Stat 330 Assignment 6 Solutions

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6.4.3

We have that

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} \Big\{ \frac{1}{\theta_2} e^{-(x_i-\theta_1)/\theta_2} \mathbb{1}(x_i \geq \theta_1) \Big\}$$

$$= \frac{1}{\theta_2^n} exp\Big\{ \frac{-1}{\theta_2} \sum_{i=1}^{n} (x_i - \theta_1) \Big\} \mathbb{1}(min(x_1, ..., x_n) \geq \theta_1)$$

$$= \frac{1}{\theta_2^n} exp\Big\{ \frac{-1}{\theta_2} \sum_{i=1}^{n} (x_i - \theta_1) \Big\} \mathbb{1}(x_{(1)} \geq \theta_1).$$

Thus θ_1's largest possible value is at $x_{(1)}$, and with a fixed θ_2^*, $L(\theta_1, \theta_2)$ is maximized at $\hat{\theta}_1 = x_{(1)}$ since $L(\theta_1, \theta_2)$ is a monotone increasing function of θ_1. To find $\hat{\theta}_2$, we have

$$l(\theta_1, \theta_2) = -nlog\theta_2 - \frac{1}{\theta_2} \sum_{i=1}^{n} (x_i - \theta_1)$$

$$\frac{\partial l}{\partial \theta_2} = \frac{-n}{\theta_2} + \frac{1}{\theta_2^2} \sum_{i=1}^{n} (x_i - \theta_1)$$

Setting this to 0, we get

$$\hat{\theta_2} = \frac{\sum_{i=1}^{n} (x_i - \hat{\theta}_1)}{n}$$

Since the mle of θ_1 is $x_{(1)}$, the mle of θ_2 is $\frac{\sum_{i=1}^{n}(x_i - x_{(1)})}{n}$.

6.5.4

The mle for θ_1 is $\bar{x}$ under H_0 and H_1. The mle for θ_2, after plugging in $\hat{\theta}_1$, is $\frac{\sum_{i=1}^{n}(x_i - \bar{x})}{n}$. Then we get the likelihood ratio

$$\Lambda = \frac{\frac{1}{(2\pi\theta_2')^{\frac{n}{2}}} exp(\frac{-n\hat{\theta}_2}{2\theta_2'})}{\frac{1}{(2\pi\hat{\theta}_2)^{\frac{n}{2}}} exp(\frac{-n\hat{\theta}_2}{2\hat{\theta}_2})} = \left(\sqrt{\frac{\hat{\theta}_2}{\theta_2'}} exp(\frac{-\hat{\theta}_2}{2\theta_2'}) \right)^n e^{\frac{n}{2}}.$$

We reject the null hypothesis when $\Lambda \leq c$ for some constant c. Since Λ is small when the value of $\frac{\hat{\theta}_2}{\theta_2'}$ is either very small or very large, it is also small when the value of $n\hat{\theta}_2 = \sum_{i=1}^{n}(x_i - \bar{x})^2$ is very small or very large. Therefore the test rejects the null hypothesis when $\sum_{i=1}^{n}(x_i - \bar{x})^2 \leq c_1$ or when $\sum_{i=1}^{n}(x_i - \bar{x})^2 \geq c_2$ for appropriately selected constants c_1 and c_2.

6.6.6

We have

$$L(\theta|\mathbf{x}) = [F(a-\theta)]^{n_2}\prod_{i=1}^{n_1} f(x_i-\theta)$$

$$L^c(\theta|\mathbf{x},\mathbf{z}) = \prod_{i=1}^{n_1} f(x_i-\theta)\prod_{i=1}^{n_2} f(z_i-\theta)$$

The conditional distribution $\mathbf{X}$ given $\mathbf{Z}$ is

$$k(\mathbf{z}|\theta,\mathbf{x}) = \frac{\prod_{i=1}^{n_1} f(x_i-\theta)\prod_{i=1}^{n_2} f(z_i-\theta)}{[F(a-\theta)]^{n_2}\prod_{i=1}^{n_1} f(x_i-\theta)} = [F(a-\theta)]^{-n_2}\prod_{i=1}^{n_2} f(z_i-\theta), \quad z_i < a \;\; i=1,...,n_2$$

