SFU

STAT 330
Tutorial 4

Mengqi (Molly) Cen
Department of Statistics & Actuarial Science

Oct 5th/ Oct 7th

Contents

Joint pmf

2.3.9. Five cards are drawn at random and without replacement from an ordinary deck of cards. Let X_1 and X_2 denote, respectively, the number of spades and the number of hearts that appear in the five cards.

(a) Determine the joint pmf of X_1 and X_2.

(b) Find the two marginal pmfs.

Review:

When X_1, X_2 are discrete, the joint pmf of (X_1, X_2) is
$p(x_1, x_2) = P(X_1 = x_1, X_2 = x_2)$

The marginal pmf of X_1 is $p_{X_1}(x_1) = \sum_{\text{all } x_2} p(x_1, x_2)$.
The marginal pmf of X_2 is $p_{X_2}(x_2) = \sum_{\text{all } x_1} p(x_1, x_2)$.

Answer:

(a) X_1, X_2 are discrete r.v.s with all the possible values $0, 1, \ldots, 5$ and $x_1 + x_2 = 5$

$$p(x_1, x_2) = P(X_1 = x_1, X_2 = x_2)$$
$$= \frac{\binom{13}{x_1}\binom{13}{x_2}\binom{26}{5-x_1-x_2}}{\binom{52}{5}},$$

for $x_1, x_2 = 0, 1, \ldots, 5$, $0 \le x_1 + x_2 \le 5$ and 0 otherwise.

(b) $$p_{X_1}(x_1) = \frac{\binom{13}{x_1}\binom{39}{5-x_1}}{\binom{52}{5}},$$ for $x_1 = 0, 1, \ldots, 5$ and 0 otherwise.

$$p_{X_2}(x_2) = \frac{\binom{13}{x_2}\binom{39}{5-x_2}}{\binom{52}{5}},$$ for $x_2 = 0, 1, \ldots, 5$ and 0 otherwise.

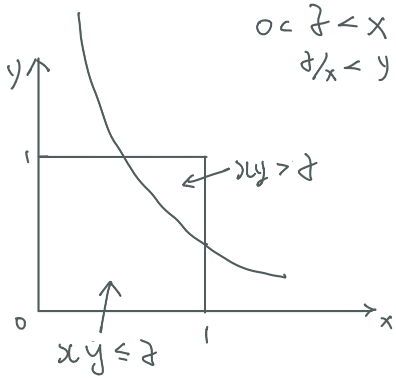
Joint pdf

2.1.7. Let X and Y have the pdf $f(x,y) = 1,\ 0 < x < 1,\ 0 < y < 1$, zero elsewhere. Find the cdf and pdf of the product $Z = XY$.

Review:

When X_1, X_2 are continuous, the joint pdf of (X_1, X_2) is $f(x_1, x_2)$ such that

$$P((X_1,X_2) \in A) = \iint_A f(x_1,x_2)\,dx_1 dx_2 .$$



$0 < z < x < 1$
$z/x < y < 1$

Answer:

$$F(z) = P(Z \le z) = P(XY \le z)$$
$$= 1 - P(XY > z)$$
$$= 1 - \int_z^1 \int_{z/x}^1 1\, dy\, dx$$
$$= 1 - \int_z^1 (1 - z/x)\, dx$$
$$= 1 - (x - z\log(x))\Big|_z^1$$
$$= z - z\log(z), \text{ for } z \in (0,1)$$

$$f(z) = \frac{\partial F}{\partial z} = 1 - 1 - \log(z) = -\log(z),$$

for $z \in (0,1)$, 0 otherwise.

Conditional Distribution and Expectation

2.3.9. Five cards are drawn at random and without replacement from an ordinary deck of cards. Let X_1 and X_2 denote, respectively, the number of spades and the number of hearts that appear in the five cards.

(c) What is the conditional pmf of X_2, given $X_1 = x_1$?

Review:

If X_1, X_2 are discrete with the joint pmf $p(x_1, x_2)$,

and marginal pmfs $p_{X_1}(x_1)$ and $p_{X_2}(x_2)$,

the conditional pmf of X_2 given $X_1 = x_1$ is

$$p_{X_2|X_1}(x_2|x_1) = P(X_2 = x_2 \mid X_1 = x_1) = \frac{P(X_1 = x_1, X_2 = x_2)}{P(X_1 = x_1)} \text{ provided } p_{X_1}(x_1) > 0,$$

the conditional pmf of X_1 given $X_2 = x_2$ is

$$p_{X_1|X_2}(x_1|x_2) = P(X_1 = x_1 \mid X_2 = x_2) = \frac{P(X_1 = x_1, X_2 = x_2)}{P(X_2 = x_2)} \text{ provided } p_{X_2}(x_2) > 0.$$

Answer:

