STAT 330
Tutorial 6

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Outline

1. **Sampling and Statistics**
   - Method of Moments Estimation
   - Maximum Likelihood Estimation

2. **Confidence Interval**

3. **Order Statistics**

4. **Hypothesis Testing**

Method of Moments Estimation

Example 5.2. Suppose that $Y_1, \ldots, Y_n$ is a random sample from an $\mathrm{Exp}(1/\theta)$ population so that their common density is

$$f(y|\theta) = \theta e^{-\theta y}, \quad y > 0,$$

where $\theta > 0$ is a parameter. Derive the method of moment estimator $\hat{\theta}_{\mathrm{MOM}}$

Review:

Suppose $X \sim F(\cdot;\theta_1, \ldots, \theta_m)$ and iid observations $X_1, \ldots, X_n$.

k^{th} population moment of X: $\mu_k = E(X^k)$

k^{th} sample moment with $X_1, \ldots, X_n$:

$$\hat{\mu}_k = \frac{1}{n}(X_1^k + X_2^k + \ldots + X_n^k)$$

Use $\hat{\mu}_k$ to estimate μ_k.

Answer:

Observe that $\mu_1 = E(Y_1) = \int_0^\infty y\theta e^{-\theta y}\,dy$

$$= \theta \int_0^\infty y e^{-\theta y}\,dy$$

$$= \theta\left[-\frac{y}{\theta}e^{-\theta y}\right]_0^\infty - \theta\int_0^\infty -\frac{1}{\theta}e^{-\theta y}\,dy$$

$$= \theta\left[-\frac{y}{\theta}e^{-\theta y}\right]_0^\infty - \theta\left[\frac{1}{\theta^2}e^{-\theta y}\right]_0^\infty$$

$$= \frac{1}{\theta}$$

and $\hat{\mu}_1 = \frac{1}{n}\sum_{i=1}^n Y_i$.

Use $\hat{\mu}_1$ to estimate μ_1,

we get $\frac{1}{\theta} = \frac{1}{n}\sum_{i=1}^n Y_i$.

Solving for θ yields $\theta = \left(\frac{1}{n}\sum_{i=1}^n Y_i\right)^{-1}$

Therefore, $\hat{\theta}_{MOM} = n\left(\sum_{i=1}^n Y_i\right)^{-1}$.

Maximum Likelihood Estimation

Suppose that the lifetime of *Badger* brand light bulbs is modeled by an exponential distribution with (unknown) parameter λ. We test 5 bulbs and find they have lifetimes of 2, 3, 1, 3, and 4 years, respectively. What is the MLE for λ?

Review:

Likelihood Function:

Let the joint distribution (pmf, or pdf) of rvs $X_1, \dots, X_n$ be $f(x_1, \dots, x_n; \theta_1, \dots, \theta_m)$.

When $x_1, \dots, x_n$ are observed values (realizations) of the rvs, the likelihood function of $\theta_1, \dots, \theta_m$ given the data is

$$L(\theta_1, \dots, \theta_m \mid \text{data}) = f(x_1, \dots, x_n; \theta_1, \dots, \theta_m).$$

Maximum Likelihood Estimator:

The MLE $\hat{\theta}_1, \dots, \hat{\theta}_m$ are the values of $\theta_1, \dots, \theta_m$ that maximize the likelihood function:

$$L(\hat{\theta}_1, \dots, \hat{\theta}_m \mid \text{data}) = \max_{\theta \in \Theta} L(\theta_1, \dots, \theta_m \mid \text{data}).$$

Answer:

Let X_i be the lifetime of the ith bulb.

Then each X_i has pdf $f(x_i) = \lambda e^{-\lambda x_i}$, where $i = 1, 2, \dots 5$ and $\lambda > 0$.

The joint pdf is:

$$f(x_1, x_2, x_3, x_4, x_5 \mid \lambda)$$
$$= (\lambda e^{-\lambda x_1})(\lambda e^{-\lambda x_2})(\lambda e^{-\lambda x_3})(\lambda e^{-\lambda x_4})(\lambda e^{-\lambda x_5})$$
$$= \lambda^5 e^{-\lambda(x_1 + x_2 + x_3 + x_4 + x_5)}, \quad \lambda > 0.$$

The likelihood function of λ is:

$$L(\lambda \mid x_1 = 2, x_2 = 3, x_3 = 1, x_4 = 3, x_5 = 4) = \lambda^5 e^{-13\lambda}.$$

Taking the logarithm of likelihood function of λ is:

$$l(\lambda \mid x_1 = 2, x_2 = 3, x_3 = 1, x_4 = 3, x_5 = 4) = 5(\log(\lambda)) - 13\lambda.$$

Using derivative of function equals to 0 to find MLE:

$\frac{5}{\lambda} - 13 = 0$. So $\lambda = \frac{5}{13}$.

The second derivative of function is $-\frac{5}{\lambda^2} < 0$

So $\lambda = \frac{5}{13}$ is the MLE.

Maximum Likelihood Estimation

Suppose our data $x_1, \ldots x_n$ are independently drawn from a uniform distribution $U(a, b)$. Find the MLE estimate for a and b.

Answer:

The density for $U(a,b)$ is $\frac{1}{b-a}$ on $[a,b]$.

Therefore, the likelihood function is

$$L(a,b \mid x_1, x_2, \ldots, x_n) = \begin{cases} \left(\frac{1}{b-a}\right)^n, & \text{if all } x_i \text{ are in the interval } [a,b] \\ 0, & \text{otherwise.} \end{cases}$$

The likelihood function is maximized when $b-a$ is as small as possible and all x_i are in the interval $[a,b]$, the MLE is

$$\hat{a} = \min(x_1, \ldots, x_n), \quad \hat{b} = \max(x_1, \ldots, x_n).$$

Confidence Interval

4.2.17. It is known that a random variable X has a Poisson distribution with parameter μ. A sample of 200 observations from this distribution has a mean equal to 3.4. Construct an approximate 90% confidence interval for μ.