Thus $\mathbf{Z}$ and $\mathbf{X}$ are independent and $Z_1,...,Z_{n_2}$ are iid with the common pdf $f(z-\theta)/F(a-\theta)$, for $z<a$. So

$$Q(\theta|\theta_0,\mathbf{x}) = E_{\theta_0}[logL^c(\theta|\mathbf{x},\mathbf{Z})] = E_{\theta_0}[\sum_1^{n_1} logf(x_i-\theta) + \sum_1^{n_2} logf(Z_i-\theta)]$$

$$= \sum_1^{n_1} logf(x_i-\theta) + n_2E_{\theta_0}[logf(Z-\theta)] = \sum_1^{n_1} logf(x_i-\theta) + n_2\int_{-\infty}^{a} logf(z-\theta)\frac{f(z-\theta_0)}{F(a-\theta_0)}dz$$

The last result is the E step of the EM algorithm. For the M step, we need the partial derivative of $Q(\theta|\theta_0,\mathbf{x})$ with respect to θ:

$$\frac{\partial Q}{\partial\theta} = -\{\sum_{i=1}^{n_1}\frac{f'(x_i-\theta)}{f(x_i-\theta)} + n_2\int_{-\infty}^{a}\frac{f'(x_i-\theta)}{f(x_i-\theta)}\frac{f(z-\theta_0)}{F(a-\theta_0)}dz\}$$

From example 6.6.1, we are given $f(x)=\phi(x)=(2\pi)^{-1/2}e^{-x^2/2}$ and $f'(x)/f(x)=-x$. So

$$\frac{\partial Q}{\partial\theta} = \sum_{i=1}^{n_1}(x_i-\theta) + n_2\int_{-\infty}^{a}(z-\theta)\frac{1}{\sqrt{2\pi}}\frac{exp\{-\frac{1}{2}(z-\theta_0)^2\}}{\Phi(a-\theta_0)}dz$$

$$= n_1(\bar{x}-\theta) + n_2\int_{-\infty}^{a}\frac{z-\theta_0}{\sqrt{2\pi}}\frac{e^{-\frac{1}{2}(z-\theta_0)^2}}{\Phi(a-\theta_0)}dz - n_2(\theta-\theta_0)$$

$$= n_1(\bar{x}-\theta) + \frac{n_2}{\Phi(a-\theta_0)}[-\Phi(a-\theta_0)] - n_2(\theta-\theta_0)$$

Setting the last equation to 0 and solving gives the M step estimates. In particular, given $\hat{\theta}^{(m)}$ is the EM estimate from the m^{th} step, the $(m+1)^{th}$ step estimate is

$$\hat{\theta}^{(m+1)} = (\frac{n_1}{n})\bar{x} + (\frac{n_2}{n})\hat{\theta}^{(m)} + \frac{(\frac{n_2}{n})(-\phi(a-\hat{\theta}^{(m)}))}{\Phi(a-\hat{\theta}^{(m)})}.$$

10.2.3

a)

Using R, the level of the test is

$$P_{H_0}(S\geq 16) = P(bin(25,0.5)\geq 16) = 1-P(bin(25,0.5)<15) = 0.1148$$

b)

The probability of success is

$$p = P(X > 0) = P(\frac{X - 0.5}{1} > \frac{0 - 0.5}{1}) = P(Z > -0.5) = P(Z < 0.5) = 0.6915$$

Therefore the power of the sign test is

$$P_{0.6915}(S \geq 16) = P(bin(25, 0.6915) \geq 16) = 1 - P(bin(25, 0.6915) < 15) = 0.7836.$$

c)

