What to study today (Sept 30, 2020)?

2. Probability and Distribution (Chp 1-3)

2.1 Probability (Chp1.1-4)

2.2 Random Variable and Distribution (Chp1.5-10)

2.3 Multivariate Distribution (Chp2)

- *2.3.1 Basic Concepts with Two Random Variables*
- **2.3.2 Conditional Distribution and Expectation**
- **2.3.3 Extension to Several Random Variables**

2.4 Some Special Distributions (Chp3)

2.3.2 Conditional Distribution and Expectation

Definition. Consider rv (X_1, X_2).

- If X_1, X_2 are discrete with the joint pmf $p(x_1, x_2)$, and marginal pmfs $p_{X_1}(x_1)$ and $p_{X_2}(x_2)$, the **conditional pmf** of X_2 given $X_1 = x_1$ is

$$p_{X_2|X_1}(x_2|x_1) = P(X_2 = x_2|X_1 = x_1) = \frac{P(X_1 = x_1, X_2 = x_2)}{P(X_1 = x_1)}$$

provided $p_{X_1}(x_1) > 0$, and the **conditional pmf** of X_1 given $X_2 = x_2$ is

$$p_{X_1|X_2}(x_1|x_2) = \frac{p(x_1, x_2)}{p_{X_2}(x_2)}, \quad \textit{provided } p_{X_2}(x_2) > 0.$$

- If X_1, X_2 are continuous with the joint pdf $f(x_1, x_2)$, and marginal pdfs $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$, the **conditional pdf** of X_2 given $X_1 = x_1$ is

$$f_{X_2|X_1}(x_2|x_1) = \frac{f(x_1, x_2)}{f_{X_1}(x_1)},$$

provided $f_{X_1}(x_1) > 0$, and the **conditional pdf** of X_1 given $X_2 = x_2$ is

Example 2.8. (cont'd) Consider to roll a die after flipping an even coin: use a six-face die with face colors R,R,B,B,Y,Y if getting a head; otherwise, use a die with face colors R,R,R,B,B,B. Let X_1 be the number of head and X_2 be 1,2,3 for colors R,B,Y from the die.

- Are X_1 and X_2 independent?
- What is the connection of $p_{X_1}(x_1)$ and $p_{X_2}(x_2)$ to $p(x_1, x_2)$?
- What i the conditional pmf $p_{X_2|X_1}(x_2|x_1) = p(x_1, x_2)/p_{X_1}(x_1)$ with $x_1 = 1, 0$?
- What is conditional pmf $p_{X_1|X_2}(x_1|x_2) = p(x_1, x_2)/p_{X_2}(x_2)$ with $x_2 = 1, 2, 3$?

Example 2.9. (Cont'd) Consider two continuous rvs X_1, X_2 with joint pdf:

$$f(x_1, x_2) = \begin{cases} x_1 + x_2, & 0 < x_1, x_2 < 1 \\ 0, & \textit{otherwise} \end{cases}$$

- The connection between the joint pdf $f(x_1, x_2)$ and the marginal pdfs $f_{X_1}(\cdot)$ and $f_{X_2}(\cdot)$?
- Are X_1 and X_2 indpt?
- What are the conditional pdfs?

2.3.2 Conditional Distribution and Expectation

Consider two rvs X and Y.
Definition. The **conditional expectation** of Y given $X = x$ is

$$E(Y|X = x) = \int y dF_{Y|X}(y|x),$$

- $dF_{Y|X}(y|x) = f_{Y|X}(y|x)dy$ with $f_{Y|X}(y|x)dx$ the conditional pdf of Y given $X = x$ when Y is continuous;
- $dF_{Y|X}(y|x) = p_{Y|X}(y|x)$ with $p_{Y|X}(y|x)$ the conditional pmf of Y given $X = x$ when Y is discrete.

Properties. $E(Y|X)$ is a function of X and satisfies

- $E\left[E(Y|X)\right] = E(Y)$,
- $Var\left[E(Y|X)\right] \leq Var(Y)$.

Remark. $Var(Y) = Var\left[E(Y|X)\right] + E\left[Var(Y|X)\right]$.