(c) Provided $0 \le x_1 \le 5$

$$p_{X_2|X_1}(x_2|x_1) = \frac{P(X_1 = x_1, X_2 = x_2)}{P(X_1 = x_1)}$$

$$= \frac{\dfrac{\binom{13}{x_1}\binom{13}{x_2}\binom{26}{5-x_1-x_2}}{\binom{52}{5}}}{\dfrac{\binom{13}{x_1}\binom{39}{5-x_1}}{\binom{52}{5}}}$$

$$= \frac{\binom{13}{x_2}\binom{26}{5-x_1-x_2}}{\binom{39}{5-x_1}}, \text{ for } 0 \le x_2 \le 5 - x_1$$

Conditional Distribution and Expectation

2.3.8. Let X and Y have the joint pdf $f(x, y) = 2\exp\{-(x+y)\}$, $0 < x < y < \infty$, zero elsewhere. Find the conditional mean $E(Y|x)$ of Y, given $X = x$.

Review :

If X_1 X_2 are continuous with joint pdf $f(x_1, x_2)$,

the marginal pdf of X_1 is $f_X(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2$,

the marginal pdf of X_2 is $f_X(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1$.

The conditional pdf of X_2 given $X_1 = x_1$ is

$$f_{X_2|X_1}(x_2|x_1) = \frac{f(x_1, x_2)}{f_{X_1}(x_1)}, \text{ provided } f_{X_1}(x_1) > 0,$$

and the conditional pdf of X_1 given $X_2 = x_2$ is

$$f_{X_1|X_2}(x_1|x_2) = \frac{f(x_1, x_2)}{f_{X_2}(x_2)}, \text{ provided } f_{X_2}(x_2) > 0,$$

Answer :

$$f_X(x) = \int_x^{\infty} 2e^{-(x+y)}\, dy$$

$$= 2e^{-2x}, \quad 0 < x < \infty.$$

$$f_{Y|X}(y|x) = \frac{2e^{-(x+y)}}{2e^{-2x}}$$

$$= e^{x-y}, \quad 0 < x < y < \infty.$$

$$E(Y|X=x) = \int_x^{\infty} y e^{x-y}\, dy$$

$$= x+1, \quad x > 0.$$

Correlation Coefficient

2.4.1. Let the random variables X and Y have the joint pmf

$$p(x,y) = \tfrac{1}{3},\ (x,y) = (0,0),(1,1),(2,0), \text{ zero elsewhere.}$$

compute the correlation coefficient of X and Y.

Review:

The covariance of two rvs X_1, X_2 is

$$Cov(X_1,X_2) = E[(X_1-\mu_1)(X_2-\mu_2)] = E(X_1X_2) - \mu_1\mu_2,$$

where $\mu_1 = E(X_1)$ and $\mu_2 = E(X_2)$.

The correlation coefficient of X_1, X_2 is

$$\rho(X_1,X_2) = \frac{Cov(X_1,X_2)}{\sqrt{Var(X_1)\,Var(X_2)}},\ -1 \le \rho \le 1.$$

Answer:

$$\mu_1 = E(X) = (0+1+2)\times\tfrac{1}{3} = 1$$

$$\mu_2 = E(Y) = (0+1+0)\times\tfrac{1}{3} = \tfrac{1}{3}$$

$$E[XY] = (0\times0 + 1\times1 + 2\times0)\times\tfrac{1}{3} = \tfrac{1}{3}$$

$$\rho(X_1,X_2) = \frac{Cov(X\ Y)}{\sqrt{Var(X)\,Var(Y)}}$$

$$= \frac{E(XY) - \mu_1\mu_2}{\sqrt{Var(X)\,Var(Y)}}$$

$$= 0$$

Extension to Several Random Variables

2.6.4. A fair die is cast at random three independent times. Let the random variable X_i be equal to the number of spots that appear on the ith trial, $i = 1, 2, 3$. Let the random variable Y be equal to $\max(X_i)$. Find the cdf and the pmf of Y.

Review:

The joint cdf of the random vector $(X_1, \ldots, X_k)$ is $F(x_1, x_2, \ldots, x_k) = P(X_1 \le x_1, X_2 \le x_2, \ldots X_k \le x_k)$

for $-\infty < x_1, x_2, \ldots, x_k < \infty$

Answer:

$$
\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P[\max(X_1, X_2, X_3) \le y] \\
&= P(X_1 \le y, X_2 \le y, X_3 \le y) \\
&= P(X_1 \le y) P(X_2 \le y) P(X_3 \le y) \\
&= P(X_1 \le y)^3 \\
&= \left(\frac{y}{6}\right)^3, \text{ for } y = 0, 1, \ldots, 6.
\end{aligned}
$$

$$
\begin{aligned}
p_Y(y) &= P(Y \le y) \\
&= F_Y(y) - F_Y(y-1) = \frac{y^3 - (y-1)^3}{6^3},
\end{aligned}
$$

for $y = 1, 2, \ldots, 6$ and 0 otherwise.

Questions