First, given $\sigma = 1$ and $n = 25$, we need to solve the following equation for k:

$$P_{H_0}[\overline{X}/(1/\sqrt{25}) \geq k] = P_{H_0}[\frac{\overline{X} - 0}{(1/\sqrt{25})} \geq k - 0] = P[Z \geq k] = 0.1148$$

where $Z \sim N(0, 1)$. Solving the equation, we get $k = 1.20$. Now, to determine the power of this test for the situation in part (b), we get

$$P_{\mu=0.5}[\overline{X}/(1/\sqrt{25}) \geq 1.20] = P_{H_0}[\frac{\overline{X} - 0.5}{(1/\sqrt{25})} \geq 1.20 - \frac{0.5}{(1/\sqrt{25})}] = P[Z \geq -1.30]$$

$$= P[Z < 1.30] = 0.9032.$$

10.3.2

a)

Assume that $\theta = 0$. Since X_i are symmetrically iid about 0,

$$\hat{\theta} = \hat{\theta}(X_1, ..., X_n) = \hat{\theta}(-X_1, ..., -X_n)$$

is iid. Let G(x) and g(x) be the cdf and pdf of $\hat{\theta}(X)$, respectively. From definition 10.1.1, $T(G_{\hat{\theta}}) = E(\hat{\theta})$ is a location functional. By theorem 10.1.1, since g(x) is symmetric about 0,

$$T(G_{\hat{\theta}}) = E(\hat{\theta}) = 0 = \theta.$$

Therefore $\hat{\theta}$ is an unbiased estimator of θ.

b)

Since $\hat{\theta}_i$ are iid, $\hat{\theta}_i^2$ are iid. Then by the Weak Law of Large numbers, with our assumption that the true $\theta = 0$,

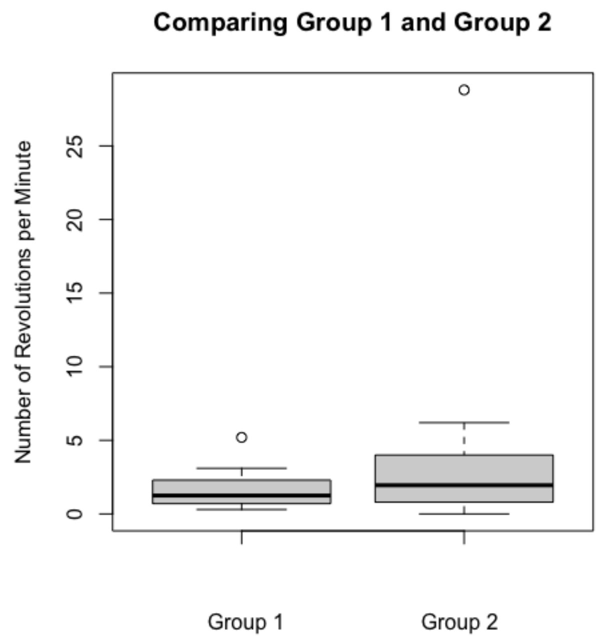
$$\frac{1}{n_s} \sum_{i=1}^{n_s} \hat{\theta}_i^2 \xrightarrow{\text{p}} E(\hat{\theta}^2) = V(\hat{\theta}) + (E(\hat{\theta}))^2 = V(\hat{\theta}) + \theta^2 = V(\hat{\theta}) + 0^2 = V(\hat{\theta})$$

since by part a), $\hat{\theta}$ is unbiased.

10.4.9

a)

We get the following boxplots:



