2.2.2 Discrete Random Variable: Definition

Definition. A rv X is **discrete**, if the set of all its possible values is countable (finite or countably infinite).

Definition. The **probability mass function** (pmf) of a discrete rv X is $p(x) = P(X = x)$ for $x = x_1, x_2, \ldots$.

Basic Properties. If $p(x)$ is the pmf of rv X,

- $p(x) \geq 0$ for $x = x_1, x_2, \ldots$.
- $\sum_{all\ x} p(x) = 1$.

For example ... ...

- Let X be the number of heads when flipping a fair coin K times: $p(x) = \binom{K}{x} \Big/ 2^K$ for $x = 0, 1, \ldots, K$.
- Toss a coin repeatedly till a head occurs. Let X be the number of trials: $p(x) = 1/2^x$ for $x = 1, 2, \ldots$

2.2.2 Discrete Random Variable: Connection between pmf and cdf

Let $F(x)$, $p(x)$ be the cdf, pmf of a discrete rv X with all possible values $x_1, x_2, \ldots$.

- $F(x)$ is a step function with jumps at most at $x_1, x_2, \ldots$: $F(x) = \sum_{all \ u \leq x} p(u)$.
- $p(x)$ is the jump size of the cdf: $p(x) = F(x) - F(x-)$.

For example, X is the number of heads when flipping a coin twice. What is its pmf? What is its cdf?

2.2.3 Continuous Random Variable: Definition

Definition. A rv X is **continuous**, if its cdf $F(x)$ is continuous for $x \in \mathcal{R}$.

Definition. The **probability density function** (pdf) of a continuous rv X with cdf $F(x)$ is $f(x) = dF(x)/dx$ for $x \in \mathcal{R}$ if $dF(x)/dx$ exists.

Basic Properties. If $f(x)$ is the pdf of rv X,

- $f(x) \geq 0$ for $x \in \mathcal{R}$.
- $\int_{-\infty}^{\infty} f(x) dx = 1$.

Further ... ...

- $F(x) = \int_{-\infty}^{x} f(u) du$.
- $P(a < X \leq b) = \int_{a}^{b} f(x) dx$.

2.2.3 Continuous Random Variable: Examples

- Let X be the waiting time (in minute) for a bus:

$$f(x) = \begin{cases} 1/10 & x \in (0, 10) \\ 0 & \text{otherwise.} \end{cases}$$

  - What is the probability with X between 3 to 8 min?
  - What is the cdf of X?

- Let the pdf of rv X be

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}), \ -\infty < x < \infty.$$

  - What is $P(X > 0)$?
  - What is $\int_{-\infty}^{\infty} xf(x)dx$?

2.2.3 Continuous Random Variable: Examples

Example 2.6 Assume r.v. $W \sim U(0,1)$, the continuous uniform distribution over $[0,1]$.

- Define a r.v. $X = aW + b$. What are the pdf and cdf of X if $a > 0$?
- Define another discrete r.v. $Y = \begin{cases} 0, & 0 \leq W < 1/2 \\ 1, & 1/2 \leq W \leq 1 \end{cases}$.

  What are the pmf and cdf of Y?

What will we study in the next class?

1. *Introduction*
2. **Probability and Distributions (Chp 1-3)**
   - *2.1 Probability (Chp1.1-4)*
   - **2.2 Random Variable and Distribution (Chp1.5-10)**
     - *2.2.1 Basic Concepts*
     - *2.2.2 Discrete Random Variable*
     - *2.2.3 Continuous Random Variable*
     - **2.2.4 Expectation and Related**

   2.3 Multivariate Distributions (Chp2)
   2.4 Some Important Distributions (Chp3)
3. *Essential Topics in Mathematical Statistics (Chp 4-6)*
4. *Further Topics, Selected from Chp 7-11*