Stat 330 Assignment 2 Solutions

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1.6.9

We have

$$P(Y=1) = P(X^2=1) = P(X=1) + P(X=-1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$P(Y=0) = P(X^2=0) = P(X=0) = \frac{1}{3}$$

Therefore

$$p(y) = \begin{cases} \frac{1}{3} & y=0 \\ \frac{2}{3} & y=1 \\ 0 & otherwise \end{cases}$$

1.7.18

We want to find an m such that $P(X > m) = 0.05$. This means that

$$1 - P(X \leq m) = 0.05 \Rightarrow P(X \leq m) = 0.95$$

$$P(X \leq m) = \int_0^m \frac{12x(1000-x)^2}{10^{12}} dx$$

The integral is

$$\frac{6m^2 1000^2 - 4m^3 2000 + 3m^4}{10^{12}} = 0.95$$

Solving the equation above for m we get

$$m = 751.40 \ or \ m = -326.19$$

Since m needs to be non-negative, we conclude that the store should have 752 gallons of ice cream on hand every day.

1.8.6

a)

$$E(X^2) = \sum_{x:x=-1,0,1} x^2 p(x) = (-1)^2 p(-1) + 0(p(0)) + (1)^2 p(1) = p(-1) + p(1)$$

Since $p(-1) + p(0) + p(1) = 1$ and $p(0) = \frac{1}{4}$,

$$p(-1) + p(1) = 1 - \frac{1}{4} = \frac{3}{4}$$

Therefore $E(X^2) = \frac{3}{4}$.

b)

$$E(X) = -1(p(-1)) + 0(p(0)) + 1(p(1)) = p(1) - p(-1)$$

Since $p(-1) + p(0) + p(1) = 1$ and $p(0) = \frac{1}{4}$,

$$p(-1) + p(1) = 1 - \frac{1}{4} = \frac{3}{4}$$

Therefore, $p(-1) = \frac{3}{4} - p(1)$, so

$$E(X) = \frac{1}{4} = p(1) - \left(\frac{3}{4} - p(1)\right) \Rightarrow 1 = p(1) + p(1) \Rightarrow p(1) = \frac{1}{2}$$

From this and the fact that $p(-1) = \frac{3}{4} - p(1)$, we get

$$p(-1) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$$

1.9.7

$$M(t) = E(e^{tX}) = \int_{-1}^{2} \frac{e^{tx}}{3} dx = \left.\frac{e^{tx}}{3t}\right|_{-1}^{2} = \frac{e^{2t} - e^{-t}}{3t} \; when \; t \neq 0$$

When $t = 0$, by L'Hopital's Rule,

$$lim_{t \to 0} \frac{e^{2t} - e^{-t}}{3t} = lim_{t \to 0} \frac{2e^{2t} + e^{-t}}{3} = 1$$

Therefore

$$M(t) = \begin{cases} \frac{e^{2t} - e^{-t}}{3t} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

1.10.2

We know Markov's Inequality:

$$P(X \geq c) \leq \frac{E(X)}{c}$$

when X is a non-negative random variable, $E(X)$ exists, and c is a positive constant. These conditions hold for our problem. X is positive as $P(X \leq 0) = 0$, and $E(X) = \mu$ exists. If we take $c = 2\mu$, c is positive since $\mu = E(X)$ is positive (X is positive). Therefore, when we take $c = 2\mu$ and $E(X) = \mu$,

$$P(X \geq 2\mu) \leq \frac{\mu}{2\mu} = \frac{1}{2}$$

2.1.7

First, we can compute $P(Z \leq z)$:

$$F(z) = P(Z \leq z) = \int_0^z \int_0^{z-x} e^{-x-y} dy dx = 1 - e^{-z} - ze^{-z} \; for \; 0 < z < \infty$$

Using the above, we have

$$P(Z \leq 0) = F(0) = 0$$

$$P(Z \leq 6) = F(6) = 1 - e^{-6} - 6e^{-6} = 1 - 7e^{-6}$$

$$f(z) = \frac{d}{dx} F(z) = \begin{cases} ze^{-z} & 0 < z < \infty \\ 0 & otherwise \end{cases}$$

