STAT 855. Lifetime Data Analysis

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What to do today?

I.2. Review of Likelihood Methods

Likelihood methods are inference procedures based on the likelihood function with a population model.

- Formulation.
  - Consider a population under consideration, quantified by a random variable Y.
  - Assume $Y \sim f(y|\theta)$, where θ is the population characteristics of interest.
    - If Y is discrete, $f(y|\theta) = P(Y = y|\theta)$, the pmf.
    - If Y is continuous, $f(y|\theta)$ is the pdf with $P(Y \leq y) = \int_{-\infty}^{y} f(u|\theta)du$.
- Sample data.
  - Collect n iid (independent and identically distributed) observations on Y: $y_1, \ldots, y_n$.
- Likelihood function of θ based on the sample data:

$$L(\theta|\mathbf{y}) = \prod_{i=1}^{n} f(y_i|\theta)$$

A measure of the possibility to have the data with the parameter θ.

I.2. Review of Likelihood Methods

Maximum Likelihood Estimation. (MLE)
The estimator $\hat{\theta}$ is the MLE of θ if

$$L(\hat{\theta}|\mathbf{y}) = \max_{\theta} L(\theta|\mathbf{y})$$

- rationale
- implementation
  - Often to obtain $\hat{\theta}$ by solving

$$\frac{\partial}{\partial\theta} \log L(\theta|\mathbf{y}) = 0.$$

  - Computational algorithms: optimization or solving equation e.g. Newton-Raphson algorithm (eg *R: nlm()*)

Part I.2 Review of Likelihood Based Approaches

likelihood related quantities:

- score function $U(\theta)$
- Fisher information $I(\theta)$

I.2. Review of Likelihood Methods

Properties of MLE

- Invariance property
  If $\phi = h(\theta)$, $\hat{\phi} = h(\hat{\theta})$ is the MLE of ϕ.

- Large sample properties (when $n >> 1$): provided with some regularity conditions,
  - MLE is consistent,
  - MLE has the asymptotic normality: $\hat{\theta} \sim N(\theta, AV_n(\theta))$ approximately
  - MLE is asymptotically most efficient,
  - MLE's variance can be estimated by

$$\hat{V}(\hat{\theta}) = -\Big(\frac{\partial^2 \log L(\theta|\mathbf{y})}{\partial \theta^2}\Big)^{-1}\Big|_{\theta=\hat{\theta}}$$

I.2. Review of Likelihood Methods

Inference with MLE when $n >> 1$

- Testing with MLE: $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$
  - Wald test:
    $Z = \frac{\hat{\theta}-\theta_0}{\sqrt{\hat{V}(\hat{\theta})}} \sim N(0,1)$ approximately under H_0
  - Score test:
    $U(\theta) = \frac{\partial \log L(\theta|\mathbf{y})}{\partial \theta}$ and $\frac{U(\theta_0)}{\sqrt{Var\big(U(\theta_0)\big)}} \sim N(0,1)$ approximately under H_0
  - Likelihood ratio test:
    $\Lambda = \frac{L(\theta_0|\mathbf{y})}{L(\hat{\theta}|\mathbf{y})}$ and $-2\log\Lambda \sim \chi^2(1)$ approxmately under H_0

I.2. Review of Likelihood Methods

Inference with MLE when $n >> 1$

- Confidence interval with MLE:
  - Wald type: CI with level of $1-\alpha$

$$\hat{\theta} \pm Z_{1-\alpha/2}\sqrt{\hat{V}(\hat{\theta})}$$

  - Score based: CI with level of $1-\alpha$

$$\left\{\theta : \left|U(\theta)/\sqrt{Var\big(U(\theta)\big)}\right| \leq Z_{1-\alpha/2}\right\}$$

  - Likelihood ratio based: CI with level of $1-\alpha$

$$\left\{\theta : -2\log\big(L(\theta|\mathbf{y})/L(\hat{\theta}|\mathbf{y})\big) \leq \chi^2_{1-\alpha}(1)\right\}$$

Part I.2 Review of Likelihood Based Approaches

Consider r.v. $Y \sim f(\cdot;\theta)$, $\theta \in \Theta$

with a set of observations: $Y_1, \ldots, Y_n$

If the observations are not indpt, $L(\theta|data) =$**?**

- assume the joint distn of $(Y_1, \ldots, Y_n)$ ...

e.g. with a frailty model?

e.g. with a copula model?

Part I.2 Review of Likelihood Based Approaches

Consider r.v. $Y \sim f(\cdot;\theta)$, $\theta \in \Theta$
with a set of observations: $Y_1, \ldots, Y_n$

If some of $Y_1, \ldots, Y_n$ are not available, $L(\theta|data) =$?

- e.g. assume the available observations form a random sample?
- e.g. use other information?

Part I.2 Review of Likelihood Based Approaches

What if consider r.v. $Y \sim F(\cdot)$, unspecified $F(\cdot)$
with a set of iid observations: $Y_1, \ldots, Y_n$

How to estm $F(\cdot)$ based on $L(F)$?

- the nonparametric MLE: empirical distn?

