Stat 330 Assignment 1 Solutions

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1.2.7

a)

$$C_k = \left\{x : 2 - \frac{1}{k} < x \leq 2\right\} \; for \; k = 1, 2, 3, ...$$

We have $C_1 = \{x : 1 < x \leq 2\}, C_2 = \{x : \frac{3}{2} < x \leq 2\}, ...$

From this we can see that the right bound always contains 2 and that the left bound approaches 2 from below as $k \to \infty$. So $\{C_k\}$ is nonincreasing.

We thus have that

$$lim_{k \to \infty} C_k = \cap_{k=1}^{+\infty} C_k = \{x : x = 2\}$$

since the intersection of all the intervals contains 2.

Another way to see this is by first taking some $y > 2$. Clearly $y > 2$ is not in any C_k. If we now take $y < 2$:

$$2 - \frac{1}{k} \to 2 \; as \; k \to \infty$$

So for some l, $y < 2 - \frac{1}{l}$, so $y \notin (2 - \frac{1}{l}, 2]$.

So the intersection of all C_k must be $\{x : x = 2\}$.

b)

$$C_k = \left\{x : 2 < x \leq 2 + \frac{1}{k}\right\} \; for \; k = 1, 2, 3, ...$$

We have $C_1 = \{x : 2 < x \leq 3\}, C_2 = \{x : 2 < x \leq \frac{5}{2}\}, ...$

From this we can see that the right bound approaches 2 from above as $k \to \infty$ while the left bound is always greater than 2. So $\{C_k\}$ is nonincreasing.

We have that $lim_{k \to \infty} C_k = \cap_{k=1}^{\infty} C_k = \emptyset$. This is because there is no such set

$$C_k = \{x : 2 < x \leq 2\}$$

c)

$$C_k = \left\{(x, y) : 0 \leq x^2 + y^2 \leq \frac{1}{k}\right\} \; for \; k = 1, 2, 3, ...$$

We have $C_1 = \{(x, y) : 0 \leq x^2 + y^2 \leq 1\}, C_2 = \{(x, y) : 0 \leq x^2 + y^2 \leq \frac{1}{2}\}, ...$

From this we can see that the left bound always contains 0 and that the right bound approaches 0 from above as $k \to \infty$. $\{C_k\}$ represents a sequence of circles centred at the origin $((x, y) = (0, 0))$ whose radius decreases from 1 to 0 as $k \to \infty$. So $\{C_k\}$ is nonincreasing.

We also have that $lim_{k \to \infty} C_k = \{(x, y) : x = 0, y = 0\}$ because all C_k only contain

$$\{(x, y) : (x, y) = (0, 0)\}$$

1.2.11(b)

$Q(c) = 0$ since $0 < x = y < 1$ is a line, which has no area.

1.3.3

$$P(C) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = \sum_{x=1}^{\infty} (\frac{1}{2})^x = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

The probability P assigns to the elements of C_1 to the probabilities of $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$. Then

$$P(C_1) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32}$$

The probability P assigns the elements of C_2 to the probabilities of $\frac{1}{32}, \frac{1}{64}$. Then

$$P(C_2) = \frac{1}{32} + \frac{1}{64} = \frac{3}{64}$$

We have that $C_1 \cap C_2 = \{c : c\, is\, TTTTH\}$. Thus

$$P(C_1 \cap C_2) = \frac{1}{32}$$

Finally we have that $C_1 \cup C_2 = \{c : c\ is\ H, TH, TTH, TTTH, TTTTH, TTTTTH\}$. Then

$$P(C_1 \cup C_2) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} = \frac{63}{64}$$

1.3.7

1. $P(C_1 \cap C_2) \leq P(C_1)$

Proof: Since $(C_1 \cap C_2) \subseteq C_1$, and

$$P(C_1) = P[(C_1 \cap C_2) \cup (C_1 \cap C_2^c)] \overset{since\ they're\ mutually\ exclusive}{=} P(C_1 \cap C_2) + P(C_1 \cap C_2^c),$$

therefore

$$P(C_1) = P(C_1 \cap C_2)\ if\ and\ only\ if\ P(C_1 \cap C_2^c) = 0$$
$$so\ P(C_1) \geq P(C_1 \cap C_2).$$

2. $P(C_1) \leq P(C_1 \cup C_2)$

Proof: Since $C_1 \subseteq (C_1 \cup C_2)$, and

$$P(C_1 \cup C_2) = P[C_1 \cup (C_1^c \cap C_2)] \overset{since\ they're\ mutually\ exclusive}{=} P(C_1) + P(C_1^c \cap C_2),$$

thus

$$P(C_1 \cup C_2) = P(C_1)\ only\ when\ P(C_1^c \cap C_2) = 0$$
$$so\ P(C_1 \cup C_2) \geq P(C_1)$$