2.1.10

a)

X_1	X_2: 0	1	2	$p(x_1)$
0	2/12	3/12	2/12	7/12
1	2/12	2/12	1/12	5/12
$p(x_2)$	4/12	5/12	3/12	1

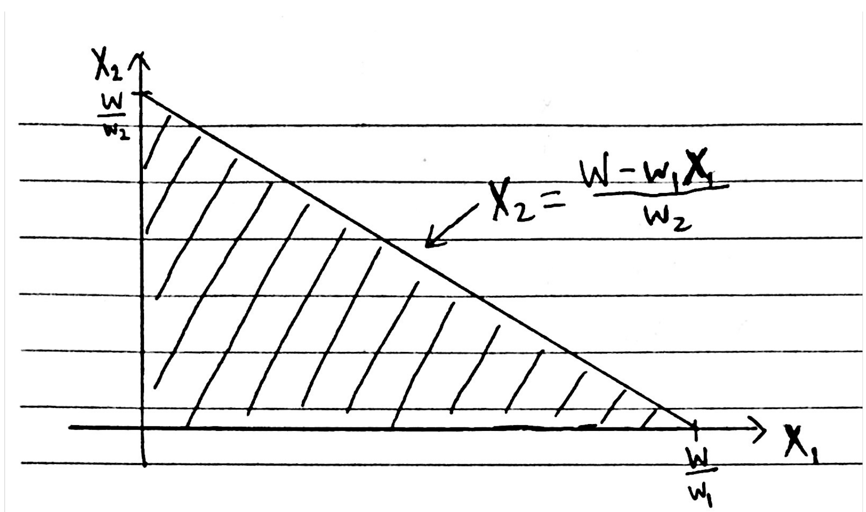
b)

$$P(X_1+X_2=1) \overset{\because X_1,X_2=0\ or\ 1}{=} P(X_1=1,X_2=0)+P(X_1=0,X_2=1)=\frac{2}{12}+\frac{3}{12}=\frac{5}{12}$$

2.2.8

a)

CDF method



$$F(w)=P(W\le w)=\int_0^{\frac{w}{w_1}}\int_0^{\frac{w-w_1x_1}{w_2}} e^{-x_1-x_2}dx_2dx_1$$

$$=1-e^{\frac{-w}{w_1}}+\frac{w_2}{w_2-w_1}e^{\frac{-w}{w_1}}-\frac{w_2}{w_2-w_1}e^{\frac{-w}{w_2}}=1-\frac{1}{w_2-w_1}(w_2e^{\frac{-w}{w_2}}-w_1e^{\frac{-w}{w_1}})\ for\ w>0$$

From this we get

$$f(w) = \frac{d}{dw}F(w) = -\frac{1}{w_2 - w_1}(-e^{\frac{-w}{w_2}} + e^{\frac{-w}{w_1}}) = \frac{1}{w_1 - w_2}(e^{\frac{-w}{w_1}} - e^{\frac{-w}{w_2}})\ for\ w > 0$$

Transformation method

Let the dummy variable $Z = w_1X_1$. Then $X_1 = \frac{Z}{w_1}$ and $X_2 = \frac{W-Z}{w_2}$. The Jacobian is

$$\mathbf{J} = \begin{bmatrix} 0 & \frac{1}{w_1} \\ \frac{1}{w_2} & \frac{-1}{w_2} \end{bmatrix} = \frac{-1}{w_1w_2}$$

Then we have

$$f(w,z) = f_{X_1X_2}\left(\frac{z}{w_1}, \frac{w-z}{w_2}\right)|J|$$

$$= \frac{e^{\frac{-z}{w_1} - \frac{(w-z)}{w_2}}}{w_1w_2}\quad for\quad \frac{z}{w_1} > 0 \to z > 0\ \ and\ \ \frac{w-z}{w_2} > 0 \to w > z$$

