Q3.1. By Q1.1, $X \sim N(\mu_x, \sigma_x^2)$.

$\because$ $\bar{x} = 2.07$, $s_x^2 = 121.63$

$$\therefore \quad \theta = P(X \geq 0) \approx P\left(Z \geq \frac{0 - 2.07}{\sqrt{121.63}}\right) \qquad Z \sim N(0,1)$$

$$= 0.57$$

Q3.2. $W_{100}$ is approximately Normal with mean $100E(Y)$, variance $100\,Var(Y)$.

$$\because \quad Y = \begin{cases} 20, & \text{if } X \geq 0 \\ 0, & \text{otherwise} \end{cases} \Rightarrow E(Y) = 20\theta$$

$$V(Y) = 20^2\,\theta(1-\theta)$$

$$\therefore \quad W_{100} \sim N(2000\,\theta,\ 40000\,\theta(1-\theta))$$

Q3.3. $$P(W_{100} > 1500) \approx P\left(Z > \frac{1500 - 2000 \cdot (0.57)}{\sqrt{40000(.57)(.43)}}\right)$$

$\uparrow$ 3.64

$$\approx 0$$

Q3.4. (a) Let $\varphi = P(W_{100} \leq 1200)$

$$\hat{\varphi} = P\left(Z \leq \frac{1200 - 2000 \cdot \hat{\theta}}{\sqrt{40000\,\hat{\theta}\,(1-\hat{\theta})}}\right)$$

with $\hat{\theta} = P\left(Z \geq \frac{0 - \bar{x}}{\sqrt{s_x^2}}\right)$.

That is, $\hat{\varphi} = \hat{\varphi}(X_1, \cdots, X_{100})$, an estimator of $\varphi$ with observations $x_1, \cdots, x_{100}$.

The following is a bootstraping procedure for constructing a 90% CI for the probabilit $\varphi = P(W_{100} \leq 1200)$:

Step 1.

Step 1.1 Select 100 observations from the study data

$\{x_1, \cdots, x_{100}\}$ randomly with replacement. Denote them by $\{x_1^*, \cdots, x_{100}^*\}$

Step 1.2. Evaluate $\hat{\varphi} = \hat{\varphi}(X_1, \cdots, X_{100})$ with $\{x_1^*, \cdots, x_{100}^*\}$.

Repeat Step 1.1 and step 1.2 B times and denote the resulting $\hat{\varphi}$ estimates by $\hat{\varphi}_1^*, \hat{\varphi}_2^*, \cdots, \hat{\varphi}_B^*$

Step 2 Let $\hat{\varphi}_L$ and $\hat{\varphi}_U$ be the 5% and 95% quantile of $\hat{\varphi}_1^*, \cdots, \hat{\varphi}_B^*$.

The interval $(\hat{\varphi}_L, \hat{\varphi}_U)$ is then the bootstrap 90% CI for $\varphi$.

(b) Note that $\sqrt{n}\left(\frac{W_n}{n} - E(Y)\right) \xrightarrow{d} N(0, Var(Y))$.

By the $\Delta$-method,

$$\sqrt{n}\left(\left(\frac{W_n}{n}\right)^{4/3} - (E(Y))^{4/3}\right) \xrightarrow{d} N(0, Q^2 \cdot Var(Y))$$

with $Q = \frac{4}{3}(E(Y))^{1/3}$.

$$\Longrightarrow \quad A_{100} = W_{100}^{4/3} \sim N(\mu_A, \sigma_A^2) \text{ approximately}$$

with $\mu_A = n^{4/3}(E(Y))^{4/3} = (2000\theta)^{4/3}$

$\sigma_A^2 = n^{5/3} Q^2 V(Y) = \frac{2}{9} \cdot 2^{20/3} \cdot 10^6 \cdot \theta^{5/3}(1-\theta)$