What to study today (Sept 23, 2020)?

2. Probability and Distributions (Chp 1-3)

2.1 Probability (Chp1.1-4)

2.2 Random Variable and Distribution (Chp1.5-10)

- *2.2.1 Basic Concepts*
- *2.2.2 Discrete Random Variable*
- *2.2.3 Continuous Random Variable*
- **2.2.4 Expectation and Related**

2.3 Multivariate Distributions (Chp2)

2.4 Some Important Distributions (Chp3)

2.2.4 Expectation and Related: Definition

Definition. The **expectation** of rv X with cdf $F(x)$ is $E(X) = \int_{-\infty}^{\infty} x dF(x)$, provided the integral exists.

- If X is continuous with pdf $f(x)$, $dF(x) = f(x)dx$ and $E(X) = \int_{-\infty}^{\infty} x f(x) dx$.
- If X is discrete with pmf $p(x)$, $dF(x) = F(x) - F(x-) = p(x)$ and $E(X) = \sum_{all\ x} x p(x)$.

Comment. $E(X)$ is an average of X's all possible values (the **population mean**): the sum of X's all possible values weighted by how likely X taking the values.

2.2.4 Expectation and Related: Examples for Expectation

- What is $E(X)$ if X is the waiting time (in minute) for a bus with the following pdf?

$$f(x) = \begin{cases} 1/10 & x \in (0, 10) \\ 0 & \text{otherwise.} \end{cases}$$

- What is $E(X)$ if X is the number of heads from flipping a fair coin twice?

2.2.4 Expectation and Related: More Definitions

More generally ...

Definition. The **expectation** of rv $Y = g(X)$ with X's cdf $F(x)$ is $E(Y) = \int_{-\infty}^{\infty} g(x)dF(x)$, provided the integral exists.

- If X is continuous with pdf $f(x)$, $dF(x) = f(x)dx$ and $E(Y) = \int_{-\infty}^{\infty} g(x)f(x)dx$.
- If X is discrete with pmf $p(x)$, $dF(x) = F(x) - F(x-) = p(x)$ and $E(Y) = \sum_{all\ x} g(x)p(x)$.

Definition. The **variance** of rv X with cdf $F(x)$ and $\mu = E(X)$ is $Var(X) = E\left[(X-\mu)^2\right]$, provided the expectation exists.

Often people use $Var(X) = E(X^2) - \mu^2$ to calculate the variance.

2.2.4 Expectation and Related: Examples for Variance

- What is $Var(X)$ if X is the waiting time (in minute) for a bus with the following pdf?

$$f(x) = \begin{cases} 1/10 & x \in (0, 10) \\ 0 & \text{otherwise.} \end{cases}$$

- What is $Var(X)$ if X is the number of heads from flipping a fair coin twice?

2.2.4 Expectation and Related: Moment Generating Function

Definition. The **moment generating function** (mgf) of rv X is $M(t) = E(e^{tX})$ for $t \in (-h, h)$, provided the expectation exists.

- $M(0) = 1$; $M'(0) = \frac{dM(t)}{dt}\Big|_{t=0} = E(X)$; $M''(0) = \frac{d^2M(t)}{dt^2}\Big|_{t=0} = E(X^2)$.
- Two rvs X and Y have the same mgf iff their cdf's are the same.

Remarks. A related function: the **characteristic function** of rv X is $\phi(t) = E(e^{itX})$ with i the imaginary unit.

- $\phi(t)$ always exists for $t \in \mathcal{R}$.
- $\phi(-it) = M(t)$.

2.2.4 Expectation and Related: Further Properties

- The expectation operator E is linear: for constants k_1, k_2,

$$E\left[k_1 g_1(X) + k_2 g_2(X)\right] = k_1 E\left[g_1(X)\right] + k_2 E\left[g_2(X)\right]$$

  provided the expectations exist.

For example, if the pdf of rv X is $f(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})$ for $-\infty < x < \infty$, what is $E(3X - 2X^2)$? [-2]

2.2.4 Expectation and Related: Further Properties

- **Markov's Inequality** If $h(x)$ is a nonnegative function, for $a > 0$, $P\left[h(X) \geq a\right] \leq \frac{E\left[h(X)\right]}{a}$.
  - A special case: **Chebyshev's Inequality.** for constant k, $P\left[|X - \mu| \geq k\sigma\right] \leq \frac{1}{k^2}$ with $\mu = E(X)$ and $\sigma^2 = Var(X)$.

- **Jensen's Inequality** If $\phi(\cdot)$ is convex downward (or concave) on rv X's support, $\phi\left[E(X)\right] \leq E\left[\phi(X)\right]$.
  - A special case: $\left[E(X)\right]^2 \leq E(X^2)$.

What will we study in the next class?

1. *Introduction*

2. **Probability and Distributions (Chp 1-3)**

   - *2.1 Probability (Chp1.1-4)*
   - *2.2 Random Variable and Distribution (Chp1.5-10)*
   - **2.3 Multivariate Distributions (Chp2)**
     - **2.3.1 Basic Concepts on Two Random Variables**
     - **2.3.2 Conditional Distribution and Expectation**
     - *2.3.3 Extension to Several Random Variables*
   - *2.4 Some Important Distributions (Chp3)*

3. *Essential Topics in Mathematical Statistics (Chp 4-6)*

4. *Further Topics, Selected from Chp 7-11*