3. $P(C_1 \cup C_2) \leq P(C_1) + P(C_2)$

$$\because P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$$
$$\therefore P(C_1 \cup C_2) = P(C_1) + P(C_2)\ when\ P(C_1 \cap C_2) = 0$$
$$so\ P(C_1 \cup C_2) \leq P(C_1) + P(C_2)$$

From parts 1,2, and 3, we have shown that

$$P(C_1 \cap C_2) \leq P(C_1) \leq P(C_1 \cup C_2) \leq P(C_1) + P(C_2)$$

1.3.22

a)

If C_1, C_2, C_3 are mutually exclusive, then

$$P\left(\cup_{i=1}^{3} C_i\right) = \sum_{i=1}^{3} P(C_i) \leq 1$$

The restriction is that $0 \leq p_1 + p_2 + p_3 \leq 1$

b)

No, since $p_1 + p_2 + p_3 = \frac{12}{10} > 1$.

1.4.12

a)

Since C_1 and C_2 are independent,

$$P(C_1 \cap C_2) = P(C_1)P(C_2) = 0.6 \times 0.3 = 0.18$$

b)

Since C_1 and C_2 are independent,

$$P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2) = 0.6 + 0.3 - 0.18 = 0.72$$

c)

$$P(C_1 \cup C_2^c) = 1 - P[(C_1 \cup C_2^c)^c] = 1 - P(C_1^c \cap C_2) = 1 - (0.4)(0.3) = 0.88$$

1.4.25

Since all three events are mutually independent,

$$\begin{aligned} P[(C_1^c \cap C_2^c) \cup C_3] &= P(C_1^c \cap C_2^c) + P(C_3) - P[(C_1^c \cap C_2^c) \cap C_3] \\ &= P(C_1^c)P(C_2^c) + P(C_3) - P(C_1^c)P(C_2^c)P(C_3) \\ &= \left(\frac{3}{4}\right)\left(\frac{3}{4}\right) + \frac{1}{4} - \left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right) = \frac{43}{64} \end{aligned}$$

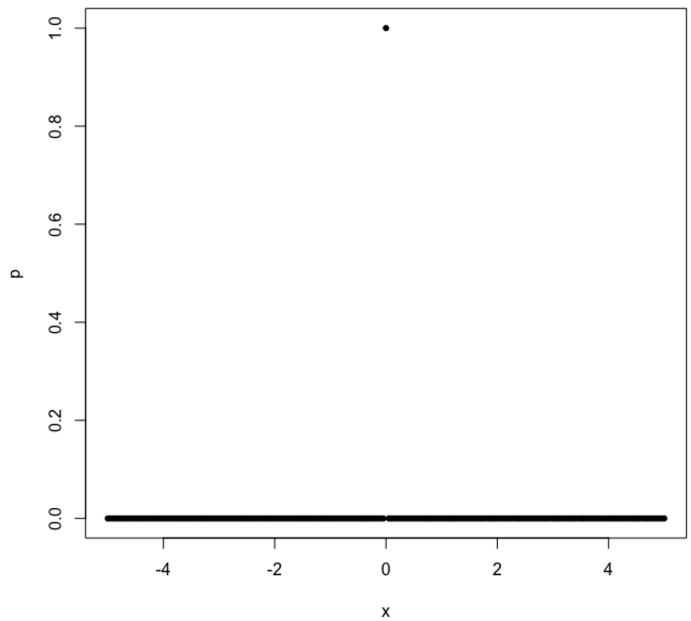
1.4.30

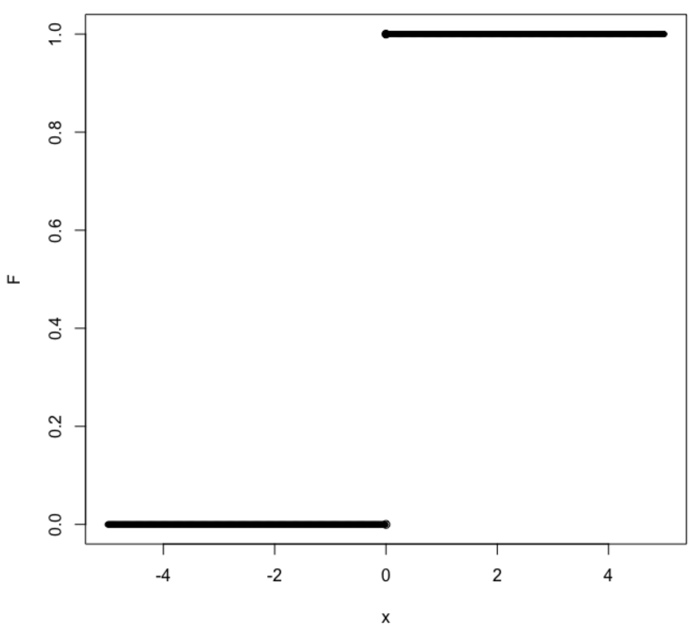
If the prize is behind curtain 1 and we condition that the contestant has chosen to switch, this means that they had either chosen curtain 2 first and Monte opened curtain 3, or that they had chosen curtain 3 first and Monte opened curtain 2. Monte could not have opened curtain 1 since the prize was behind it. In both cases, the contestant would win since they would switch to curtain 1. They would only lose if they chose curtain 1 first. Therefore the conditional probability of the contestant winning given that they switch is $\frac{2}{3}$. Similarly, the same conditional probability would be obtained if the prize was behind curtain 2 or curtain 3. Therefore the contestant should switch curtains.