- local likelihood estimation: local polynomial approximation?

Part I.2 Review of Likelihood Based Approaches

What if

- consider $Y|X = x \sim f(y|x;\theta)$, $\theta \in \Theta$?

  with a set of observations: $(Y_1, X_1), \ldots, (Y_n, X_n)$
  $L(\theta|data) =?$

- $f(y|x;\theta)$ is unspecified but assume $E(Y|X = x) = \mu(x;\theta)$ and $V(Y|X = x) = \sigma^2(x;\theta)$?

conditional likelihood, quasi-likelihood, pseudo-likelihood, partial likelihood, composite likelihood, ...

Part II. Parametric Inference in LIDA

Part II.1 Some Basic Concepts

Consider continuous r.v. $T \geq 0$, time to an event

- probability density function (pdf) $f(t), t \geq 0$
- cumulative distribution function (cdf) $F(t) = P(T \leq t), t \geq 0$
- survivor function $S(t) = P(T \geq t), t \geq 0$

Part II.1 Some Basic Concepts

Consider continuous r.v. $T \geq 0$, time to an event

- hazard function

$$h(t) = \lim_{\Delta t \to 0+} \frac{P\big(T \in [t, t + \Delta t) \big| T \geq t\big)}{\Delta t}$$

  interpretation, analytic properties, ...

- cumulative hazard function $H(t) = \int_0^t h(u)du, t \geq 0$

Part II.2 Commonly Used Parametric Distributions

Exponential distribution $T \sim NE(\lambda), \lambda > 0$

$$f(t;\lambda) = \lambda \exp(-\lambda t), \quad t > 0$$

(or $f(t;\theta) = \frac{1}{\theta}\exp(-t/\theta),\ t > 0$ with $\theta > 0$)

- $E(T)$
- $S(t)$
- $h(t)$
- memoryless property: $P(T > a + b | T > a) = P(T > b)$
- the central role

Part II.2 Commonly Used Parametric Distributions

Weibull distribution $T \sim Weibull(k, \theta)$:

$$f(t;\theta,k) = \frac{k}{\theta}\Big(\frac{t}{\theta}\Big)^{k-1} \exp\big(-(t/\theta)^k\big), \quad t > 0$$

with $\theta > 0$ and $k > 0$.

- $E(T) = \theta\Gamma(1 + 1/k)$, $Var(T) = \theta^2\Big[\Gamma(1+\frac{2}{k}) - \Gamma(1+\frac{1}{k})^2\Big]$
- $S(t)$
- $h(t) = \frac{k}{\theta}\Big(\frac{t}{\theta}\Big)^{k-1}$
- $T^k \sim NE(1/\theta)$

A translated Weibull distribution: $T \sim Weibull(k, \theta, \alpha)$:

$$f(t;\theta,k) = \frac{k}{\theta}\Big(\frac{t-\alpha}{\theta}\Big)^{k-1} \exp\Big\{-\Big(\frac{t-\alpha}{\theta}\Big)^k\Big\}, \quad t > \alpha$$

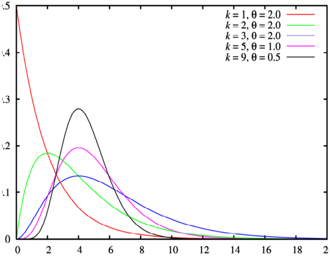
Part II.2 Commonly Used Parametric Distributions

Gamma distribution $T \sim \Gamma(k, \theta)$:

$$f(t; \theta, k) = \frac{1}{\theta\Gamma(k)}\left(\frac{t}{\theta}\right)^{k-1} \exp\big(-t/\theta\big), \quad t > 0$$

with $\theta > 0$ and $k > 0$.

- $E(T) = k\theta$, $Var(T) = k\theta^2$
- $S(t)$
- $h(t)$
- T_1, T_2 indpt and $T_j \sim \Gamma(\alpha_j, \theta)$: $T_1 + T_2 \sim \Gamma(\alpha_1 + \alpha_2, \theta)$
- $\frac{2}{\theta} T \sim \chi^2(2k)$



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Part II.2 Commonly Used Parametric Distributions

Log-normal distribution $T \sim logN(\mu, \sigma^2)$, ie $logT \sim N(\mu, \sigma^2)$:

$$T \sim \frac{1}{t\sigma\sqrt{2\pi}} \exp\big(-\frac{(logt-\mu)^2}{2\sigma^2}\big),\ \ t > 0$$

with $\sigma > 0$.

- $E(T) = \exp(\mu + \sigma^2/2)$
- $S(t)$
- $h(t)$

Part II.2 Commonly Used Parametric Distributions

- **Extreme value distribution**
- **Gumbel distribution**
- **... ...**

See books on reliability, such as Lawless (2003), for more examples of parametric models for event time

What to study next?

Part I. Preliminaries

- *I.1 Introduction*
- *I.2 Review of Likelihood Based Approaches*

Part II. Parametric Inference

- *II.1 Some basic concepts*
- *II.2 Commonly used parametric distributions*
- **II.3 Various incomplete data structures**
- **II.4 Parametric analysis with right-censored data**