STAT 330 Tutorial 5

Mengqi (Molly) Cen
Department of Statistics & Actuarial Science

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Outline

1. Discrete Distribution

Binomial Distribution

Poisson Distribution

2. Continuous Distribution

Normal Distribution

Exponential Distribution

3. Multivariate Distribution

Multivariate Normal Distribution

4. Distributions Induced from Others

χ^2 Distribution

Student's t-Distribution

F-Distribution

Mixture Distribution

Binomial Distribution

3.1.6. Let Y be the number of successes throughout n independent repetitions of a random experiment with probability of success $p = \frac{1}{4}$. Determine the smallest value of n so that $P(1 \leq Y) \geq 0.70$.

Review:

The distribution of a r.v. X is called a binomial distribution if its pmf is

$$p(x) = P(X=x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$$

for $x = 0, 1, \ldots, n$, denoted by $X \sim B(n, \theta)$.

Answer:

$$\begin{aligned}
P(Y \geq 1) &= 1 - P(Y=0) \\
&= 1 - \binom{n}{0}\left(\tfrac{1}{4}\right)^0\left(1-\tfrac{1}{4}\right)^{n-0} \\
&= 1 - \left(\tfrac{3}{4}\right)^n
\end{aligned}$$

$\because P(Y \geq 1) \geq 0.70$

$\therefore 1 - \left(\frac{3}{4}\right)^n \geq 0.70$

$\therefore \left(\frac{3}{4}\right)^n \leq 0.3$

$\therefore n \log\left(\frac{3}{4}\right) \leq \log(0.3)$

$\therefore n \geq \dfrac{\log(0.3)}{\log\left(\frac{3}{4}\right)} \approx 4.185$

$\therefore n = 5$

Poisson Distribution

3.2.8. Let the number of chocolate chips in a certain type of cookie have a Poisson distribution. We want the probability that a cookie of this type contains at least two chocolate chips to be greater than 0.99. Find the smallest value of the mean that the distribution can take.

Review :

A r.v. X has a Poisson distribution, denoted by $X \sim \text{Poisson}(\lambda)$, if its pmf is

$$P(X=x) = p(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$$

Answer :

Let X be the number of chocolate chips contained in a certain type of cookie.

We want $P(X \geq 2) > 0.99$.

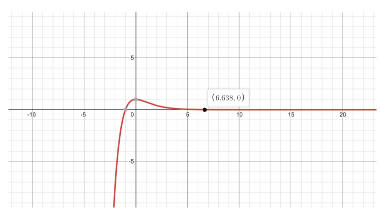
$$P(X \geq 2) = 1 - P(X < 2)$$
$$= 1 - P(X=0) - P(X=1)$$
$$= 1 - \frac{\lambda^0 e^{-\lambda}}{0!} - \frac{\lambda^1 e^{-\lambda}}{1!}$$

If $P(X \geq 2) > 0.99$

$$\therefore \; 1 - \frac{\lambda^0 e^{-\lambda}}{0!} - \frac{\lambda^1 e^{-\lambda}}{1!} > 0.99$$

$$\therefore \; e^{-\lambda} + \lambda e^{-\lambda} < 0.01$$

$$\therefore \; \lambda > 6.638$$



Normal Distribution

3.4.6. If X is $N(\mu, \sigma^2)$, show that $E(|X-\mu|) = \sigma\sqrt{2/\pi}$.

Review:

A r.v. X has a normal distribution, denoted by $X \sim N(\mu, \sigma^2)$, if its pdf is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\},$$

$-\infty < x < \infty$, where $\sigma > 0$.

Answer:

Note that $E(|X-\mu|) = \int_{-\infty}^{\infty} |x-\mu| \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} dx$

Since we want to prove $E(|X-\mu|) = \sigma\sqrt{2/\pi}$,

We can rewite it as

$$E(|X-\mu|) = \sigma\sqrt{2/\pi} \int_{-\infty}^{\infty} \frac{|X-\mu|}{2\sigma^2} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} dx$$

Let $y = \frac{x-\mu}{\sigma}$, $dy = dx/\sigma$.

