STAT 330
Tutorial 10

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Nov 23rd/ Nov 25th

Outline

Maximum Likelihood Methods

- Likelihood-Based Tests
- Multiparameter cases
- EM Algorithm

Likelihood-Based Tests

6.3.8. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with mean $\theta > 0$.

(a) Show that the likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is based upon the statistic $Y = \sum_{i=1}^n X_i$. Obtain the null distribution of Y.

(b) For $\theta_0 = 2$ and $n = 5$, find the significance level of the test that rejects H_0 if $Y \leq 4$ or $Y \geq 17$.

a) The pmf is $f(x,\theta) = e^{-\theta}\frac{\theta^x}{x!}$, $\theta > 0$.

The Likelihood function is

$$L(\theta) = \prod_{i=1}^n f(x_i,\theta) = \prod_{i=1}^n e^{-\theta}\frac{\theta^{x_i}}{x_i!} = e^{-n\theta}\frac{\theta^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}$$

$$l(\theta) = \ln(L(\theta)) = -n\theta + \sum_{i=1}^n x_i \ln(\theta) - \sum_{i=1}^n \ln(x_i!)$$

Using derivative of function equals 0 to find MLE:

$$\frac{\partial}{\partial\theta} l(\theta) = -n + \frac{1}{\theta}\sum_{i=1}^n x_i = 0.$$

We get: $\hat{\theta} = \frac{1}{n}\sum_{i=1}^n x_i$.

Hence, the likelihood ratio is:

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{e^{-n\theta_0}\frac{\theta_0^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}}{e^{-\sum_{i=1}^n x_i}\frac{\left(\frac{\sum x_i}{n}\right)^{\sum x_i}}{\prod_{i=1}^n x_i!}}$$

Given $Y = \sum_{i=1}^n X_i$, we have $\Lambda = \frac{e^{-n\theta_0}\theta_0^{Y}}{e^{-Y}\left(\frac{Y}{n}\right)^{Y}}$,

and $Y \sim \text{Poisson}(n\theta)$.

b) $\alpha = P(\text{Reject } H_0 \mid H_0)$

$= P(Y \leq 4 \text{ or } Y \geq 17 \mid n\theta_0 = 10)$

$= P(Y \leq 4 \mid n\theta_0 = 10) + P(Y \geq 17 \mid n\theta_0 = 10)$

$= P(Y \leq 4 \mid n\theta_0 = 10) + [1 - P(Y \leq 16 \mid n\theta_0 = 10)]$

$= \sum_{y=0}^{4} \frac{e^{-10}10^y}{y!} + \left[1 - \sum_{y=0}^{16}\frac{e^{-10}10^y}{y!}\right]$

$\approx 0.029 + 1 - 0.973$

$= 0.056.$

Likelihood-Based Tests

6.3.15. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pmf $p(x;\theta) = \theta^x(1-\theta)^{1-x}$, $x = 0, 1$, where $0 < \theta < 1$. We wish to test $H_0: \theta = 1/3$ versus $H_1: \theta \neq 1/3$.

(a) Find Λ and $-2\log\Lambda$.

(b) Determine the Wald-type test.

a) The pmf is $f(x;\theta) = \theta^x(1-\theta)^{1-x}$, $x = 0, 1$, $0<\theta<1$.

The likelihood function is :

$$L(\theta) = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} = \theta^{n\bar{x}}(1-\theta)^{n-n\bar{x}}$$

When under $H_0: \theta = 1/3$, the likelihood function is : $L(\theta_0) = (\frac{1}{3})^{\sum_{i=1}^{n} x_i}(\frac{2}{3})^{n-\sum_{i=1}^{n} x_i}$

When $0<\theta<1$, the MLE of θ is $\bar{x}$.

The likelihood function is $L(\hat{\theta}) = (\bar{x})^{n\bar{x}}(1-\bar{x})^{n-n\bar{x}}$.

Then, $\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{(\frac{1}{3})^{\sum_{i=1}^{n} x_i}(\frac{2}{3})^{n-\sum_{i=1}^{n} x_i}}{(\bar{x})^{n\bar{x}}(1-\bar{x})^{n-n\bar{x}}} = (\frac{1}{3\bar{x}})^{n\bar{x}}(\frac{2}{3(1-\bar{x})})^{n-n\bar{x}}$.