Example 2.8. (cont'd) Consider to roll a die after flipping an even coin: use a six-face die with face colors R,R,B,B,Y,Y if getting a head; otherwise, use a die with face colors R,R,R,B,B,B. Let X_1 be the number of head and X_2 be 1,2,3 for colors R,B,Y from the die.
What are the conditional pmfs? What is $E(X_2|X_1)$? [2, 3/2 for x_1=1,0 while $E(X_2)$=7/4] What is $Var\left[E(X_2|X_1)\right]$? [1/16 while $Var(X_2)$=25/48]

Example 2.9. (Cont'd) Consider two continuous rvs X_1, X_2 with joint pdf:

$$f(x_1, x_2) = \begin{cases} x_1 + x_2, & 0 < x_1, x_2 < 1 \\ 0, & \textit{otherwise} \end{cases}$$

What are the conditional pdfs? What are $E(X_2|X_1)$ and $Var\left[E(X_2|X_1)\right]$?

2.3.3 Extension to Several Random Variables: General Issues

Consider K rvs $X_1, X_2, \ldots, X_K$:

Definition. The **joint cdf** of the random vector $(X_1, X_2, \ldots, X_K)$ is $F(x_1, x_2, \ldots, x_k) = P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_K \leq x_k)$ for $-\infty < x_1, x_2, \ldots, x_K < \infty$.

- In general $X_1 \sim F_{X_1}(x_1) = F(x_1, \infty, \ldots, \infty)$, $X_2 \sim F_{X_2}(x_2) = F(\infty, x_2, \ldots, \infty)$, etc.
- When $X_1, \ldots, X_K$ are discrete, the **joint pmf** of $(X_1, \ldots, X_K)$ is $p(x_1, \ldots, x_K) = P(X_1 = x_1, \ldots, X_K = x_K)$; when $X_1, \ldots, X_K$ are continuous, the **joint pdf** of $(X_1, \ldots, X_K)$ is $f(x_1, \ldots, x_K)$ such that $P\big((X_1, \ldots, X_K) \in A\big) = \int \ldots \int_A f(x_1, \ldots, x_K) dx_1 \ldots dx_K$ for $A \in \mathcal{R}^K$.
- K rvs $X_1, \ldots, X_K$ are **independent** iff $F(x_1, \ldots, x_K) = F_{X_1}(x_1) \ldots F_{X_K}(x_K)$ for $-\infty < x_1, \ldots, x_K < \infty$.
- The expectation of $Y = g(X_1, \ldots, X_K)$ is $E(Y) = \int g(x_1, \ldots, x_K) dF(x_1, \ldots, x_K)$.

2.3.3 Extension to Several Random Variables: Linear Combination

Consider linear combinations of rvs $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$: $T = \sum_{i=1}^{n} a_i X_i$ and $W = \sum_{j=1}^{m} b_j Y_j$.

- $E(T) = \sum_{i=1}^{n} a_i E(X_i)$; $E(Y) = \sum_{j=1}^{m} a_j E(Y_j)$
- $V(T) = \sum_{i=1}^{n} a_i^2 V(X_i) + 2\sum_{i<j} a_i a_j Cov(X_i, X_j)$
- $Cov(T, W) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j Cov(X_i, Y_j)$.

In the special case with rvs $X_1, \ldots, X_n$ indept and identically distributed (iid) and $E(X_i) = \mu$, $Var(X_i) = \sigma^2$,

- $E(T) = \left[\sum_{i=1}^{n} a_i\right]\mu$,
- $V(T) = \left[\sum_{i=1}^{n} a_i^2\right]\sigma^2$,
- If $X_1, \ldots, X_n$ are indpt of $Y_1, \ldots, Y_m$, $Cov(T, W) = 0$.

What will we study in the next class?

1. *Introduction*
2. **Probability and Distribution (Chp 1-3)**
   - *2.1 Probability (Chp1.1-4)*
   - *2.2 Random Variable and Distribution (Chp1.5-10)*
   - *2.3 Multivariate Distribution (Chp2)*
   - **2.4 Some Important Distributions (Chp3)**
3. *Essential Topics in Mathematical Statistics (Chp 4-6)*
4. *Further Topics, Selected from Chp 7-11*