then $E(|X-\mu|) = \sigma\sqrt{2/\pi} \int_{-\infty}^{\infty} \frac{|y|}{2} e^{-\frac{y^2}{2}} dy$

$$= \sigma\sqrt{2/\pi} \int_0^{\infty} \frac{y}{2} e^{-\frac{y^2}{2}} dy + \sigma\sqrt{2/\pi} \int_{-\infty}^{0} -\frac{y}{2} e^{-\frac{y^2}{2}} dy$$

$$= \sigma\sqrt{2/\pi} \left(-\frac{1}{2} e^{-\frac{y^2}{2}}\right)\Big|_0^{\infty} + \sigma\sqrt{2/\pi} \left(\frac{1}{2} e^{-\frac{y^2}{2}}\right)\Big|_{-\infty}^{0}$$

$$= \sigma\sqrt{2/\pi} \times \frac{1}{2} + \sigma\sqrt{2/\pi} \times \frac{1}{2}$$

$$= \sigma\sqrt{2/\pi}$$

Exponential Distribution

On the average, a certain computer part lasts ten years. The length of time the computer part lasts is exponentially distributed. What is the probability that a computer part lasts more than 7 years?

Review:

A r.v. X has an exponential distribution with $\lambda > 0$, denoted by

$X \sim \text{Exponential}(\lambda)$, if its pdf is

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0.$$

The cdf is $F(x) = 1 - e^{-\lambda x}$ for $x > 0$

Answer:

Let X be the number of years a certain computer part lasts.

Since a certain computer part lasts ten years on the average,

$$\lambda = \frac{1}{10}$$

$$\begin{aligned} P(X > 7) &= 1 - P(X \le 7) \\ &= 1 - \left(1 - e^{-\frac{1}{10}\cdot 7}\right) \\ &= e^{-\frac{7}{10}} \\ &\approx 0.496 \end{aligned}$$

Multivariate Normal Distribution

3.5.8. Let

$$f(x,y) = (1/2\pi)\exp\left[-\frac{1}{2}(x^2+y^2)\right]\left\{1 + xy\exp\left[-\frac{1}{2}(x^2+y^2-2)\right]\right\},$$

where $-\infty < x < \infty$, $-\infty < y < \infty$. If $f(x,y)$ is a joint pdf, it is not a normal bivariate pdf. Show that $f(x,y)$ actually is a joint pdf and that each marginal pdf is normal. Thus the fact that each marginal pdf is normal does not imply that the joint pdf is bivariate normal.

Review:

r.v. $(X,Y)'$ follows a bivariate normal distribution, denoted by $(X,Y) \sim BN(\mu,\Sigma)$, if its pdf is, for $x,y \in \mathbb{R}$,

$$f_{X,Y}(x,y) = \frac{1}{(2\pi)^{2/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}\begin{pmatrix} x-\mu_1 \\ y-\mu_2\end{pmatrix}' \Sigma^{-1} \begin{pmatrix} x-\mu_1 \\ y-\mu_2\end{pmatrix}\right\}.$$

Answer:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \frac{1}{2\pi}\exp\left(-\frac{x^2}{2}\right)\int_{-\infty}^{\infty} \frac{1}{2\pi}\exp\left(-\frac{y^2}{2}\right)\left\{1+xy\exp\left[-\frac{1}{2}(x^2+y^2-2)\right]\right\}$$

$$= \frac{1}{2\pi}\exp\left(-\frac{x^2}{2}\right)\left\{\int_{-\infty}^{\infty}\frac{1}{2\pi}\exp\left(-\frac{y^2}{2}\right)dy + \int_{-\infty}^{\infty}\frac{1}{2\pi}\exp\left(-\frac{y^2}{2}\right)xy\exp\left[-\frac{1}{2}(x^2+y^2-2)\right]dy\right\}$$

Since $\frac{1}{2\pi}\exp\left(-\frac{y^2}{2}\right)xy\exp\left[-\frac{1}{2}(x^2+y^2-2)\right]d$ is an odd function

$$\int_{-\infty}^{\infty}\frac{1}{2\pi}\exp\left(-\frac{y^2}{2}\right)xy\exp\left[-\frac{1}{2}(x^2+y^2-2)\right]dy = 0$$

$$\therefore f_X(x) = \frac{1}{2\pi}\exp\left(-\frac{x^2}{2}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{y^2}{2}\right)dy$$

$$= \frac{1}{2\pi}\exp\left(-\frac{x^2}{2}\right)\sqrt{2\pi}$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$$

Similarly, $f_Y(y) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)$

$f_X(x)$ and $f_Y(y)$ are both pdf of standard normal distribution

χ^2-Distribution

3.3.17. Find the uniform distribution of the continuous type on the interval (b, c) that has the same mean and the same variance as those of a chi-square distribution with 8 degrees of freedom. That is, find b and c.

Review:

① If $Z_1, \dots, Z_n$ are independent and all follow $N(0,1)$, the distribution of $V = Z_1^2 + Z_2^2 + \dots + Z_r^2$ is the χ^2-distribution with the degrees of freedom r, denoted by $V \sim \chi^2(r)$.