Thus, $-2\log(\Lambda) = -2n\bar{x}\log(\frac{1}{3\bar{x}}) - 2(n-n\bar{x})\log(\frac{2}{3(1-\bar{x})})$.

b)

First we compute $FI(\theta)$:

$f(x;\theta) = \theta^x(1-\theta)^{1-x}$, $x = 0, 1$, $0<\theta<1$.

$l(\theta) = x\log\theta + (1-x)\log(1-\theta)$.

$$\frac{\partial}{\partial\theta} l(\theta) = \frac{x}{\theta} - \frac{1-x}{1-\theta}.$$

$$\frac{\partial}{\partial^2\theta} l(\theta) = -\frac{x}{\theta^2} - \frac{1-x}{(1-\theta)^2}.$$

$$FI(\theta) = -E\left[\frac{\partial}{\partial^2\theta} l(\theta)\right] = \frac{E[x]}{\theta^2} + \frac{1-E[x]}{(1-\theta)^2}$$

$$= \frac{\theta}{\theta^2} + \frac{1-\theta}{(1-\theta)^2} = \frac{1}{\theta(1-\theta)}$$

Then the Wald-type test is :

$$\chi_W^2 = \left(\sqrt{n\,FI(\theta)}\,(\hat{\theta}-\theta_0)\right)^2$$

$$= \left(\sqrt{\frac{n}{\theta(1-\theta)}}\,(\bar{x}-1/3)\right)^2.$$

Multiparameter Cases

6.4.5. Let $Y_1 < Y_2 < \cdots < Y_n$ be the order statistics of a random sample of size n from the uniform distribution of the continuous type over the closed interval $[\theta - \rho, \theta + \rho]$. Find the maximum likelihood estimators for θ and ρ. Are these two unbiased estimators?

Given $Y_1 < Y_2 < \cdots < Y_n$ be the order statistics of a random sample of size n from $U(\theta-\rho, \theta+\rho)$,

so $Y_1 \geq \theta-\rho$, $Y_n \leq \theta+\rho$.

The pdf is $f(x;\theta,\rho) = \dfrac{1}{\theta+\rho-(\theta-\rho)}$

The likelihood function is:

$$L(\theta,\rho) = \prod_{i=1}^{n}\left(\frac{1}{\theta+\rho-(\theta-\rho)}\right)$$

$$= \left(\frac{1}{\theta+\rho-(\theta-\rho)}\right)^n$$

Thus, the likelihood function is maximized when the difference between $(\theta-\rho)$ and $(\theta+\rho)$ as small as possible, and $Y_1 \geq \theta-\rho$, $Y_n \leq \theta+\rho$.

Hence, $Y_1 = \theta-\rho$. $Y_n = \theta+\rho$.

Solve the equations we get

$$\hat{\theta} = \frac{Y_1+Y_n}{2}, \quad \hat{\rho} = \frac{Y_n-Y_1}{2}.$$

Multiparameter Cases

6.5.6. Consider the two uniform distributions with respective pdfs

$$f(x;\theta_i) = \begin{cases} \frac{1}{2\theta_i} & -\theta_i < x < \theta_i, -\infty < \theta_i < \infty \\ 0 & \text{elsewhere,} \end{cases}$$

for $i = 1, 2$. The null hypothesis is $H_0 : \theta_1 = \theta_2$, while the alternative is $H_1 : \theta_1 \neq \theta_2$. Let $X_1 < X_2 < \cdots < X_{n_1}$ and $Y_1 < Y_2 < \cdots < Y_{n_2}$ be the order statistics of two independent random samples from the respective distributions. Using the likelihood ratio Λ, find the statistic used to test H_0 against H_1.

Similar to the previous question, the likelihood function is maximized when the value of θi as small as possible, and $|x_i| < \theta_1$, $|y_j| < \theta_2$ for $i=1,\dots,n_1$, $j=1,\dots,n_2$.

Under $H_0: \theta_1 = \theta_2$, the MLE of $\hat{\theta}$ is :

$$\hat{\theta} = \hat{\theta}_1 = \hat{\theta}_2 = \max\{|x_i|, |y_j|\},$$

where $i=1,\dots,n_1$, $j=1,\dots,n_2$.

The likelihood function is :

$$L(\hat{\theta}) = \left(\frac{1}{2\theta}\right)^{n_1}\left(\frac{1}{2\theta}\right)^{n_2} = \left(\frac{1}{2\max\{|x_i|,|y_j|\}}\right)^{n_1+n_2}$$

where $i=1,\dots,n_1$, $j=1,\dots,n_2$.