1.5.4

a)

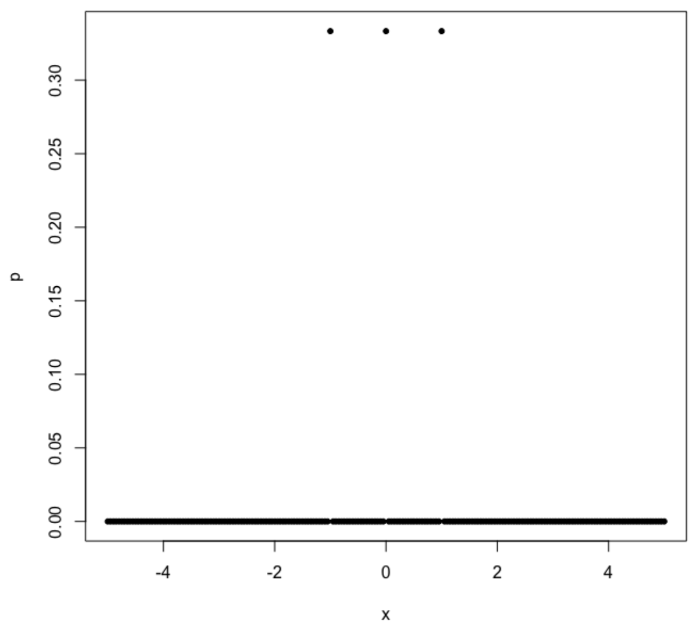
$$F(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

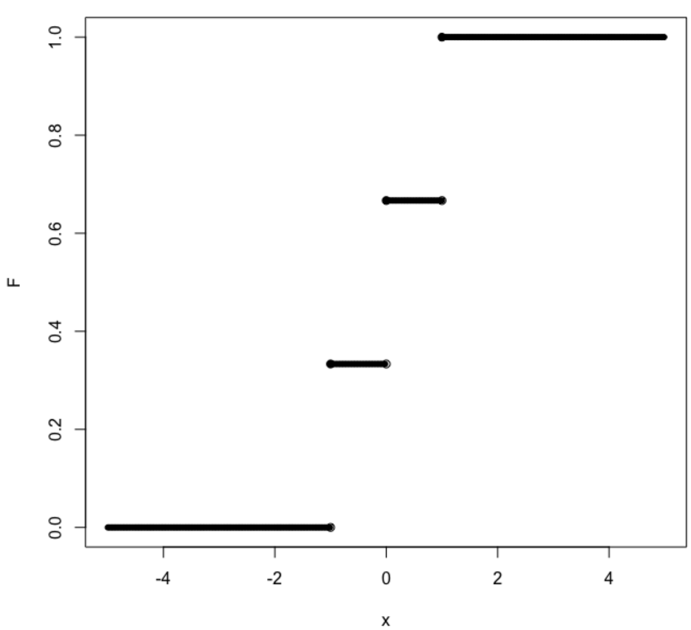




b)

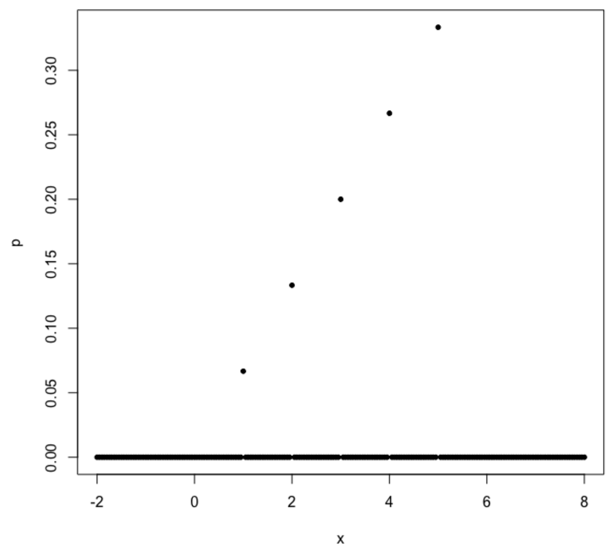
$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{3} & -1 \leq x < 0 \\ \frac{2}{3} & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

