What to do today (Nov 16, 2020)?

1. *Introduction*

2. *Probability and Distribution (Chp 1-3)*

3. **Essential Topics in Mathematical Statistics**

   3.1 Elementary Statistical Inferences (Chp 4)

   3.2 Consistency and Limiting Distributions (Chp 5)

   3.3 Maximum Likelihood Methods (Chp 6)

   - *3.3.1 Maximum Likelihood Estimation*
   - **3.3.2 Likelihood-Based Tests**
   - *3.3.3 EM Algorithm*

4. *Further Topics, Selected from Chp 7-11*

3.3.2 Likelihood-Based Tests:

Consider $H_0 : \theta \in \Omega_0$ vs $H_1 : \theta \in \Omega_1$ using a random sample $X_1, \ldots, X_n \sim f(x; \theta)$ for $\theta \in \Omega$ with $\Omega_0, \Omega_1 \subset \Omega$ and $\Omega_0 \bigcap \Omega_1 = \emptyset$.

- Here $\Omega \subseteq \mathcal{R}^K$ with K positive integer: $K = 1$ leads to θ a scalar.
- If $\Omega_0 = \{\theta_0\}$ (or $\Omega_1 = \{\theta_1\}$): a simple null (alternative) hypothesis; othewise, a composite null (alternative) hypothesis.
- In the case of $K = 1$, if $\Omega_0 = \{\theta_0\}$ and $\Omega_1 = \{\theta : \theta \neq \theta_0\} \Longrightarrow$ two-sided alternative.

Any systematic procedure for testing hypotheses on a general θ?

Consider likelihood-based testing procedures ...

3.3.2 Likelihood-Based Tests: Wald-Type Test

Consider $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$.

- If θ is scalar and the MLE $\widehat{\theta}$ exists,

$$Z = \frac{\widehat{\theta} - \theta_0}{\sqrt{AV(\widehat{\theta})}} \underset{H_0}{\sim} N(0,1)$$

approximately when $n >> 1$. $AV(\widehat{\theta}) = [n\text{FI}(\theta_0)]^{-1}$ is the *asymptotic variance* of $\widehat{\theta}$. (Assume the same regularity conditions for the asymptotic normality of MLE.) *the Z-test!*
$\Longrightarrow$ Rejection region with level of α: $\{\mathbf{x} : |Z| > c\}$, the critical value c is $Z_{1-\alpha/2}$ of the standard normal distn.

- If $K > 1$, $W = (\widehat{\theta} - \theta_0)' AV(\widehat{\theta})^{-1}(\widehat{\theta} - \theta_0)$ is

$$W = (\widehat{\theta} - \theta_0)' \left[n\text{FI}(\theta_0 \right] (\widehat{\theta} - \theta_0) \underset{H_0}{\sim} \chi^2(K)$$

approximately when $n >> 1$. (Assume the same regularity conditions for the asymptotic normality of MLE.) *the Wald test!*
$\Longrightarrow$ Rejection region with level of α: $\{\mathbf{x} : W > c\}$, the critical value c is $\chi^2(K)_{1-\alpha}$ of the $\chi^2(K)$-distn.

3.3.2 Likelihood-Based Tests: Wald-Type Test

Comments on Wald-Type Test:

- The construction of the test statistic is intuitive, and the procedure is straightforward.
- What if it is not easy to obtain $\widehat{\theta}$, $\mathsf{FI}(\theta)$?
- What if the null hypothesis is not a simple hypothesis?
- It requires $n >> 1$ in general: a *large sample* based procedure. What if the sample size is not large enough?

3.3.2 Likelihood-Based Tests: Likelihood Ratio Test (LRT)

Consider $H_0 : \theta \in \Omega_0$ vs $H_1 : \theta \notin \Omega_0$

$$\Lambda = \frac{\max_{\theta \in \Omega_0} L(\theta)}{\max_{\theta \in \Omega} L(\theta)}$$

Provided the same regularity conditions for the asymptotic normality of the MLE, $-2\log(\Lambda) \underset{H_0}{\sim} \chi^2(K-q)$ approximately when $n >> 1$, if the number of indpt parameters under H_0 is q.

$\Longrightarrow$ Rejection region with level of α: $\{\mathbf{x} : -2\log(\Lambda) > c\}$, the critical values c is $\chi^2(K-q)_{1-\alpha}$ of the $\chi^2(K-q)$-distn.

3.3.2 Likelihood-Based Tests: Likelihood Ratio Test (LRT)

Comments on LRT:

- The test statistic is also constructed intuitively.
- It is easy to carry out, easier than the Wald-type but has "no direction".
- It's also a large sample based procedure but requires relatively less sample size compared to the Wald-type.
- What if $H_1 : \theta \in \Omega_1$ with $\Omega_1 \subseteq \Omega_0^c$ but $\neq \Omega_0^c$?
- A 3rd likelihood based test: Score Test.

Example 3.12 Suppose $X_1, \ldots, X_n$ are iid with pdf $f(x;\theta) = \theta^{-1}\exp(-x/\theta)$ for $x, \theta > 0$. Test $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$.

Example 3.13 Suppose $X_1, \ldots, X_n$ are iid from $N(\mu, \sigma^2)$. Test $H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$.

What will we do next?

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