$E(V) = r$ and $Var(V) = 2r$.

② A r.v. X has a Uniform (a,b) distribution if its pdf is

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

$E(X) = \frac{a+b}{2}$ and $Var(X) = \frac{(b-a)^2}{12}$

Answer:

Let V has a chi-square with the degrees of freedom 8.

$E(V) = 8$ and $Var(V) = 2 \times 8 = 16$.

Let X has a uniform (b,c) distribution.

$E(X) = \frac{b+c}{2}$ and $Var(X) = \frac{(c-b)^2}{12}$

If X and V have the same mean and the same variance, we have

$$\begin{cases} \frac{b+c}{2} = 8 \\ \frac{(c-b)^2}{12} = 16 \end{cases}.$$

Thus,

$$\begin{cases} b = 8 - \frac{1}{3}\sqrt{3} \\ c = 8 + \frac{1}{3}\sqrt{3} \end{cases}.$$

Student's t-Distribution

3.6.11. Show that the t-distribution with $r = 1$ degree of freedom and the Cauchy distribution are the same.

Review:

If $W \sim N(0,1)$, $V \sim \chi^2(r)$, and W, V are independent, the distribution of $T = \frac{W}{\sqrt{V/r}}$ is the student's t-distribution with df r.

The student's t-distribution has probability density function

$$f(x) = \frac{\Gamma(\frac{r+1}{2})}{\sqrt{r\pi}\,\Gamma(\frac{r}{2})(\frac{x^2}{r}+1)^{(r+1)/2}}, \quad -\infty < x < \infty.$$

Answer:

When $r = 1$.

$$f(x) = \frac{\Gamma(1)}{\sqrt{\pi}\,\Gamma(\frac{1}{2})(x^2+1)}.$$

Since $\Gamma(1) = 1$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

$$f(x) = \frac{1}{\pi(1+x^2)}.$$

$$-\infty < x < \infty,$$

which is the probability density function of the standard Cauchy distribution.

F-Distribution

3.6.8. Let F have an F-distribution with parameters r_1 and r_2. Argue that $1/F$ has an F-distribution with parameters r_2 and r_1.

Review :

If $U \sim \chi^2(r_1)$, $V \sim \chi^2(r_2)$, and U, V are independent, the distribution of $W = \dfrac{U/r_1}{V/r_2}$ is the F-distribution with the degrees of freedom r_1, r_2, denoted by $W \sim F(r_1, r_2)$.

Answer :

$$F = \frac{U/r_1}{V/r_2}$$

$$\frac{1}{F} = \frac{1}{\dfrac{U/r_1}{V/r_2}} = \frac{V/r_2}{U/r_1}$$

Thus, $1/F$ has the F-distribution with parameters r_2 and r_1.

Mixture Distribution

3.7.4. Let X have the conditional geometric pmf $\theta(1-\theta)^{x-1}$, $x = 1, 2, \ldots$, where θ is a value of a random variable having a beta pdf with parameters α and β. Show that the marginal (unconditional) pmf of X is

$$\frac{\Gamma(\alpha+\beta)\Gamma(\alpha+1)\Gamma(\beta+x-1)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+x)}, \quad x = 1, 2, \ldots.$$

Review:

If W is continuous r.v. with pdf $f_W(w)$ for $w \in \mathbb{R}$, X's distribution is a mixture of the distribution's $\{F(x|w) : w \in \mathbb{R}\}$:

$$X \sim \int F(x|w) f_W(w)\,dw$$

Answer:

We have $p_{X|\Theta}(x|\theta) = \theta(1-\theta)^{x-1}$, $x = 1, 2, \ldots$.

Also, $f_\Theta(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}$, $\theta \in [0,1]$.

So,
$$\begin{aligned} f_{X,\Theta}(x,\theta) &= f_{X|\Theta}(x|\theta) f_\Theta(\theta) \\ &= \theta(1-\theta)^{x-1} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}. \\ &= \theta^{\alpha}(1-\theta)^{\beta+x-2} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}, \end{aligned}$$
$\theta \in [0,1]$, $x = 1, 2, \ldots$

Integrating out θ, we have

$$\begin{aligned} p_X(x) &= \int_0^1 \theta^{\alpha}(1-\theta)^{\beta+x-2} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\,d\theta \\ &= \frac{\Gamma(\alpha+1)\Gamma(\beta+x-1)}{\Gamma(\alpha+\beta+x)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \end{aligned}$$

Questions