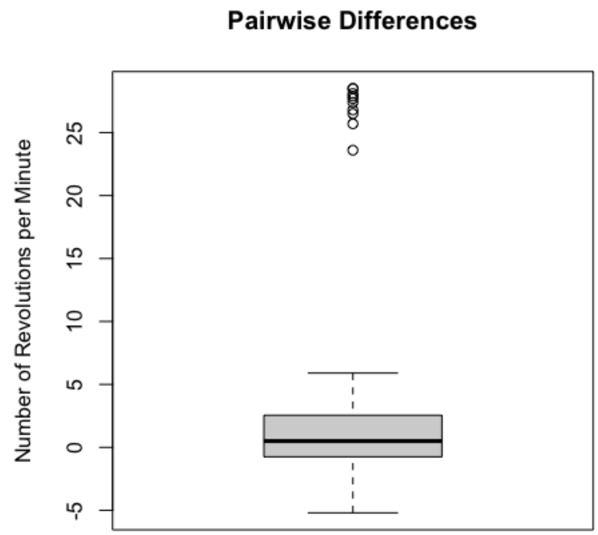
b)

Showing that the difference in sample means is 3.11 is easily done using R:

```
x=c(2.3, 0.3, 5.2, 3.1, 1.1, 0.9, 2.0, 0.7, 1.4, 0.3)
y=c(0.8, 2.8, 4.0, 2.4, 1.2, 0.0, 6.2, 1.5, 28.8, 0.7)
mean(y)-mean(x)
```

```
[1] 3.11
```

This is much larger than the MWW estimate of shift, which is 0.50, as shown in example 10.4.3 of the textbook.



We can see from the boxplot above that there are many outliers in the pairwise differences, which would explain the discrepancy. A solution for this would be to consider the median of the differences instead, since that would be more robust to outliers.

c)

We can get the confidence interval using R:

```
library(distributions3)
T_18 <- StudentsT(df = 18)
Sp2=(9*var(x)+9*var(y))/18
c1 = mean(y)-mean(x) - (quantile(T_18, 0.975) * sqrt(Sp2) *(1/ sqrt(5)))
c2 = mean(y)-mean(x) + (quantile(T_18, 0.975) * sqrt(Sp2) *sqrt(1/5))
c(c1,c2)
```

```
[1] -2.701728  8.921728
```

This is much larger than the MWW confidence interval, which is (-0.80,2.90), as shown in example 10.4.3 of the textbook. This is also due to the outliers, since the confidence interval for t is not robust to outliers, while the MWW confidence interval is.

d)

We can find the value of the t-test statistic using R:

```
tstat = (mean(y)-mean(x))/(sqrt(Sp2)*sqrt(1/5))
tstat
```

```
[1] 1.124256
```

We can also find the p-value with R:

```
p_value=2*pt(-abs(tstat),df=18)
p_value
```

```
[1] 0.2756746
```

Considering the boxplots above, the p-value is lower than warranted. This is because the outliers impair the t-test.

11.1.1

We have

$$P(\theta = 0.3|Y = 9) = \frac{P(\theta = 0.3, Y = 9)}{P(Y = 9)}$$

$$= \frac{\binom{20}{9}(0.3)^9(0.7)^{11}(\frac{2}{3})}{(\frac{2}{3})\binom{20}{9}(0.3)^9(0.7)^{11} + (\frac{1}{3})\binom{20}{9}(0.5)^9(0.5)^{11}} = 0.449$$

and

$$P(\theta = 0.5|Y = 9) = \frac{P(\theta = 0.5, Y = 9)}{P(Y = 9)}$$

$$= \frac{\binom{20}{9}(0.5)^9(0.5)^{11}(\frac{1}{3})}{(\frac{2}{3})\binom{20}{9}(0.3)^9(0.7)^{11} + (\frac{1}{3})\binom{20}{9}(0.5)^9(0.5)^{11}} = 0.55.$$

11.1.4

We have $Y = \sum X_i \sim Poisson(n\theta)$, and

$$h(\theta) = \frac{\theta^{\alpha-1}e^{-\theta/\beta}}{\Gamma(\alpha)\beta^\alpha} \Rightarrow \Theta \sim \Gamma(\alpha, \beta).$$

Then

$$g(y|\theta) = \frac{e^{-n\theta}(n\theta)^y}{y!}$$

so we get

$$k(\theta|y) = \frac{g(y|\theta)h(\theta)}{h(y)} = \frac{\theta^{y+\alpha-1}n^y e^{-n\theta-\theta/\beta}}{y!\Gamma(\alpha)\beta^\alpha h(y)}.$$