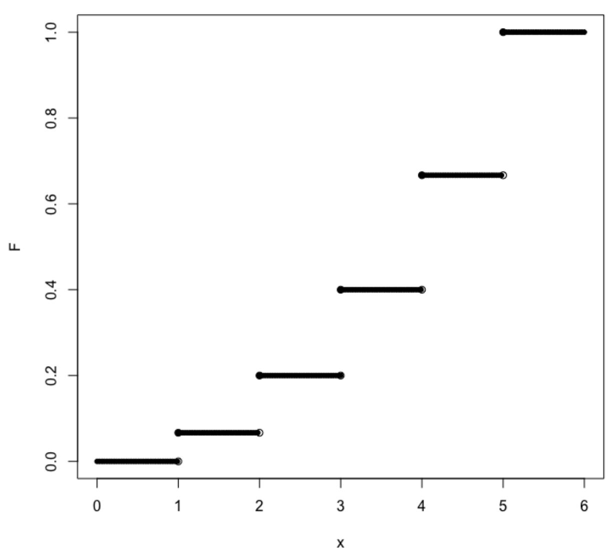




c)

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{15} & 1 \leq x < 2 \\ \frac{3}{15} & 2 \leq x < 3 \\ \frac{6}{15} & 3 \leq x < 4 \\ \frac{10}{15} & 4 \leq x < 5 \\ 1 & x \geq 5 \end{cases}$$

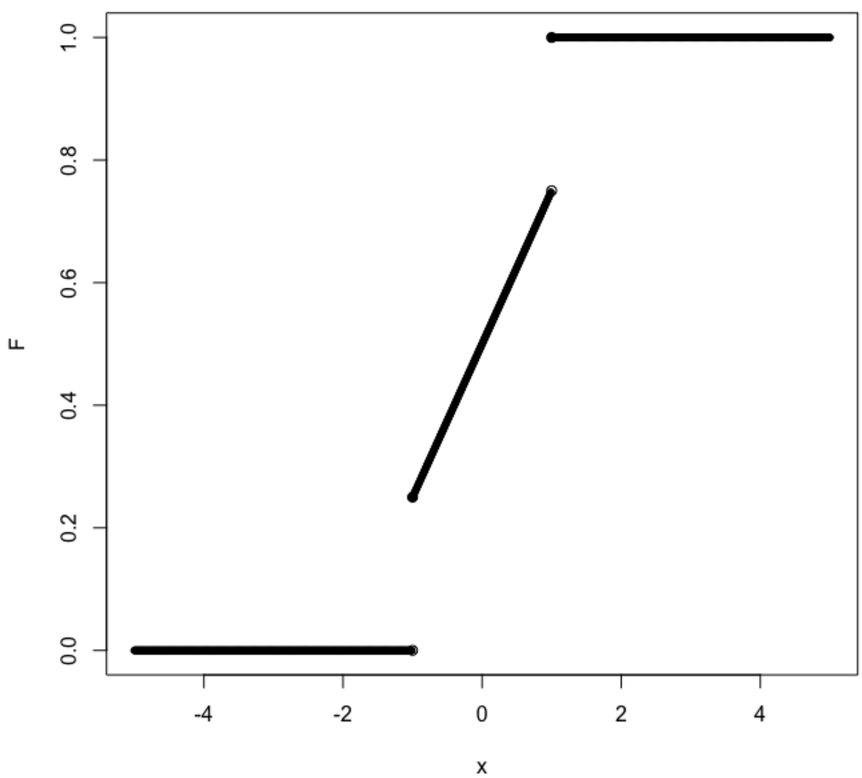




1.5.8

The plot could have either generated using R or drawn by hand (both are accepted). One possible way to plot F(x) by R is shown below:

```
x <- seq(-5, 5, 0.01)
F <- (x < -1) * 0 +
  (x >= 1) * 1 +
  (x >= -1 & x < 1) * (x/4+1/2)
plot(x, F, pch=20)
points(-1, y=1/4, pch=19)
points(1, y=1, pch=19)
points(-1, y=0, pch=21)
points(1,y=3/4,pch=21)
```



a)

$$P\left(-\frac{1}{2} < X \leq \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(-\frac{1}{2}\right) = \frac{5/2}{4} - \frac{3/2}{4} = \frac{1}{4}$$

b)

Since F(x) is continuous for $-1 < x < 1$,

$$P(X = 0) = F(0) - F(0^-) = 0$$

c)

$$P(X = 1) = F(1) - F(1^-) = 1 - \frac{3}{4} = \frac{1}{4}$$

d)

$$P(2 < X \leq 3) = F(3) - F(2) = 1 - 1 = 0$$