From this we get

$$f(w) = \int_0^w \frac{e^{\frac{-z}{w_1} - \frac{(w-z)}{w_2}}}{w_1w_2}dz = \frac{1}{w_1 - w_2}(e^{\frac{-w}{w_1}} - e^{\frac{-w}{w_2}})\ for\ w > 0$$

Therefore

$$f_W(w) = \begin{cases} \frac{1}{w_1-w_2}(e^{\frac{-w}{w_1}} - e^{\frac{-w}{w_2}}) & w > 0 \\ 0 & elsewhere \end{cases}$$

b)

Case 1: $w_1 > w_2$

We have that $\frac{1}{w_1-w_2} > 0$ and $e^{\frac{-w}{w_1}} - e^{\frac{-w}{w_2}} > 0$ so $f(w) > 0$.

Case 2: $w_1 < w_2$

We have that $\frac{1}{w_1-w_2} < 0$ and $e^{\frac{-w}{w_1}} - e^{\frac{-w}{w_2}} < 0$ so $f(w) > 0$.

Therefore $f(w) > 0$ when $w > 0$.

c)

Since $h = w_1 - w_2$, $w_2 = w_1 - h$, and when $w_1 = w_2$, $h = 0$, by L'Hopital's Rule

$$lim_{h\to 0}\frac{1}{h}(e^{\frac{-w}{w_1}} - e^{\frac{-w}{w_1-h}}) = lim_{h\to 0}\frac{\frac{w}{(w_1-h)^2}e^{\frac{-w}{w_1-h}}}{1} = \frac{w}{w_1^2}e^{\frac{-w}{w_1}}\ when\ w_1 = w_2$$

Therefore

$$f_W(w) = \begin{cases} \frac{w}{w_1^2}e^{\frac{-w}{w_1}} & w > 0 \\ 0 & elsewhere \end{cases}$$

2.3.6

a)

$$f(x) = \int_0^\infty \frac{2}{(1+x+y)^3}dy = \frac{1}{(1+x)^2}\ for\ x > 0$$

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{2(1+x)^2}{(1+y+x)^3}\ for\ x > 0, y > 0$$

b)

$$E(1 + X + Y|X = x) = 1 + x + E(Y|X = x)$$
$$= 1 + x + \int_0^\infty yf(x|y)dy = 1 + x + 1 + x = 2(1 + x)\ for\ x > 0$$

From this we have already calculated

$$E(Y|X = x) = 1 + x\ for\ x > 0$$

2.5.2

a)

$$\mu_1 = \frac{1}{15}(1(2) + 1(4) + 1(3) + 2(1) + 2(1) + 2(4)) = 1.4$$
$$\mu_2 = \frac{1}{15}(1(2) + 2(4) + 3(3) + 1(1) + 2(1) + 3(4)) = 2.2\bar{6}$$
$$\sigma_1^2 = \frac{1}{15}(1(2) + 1(4) + 1(3) + 2^2(1) + 2^2(1) + 2^2(4)) - \mu^2 = \frac{33}{15} - 1.4^2 = 0.24$$
$$\sigma_2^2 = \frac{1}{15}(1(2) + 2^2(4) + 3^2(3) + 1(1) + 2^2(1) + 3^2(4)) - \mu^2 = \frac{86}{15} - 2.27^2 = \frac{134}{225}$$

We also have

$$E(XY) = \frac{1}{15}(1(1)(2) + 1(2)(4) + 1(3)(3) + 2(1)(1) + 2(2)(1) + 2(3)(4)) = \frac{49}{15}$$

so

$$Cov(X, Y) = E(XY) - \mu_1\mu_2 = \frac{49}{15} - (1.4)(2.2\bar{6})$$

Therefore

$$\rho = \frac{Cov(X, Y)}{\sigma_1\sigma_2} = \frac{\frac{49}{15} - (1.4)(2.2\bar{6})}{\sqrt{0.24}\sqrt{\frac{134}{225}}} = 0.25$$

b)