Review:

Suppose $X_1, \dots, X_n$ is a random sample on a random variable X with mean μ.

① $X \sim N(\mu, \sigma^2)$ with σ^2 known, an approximation CI of $1-\alpha$ level is $\bar{X} \pm z_{1-\alpha/2} \sqrt{\frac{\sigma^2}{n}}$.

② $X \sim N(\mu, \sigma^2)$ with σ^2 unknown, an approximation CI of $1-\alpha$ level is $\bar{X} \pm t_{1-\alpha/2} \sqrt{\frac{s^2}{n}}$.

③ $X \sim F(x;\theta)$ with $\theta = \mu$, the population mean. To estimate $\theta = \mu$ when $n >> 1$.

a. an approximation CI of $1-\alpha$ level when σ^2 is known is $\bar{X} \pm z_{1-\alpha/2} \sqrt{\frac{\sigma^2}{n}}$.

b. an approximation CI of $1-\alpha$ level when σ^2 is unknown is $\bar{X} \pm t_{1-\alpha/2} \sqrt{\frac{s^2}{n}}$

$\approx \bar{X} \pm z_{1-\alpha/2} \sqrt{\frac{s^2}{n}}$.

Answer:

Since X has the Poisson distribution, the mean of X is equal to the variance of X.

The approximation CI of 90% level when σ^2 is known when $n >> 1$ is

$$\bar{X} \pm z_{1-\alpha/2} \sqrt{\sigma^2/n}$$

$$= 3.4 \pm z_{0.95} \sqrt{3.4/200}$$

$$= 3.4 \pm 1.645 \sqrt{3.4/200}.$$

Therefore, the 90% confidence interval for μ is $(3.19, 3.61)$.

Order Statistics

4.4.24. Let Y_n denote the nth order statistic of a random sample of size n from a distribution of the continuous type. Find the smallest value of n for which the inequality $P(\xi_{0.9} < Y_n) \geq 0.75$ is true.

Review:

Consider a collection of rvs $X_1, \dots, X_n$.

The order statistics of the rvs are

$Y_1 = \min(X_1, \dots, X_n) \leq \dots \leq Y_n = \max(X_1, \dots, X_n)$.

Consider a random sample $X_1, \dots, X_n$

from a population with cdf $F(x)$.

The population p quantile of $F(x)$

is $\xi_p = F^{-1}(p)$ for $0 < p < 1$.

Answer:

For the n^{th} order statistic to be greater than $\xi_{0.9}$, fewer than n of X values are less than $\xi_{0.9}$.

To put this in the context of a binomial distribution,

the probability of success is $P(X < \xi_{0.9}) = F(\xi_{0.9}) = 0.9$.

$$\begin{aligned} P(y < Y_n) &= 1 - P(Y_n \leq y) \\ &= 1 - P(Y_1 \leq y, Y_2 \leq y, \dots, Y_n \leq y) \\ &= 1 - P(Y_1 \leq y) P(Y_2 \leq y) \cdots P(Y_n \leq y) \\ &= 1 - F^n(y) \end{aligned}$$

so $P(\xi_{0.9} < Y_n) = 1 - F^n(\xi_{0.9})$

$= 1 - 0.9^n$

We need to find the smallest value of n which

the inequality $P(\xi_{0.9} < Y_n) \geq 0.75$ is true,

so $1 - 0.9^n \geq 0.75$.

Solve the inequality we have

$n \leq \log_{0.9} 0.25 \approx 13.1575$.

Thus, the smallest value of n is 14.

Hypothesis Testing

4.5.10. Let Y have a binomial distribution with parameters n and p. We reject $H_0 : p = \frac{1}{2}$ and accept $H_1 : p > \frac{1}{2}$ if $Y \geq c$. Find n and c to give a power function $\gamma(p)$ which is such that $\gamma(\frac{1}{2}) = 0.10$ and $\gamma(\frac{2}{3}) = 0.95$, approximately.

Review:

Let $D = \{$all possible realizations of $(X_1, \dots, X_n)\}$.

Set $C \subseteq D$ is called the rejection region for a hypothesis test if test's decision rule is as follows:

Reject H_0 (Accept H_1) if $(x_1, \dots, x_n) \in C$;

Accept H_0 (Reject H_1) if $(x_1, \dots, x_n) \notin C$.

We say the rejection region C is of size α if $\alpha = \max_{\theta \in \Omega_0} P_\theta\{(X_1, \dots, X_n) \in C\}$.

The power function of the test is

power $(\theta) = P_\theta\{(X_1, \dots, X_n) \in C\}$ for $\theta \in \Omega_1$.

Answer:

The binomial distribution has mean np and standard deviation $\sqrt{np(1-p)}$.

We can approximate binomial distribution by normal distribution when n is large.

The power function of the test is

$$\gamma(p) = P(Y \geq c)$$
$$= P\left(\frac{Y - np}{\sqrt{np(1-p)}} \geq \frac{c - np}{\sqrt{np(1-p)}}\right)$$
$$= P\left(Z \geq \frac{c - np}{\sqrt{np(1-p)}}\right)$$

Then we have
$$\begin{cases} \dfrac{c - n/2}{\sqrt{n}/2} = z_{0.1} = 1.282 \\[2ex] \dfrac{c - 2n/3}{\sqrt{2n}/3} = -z_{0.05} = -1.645 \end{cases}$$

Solving the equation we obtain

$c = 41$ and $n = 72$.

Questions