Under $H_1: \theta_1 \neq \theta_2$, the MLE of $\hat{\theta}$ is

$\hat{\theta}_1 = \max|x_i|$, where $i=1,\dots,n_1$.

$\hat{\theta}_2 = \max|y_j|$, where $j=1,\dots,n_2$.

The likelihood function is :

$$L(\hat{\theta}_1,\hat{\theta}_2) = \left(\frac{1}{2\hat{\theta}_1}\right)^{n_1}\left(\frac{1}{2\hat{\theta}_2}\right)^{n_2} = \left(\frac{1}{2\max|x_i|}\right)^{n_1}\left(\frac{1}{2\max|y_j|}\right)^{n_2}$$

Thus, $$\Lambda = \frac{\left(\frac{1}{\max\{|x_i|,|y_j|\}}\right)^{n_1+n_2}}{\left(\frac{1}{\max|x_i|}\right)^{n_1}\left(\frac{1}{\max|y_j|}\right)^{n_2}} = \frac{(\max|x_i|)^{n_1}(\max|y_j|)^{n_2}}{(\max\{|x_i|,|y_j|\})^{n_1+n_2}}$$ for $i=1,\dots,n_1$, $j=1,\dots,n_2$.

EM Algorithm

6.6.5. Suppose $X_1, X_2, \ldots, X_{n_1}$ is a random sample from a $N(\theta, 1)$ distribution. Besides these n_1 observable items, suppose there are n_2 missing items, which we denote by $Z_1, Z_2, \ldots, Z_{n_2}$. Show that the first-step EM estimate is

$$\widehat{\theta}^{(1)} = \frac{n_1 \overline{x} + n_2 \widehat{\theta}^{(0)}}{n},$$

where $\widehat{\theta}^{(0)}$ is an initial estimate of θ and $n = n_1 + n_2$. Note that if $\widehat{\theta}^{(0)} = \overline{x}$, then $\widehat{\theta}^{(k)} = \overline{x}$ for all k.

The complete likelihood is

$$L^c(\theta \mid X, Z) \propto \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{n_1}(X_i-\theta)^2 + \sum_{i=1}^{n_2}(Z_i-\theta)^2\right]\right\}.$$

The observed likelihood is

$$L(\theta \mid X) \propto \exp\left\{-\frac{1}{2}\sum_{i=1}^{n_1}(x_i-\theta)^2\right\}.$$

The conditional pmf $k(Z \mid \theta, X) = \dfrac{L^c(\theta \mid X, Z)}{L(\theta \mid X)}$

$$\propto \exp\left\{-\frac{1}{2}\sum_{i=1}^{n_2}(Z_i-\theta)^2\right\}$$

Hence, $Z_1, \ldots, Z_{n_2} \sim N(\theta, 1)$.

E-step

$$Q(\theta \mid \hat{\theta}^{(0)}, \text{data}) = E_{\hat{\theta}^{(0)}} L^c(\theta \mid X, Z)$$

$$= -\frac{1}{2}\left[\sum_{i=1}^{n_1}(X_i-\theta)^2 + \sum_{i=1}^{n_2} E(Z_i-\theta)^2\right]$$

$$= -\frac{1}{2}\left[\sum_{i=1}^{n}(X_i-\theta)^2 + \sum_{i=1}^{n_2} E[(Z_i-\hat{\theta}^{(0)}) + (\hat{\theta}^{(0)}-\theta)]^2\right]$$

$$= -\frac{1}{2}\left[\sum_{i=1}^{n}(X_i-\theta)^2 + \sum_{i=1}^{n_2}[1 + (\hat{\theta}^{(0)}-\theta)^2]\right]$$

$$= -\frac{1}{2}\left[\sum_{i=1}^{n}(X_i-\theta)^2 + n_2 + n_2(\hat{\theta}^{(0)}-\theta)^2\right].$$

M-step

Then set derivative respect to θ to 0,

we get $\sum_{i=1}^{n_1}(X_i-\theta) + n_2(\hat{\theta}^{(0)}-\theta) = 0$.

Thus, $\hat{\theta}^{(1)} = \dfrac{n_1\bar{x} + n_2\hat{\theta}^{(0)}}{n}$.

Questions