We have that

$$p(x = 1) = \frac{1}{15}(2 + 4 + 3) = \frac{9}{15}$$
$$p(x = 2) = \frac{1}{15}(1 + 1 + 4) = \frac{6}{15}$$

so

$$E(Y|X = 1) = \sum_{y=1}^{3} yp(y|x = 1) = \sum_{y=1}^{3} \frac{yp(1, y)}{p(x = 1)} = \frac{19}{9}$$
$$E(Y|X = 2) = \sum_{y=1}^{3} yp(y|x = 2) = \sum_{y=1}^{3} \frac{yp(2, y)}{p(x = 2)} = \frac{15}{6}$$

The line

$$\mu_2 + \rho(\frac{\sigma_2}{\sigma_1})(x - \mu_1) = 2.2\bar{6} + 0.25\left(\frac{\sqrt{\frac{134}{225}}}{\sqrt{0.24}}\right)(x - 1.4)$$

Plugging in $x = 1$ and $x = 2$, we find that

$$E(Y|X=1) = 2.2\bar{6} + 0.25\left(\frac{\sqrt{\frac{134}{225}}}{\sqrt{0.24}}\right)(1 - 1.4)$$

$$E(Y|X=2) = 2.2\bar{6} + 0.25\left(\frac{\sqrt{\frac{134}{225}}}{\sqrt{0.24}}\right)(2 - 1.4)$$

Therefore the points $[k, E(Y|X = k)], k = 1, 2$ lie on this line.

2.6.3

$P(Y \leq y) = P(min(X_1, ..., X_4) \leq y) = 1 - P(min(X_1, ..., X_4) > y) = 1 - P(X_1 > y, ..., X_4 > y)$

Since X_1, X_2, X_3, X_4 are independent,

$$P(Y \leq y) = 1 - [1 - P(X_i \leq y)]^4 = 1 - [1 - F_{X_i}(y)]^4$$

We have that

$$F(x) = \int_0^x 3(1-x)^2 dx = 1 - (1-x)^3 \ for\ 0 < x < 1$$

so

$$F(y) = 1 - [1 - (1 - (1-y)^3)]^4 = 1 - (1-y)^{12} \ for\ 0 < y < 1$$

Therefore

$$F(y) = \begin{cases} 1 - (1-y)^{12} & 0 < y < 1 \\ 0 & elsewhere \end{cases}$$

From this we get

$$f(y) = \frac{d}{dy}F(y) = 12(1-y)^{11} \ for\ 0 < y < 1$$

Therefore

$$f(y) = \begin{cases} 12(1-y)^{11} & 0 < y < 1 \\ 0 & elsewhere \end{cases}$$

3.1.7

$P(X_1 = X_2) = P(X_1 = 0, X_2 = 0) + P(X_1 = 1, X_2 = 1) + P(X_1 = 2, X_2 = 2) + P(X_1 = 3, X_2 = 3)$

Since X_1 and X_2 are independent, this is equal to

$P(X_1 = 0)P(X_2 = 0) + P(X_1 = 1)P(X_2 = 1) + P(X_1 = 2)P(X_2 = 2) + P(X_1 = 3)P(X_2 = 3)$

We calculate the probabilities in the above formula using the binomial pmf to get

$$P(X_1 = X_2) = 0.0023 + 0.0556 + 0.1667 + 0.074 = 0.2986$$

3.2.17

$$M_Y(t) = e^{\mu(e^t - 1)} = E(e^{t(X_1 + X_2)}) = e^{\mu_1(e^t - 1)}M_{X_2}(t)$$

since X_1 and X_2 are independent. From this we get

$$M_{X_2}(t) = e^{(\mu - \mu_1)(e^t - 1)}$$

By the uniqueness of mgfs, $X_2 \sim Poisson(\mu - \mu_1)$.