Also,

$$h(y) = \frac{n^y}{y!\Gamma(\alpha)\beta^\alpha}\int_0^\infty \theta^{y+\alpha-1}e^{-n\theta-\theta/\beta}d\theta = \frac{n^y}{y!\Gamma(\alpha)\beta^\alpha}\int_0^\infty \theta^{y+\alpha-1}e^{-\theta(\frac{n\beta+1}{\beta})}d\theta$$

$$= \frac{n^y\Gamma(y+\alpha)(\frac{\beta}{n\beta+1})^{(y+\alpha)}}{y!\Gamma(\alpha)\beta^\alpha}.$$

Therefore

$$k(\theta|y) = \frac{\theta^{y+\alpha-1}e^{-\theta(n+\frac{1}{\beta})}}{\Gamma(y+\alpha)(\frac{\beta}{n\beta+1})^{\alpha+y}} \sim \Gamma(y+\alpha, \frac{\beta}{n\beta+1}).$$

From this, we get that the Bayesian point estimate $\delta(y)$ is the mean of the $\Gamma(y+\alpha, \frac{\beta}{n\beta+1})$ distribution, so

$$\delta(y) = (y+\alpha)(\frac{\beta}{n\beta+1}).$$

11.1.5

We have that

$$F(y_n) = F(x)^n = (\int_0^y \frac{1}{\theta}dx)^n = (\frac{y}{\theta})^n \Rightarrow f(y_n|\theta) = \frac{ny^{n-1}}{\theta^n}.$$

Then

$$k(\theta|y_n) = \frac{ny^{n-1}\beta\alpha^\beta\theta^{-n-(\beta+1)}}{h(y)}$$

where

$$h(y) = ny^{n-1}\beta\alpha^\beta\int_\alpha^\infty \theta^{-(n+\beta+1)}d\theta = \frac{ny^{n-1}\beta\alpha^\beta}{(n+\beta)\alpha^{(n+\beta)}}.$$

Then

$$k(\theta|y_n) = \frac{(n+\beta)\alpha^{n+\beta}}{\theta^{n+\beta+1}} \sim Pareto(\alpha, n+\beta).$$

Taking the mean of the $Pareto(\alpha, n+\beta)$ distribution, we get

$$\delta(y_n) = \frac{(n+\beta)\alpha}{n+\beta-1}.$$

11.1.8

a)

We have that $\Theta \sim beta(10,5)$ and $Y = \sum X_i \sim bin(30,\theta)$. Then

$$E[(\theta - \frac{10+Y}{45})^2] = E[\theta^2 - 2\theta(\frac{10+Y}{45}) + (\frac{10+Y}{45})^2]$$

$$= \theta^2 - \frac{2\theta}{45}E(10+Y) + E[(\frac{10}{45})^2 + 2(\frac{10}{45})(\frac{Y}{45}) + \frac{Y^2}{45^2}]$$

$$= \theta^2 - \frac{2\theta}{45}(10+30\theta) + (\frac{10}{45})^2 + \frac{20}{45^2}(30\theta) + \frac{1}{45^2}(300(1-\theta) + 30^2\theta^2)$$

$$= (\theta - \frac{10+30\theta}{45})^2 + (\frac{1}{45})^2 30\theta(1-\theta).$$

b)

We need to find values of θ for which

$$E[(\theta - \frac{10+Y}{45})^2] < \frac{\theta(1-\theta)}{30} \Rightarrow (\theta - \frac{10+30\theta}{45})^2 + (\frac{1}{45})^2 30\theta(1-\theta) < \frac{\theta(1-\theta)}{30}$$

$$\Rightarrow (\frac{\theta}{3} - \frac{2}{9})^2 + (\frac{30}{45^2} - \frac{1}{30})\theta(1-\theta) < 0$$

$$\Rightarrow 0.463 < \theta < 0.823.$$