3.4.17

Skewness

First, we have that

$$E(X^3) = \sum_{j=0}^{3} \binom{3}{j} \sigma^j E(Z^j)\mu^{3-j} = \mu^3 + 3\sigma\mu^2 E(Z) + 3\sigma^2 \mu E(Z^2) + \sigma^3 E(Z^3) = \mu^3 + 3\sigma^2\mu$$

So the measure of skewness is

$$\frac{E((X-\mu)^3)}{\sigma^3} = \frac{1}{\sigma^3} E(X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3) = \frac{1}{\sigma^3}(E(X^3) - 3\mu E(X^2) + 3\mu^2 E(X) - \mu^3)$$

$$= \frac{1}{\sigma^3}(\mu^3 + 3\sigma^2\mu - 3\mu(\mu^2+\sigma^2) + 2\mu^3) = 0$$

Kurtosis

First, we have that

$$E(X^4) = \sum_{j=0}^{4} \binom{4}{j} \sigma^j E(Z^j)\mu^{4-j} = \mu^4 + 4\sigma\mu^3 E(Z) + 6\sigma^2\mu^2 E(Z^2) + 4\sigma^3\mu E(Z^3) + \sigma^4 E(Z^4)$$

$$= \sigma^4 + 6\sigma^2\mu^2 + 3\sigma^4$$

So the measure of kurtosis is

$$\frac{E((X-\mu)^4)}{\sigma^4} = \frac{1}{\sigma^4} E(X^4 - 4\mu X^3 + 6\mu^2 X^2 - 4\mu^3 X + \mu^4)$$

$$= \frac{1}{\sigma^4}(E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 4\mu^3 E(X) + \mu^4)$$

$$= \frac{1}{\sigma^4}(\mu^4 + 6\sigma^2\mu^2 + 3\sigma^4 - 4\mu(\mu^3 + 3\mu\sigma^2) + 6\mu^2(\mu^2+\sigma^2) - 4\mu^4 + \mu^4) = 3$$

3.4.16 (7th edition of textbook)

$$P(exactly\ 2\ of\ 3\ random\ variables\ are\ < 0)$$

$$= P(X_1, X_2 < 0, X_3 > 0) + P(X_1, X_3 < 0, X_2 > 0) + P(X_2, X_3 < 0, X_1 > 0)$$

$$= P(X_1 < 0)P(X_2 < 0)P(X_3 < 0) + ... = \Phi(\frac{0-0}{1})\Phi(\frac{0-2}{\sqrt{4}})\Phi(\frac{0+1}{1}) + ... = 0.433$$

3.6.16

a)

We have that $X_1 = \frac{Y_1 Y_2}{1+Y_1}$ and $X_2 = \frac{Y_2}{1+Y_1}$.

Calculating the Jacobian, we get $J = \frac{Y_2}{(1+Y_1)^2}$. Then we have

$$f_{Y_1Y_2}(y_1,y_2) = |J| f_{X_1,X_2}(\frac{y_1y_2}{1+y_1}, \frac{y_2}{1+y_1}) = \frac{y_1^{\frac{r_1}{2}-1}}{(1+y_1)^{\frac{r_1+r_2}{2}}} \times \frac{y_2^{\frac{r_1+r_2}{2}-1} e^{\frac{-y_2}{2}}}{2^{\frac{1}{2}(r_1+r_2)}\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})} = f(y_1)\times f(y_2)$$

Since $f(y_1,y_2) = f(y_1)f(y_2)$, we have shown that Y_1 and Y_2 are independent. Also, since $f(y_2)$ is the pdf of $\chi^2_{(r_1+r_2)}$, we have that $Y_2 \sim \chi^2_{(r_1+r_2)}$.

b)

Let $W = \frac{X_1/r_1}{X_2/r_2}$. Since X_1 and X_2 are independent chi-square variables, $W \sim F(r_1, r_2)$.

Let $Z = \frac{X_3/r_3}{(X_1+X_2)/(r_1+r_2)}$. Since $X_1 + X_2 \sim \chi^2_{(r_1+r_2)}$, $Z \sim F(r_3, r_1 + r_2)$.

1

In part a), we have shown that $W = Y_1(\frac{r_2}{r_1})$ and $Y_2 = X_1 + X_2$ are independent. Therefore W and $\frac{1}{Y_2/(r_1+r_2)}$ are independent.

2

Since X_3 is independent of X_1 and X_2, W must be independent of X_3/r_3.

3

Combining (1) and (2), we find that W is independent of $\frac{X_3/r_3}{Y_2/(r_1+r_2)}$, which means that W and Z are independent.