Lecture 2: Equivalence

At the beginning of the recent film, *Sense and Sensibility*, Mr John Dashwood and his avaricious wife, Mrs John Dashwood, are debating how he can most cheaply discharge his obligations towards his half-sister, the recently widowed Mrs Henry Dashwood and her three daughters, Elinor, Marianne and Margaret. He first considers giving her a lump sum of $1,500; then, since $1,500 seems a lot to part with in one lump, he considers paying her an annual sum of $100 for as long as she lives. But, objects his wife, although Mrs Dashwood is old, such an arrangement will encourage her to cling to life for an unreasonable time; and, should she survive for more than 15 years, her half-brother will lose money by the arrangement. [Actually, Mrs Dashwood is only 40, but Jane Austen evidently considers this to be near senility; Mrs Dashwood herself says at one point in the novel that she can scarcely expect to survive another fifteen years.]

It is clear from this episode that Mrs John Dashwood cannot have studied engineering economics; for in fact, Mr Dashwood *would* save money by the latter alternative, even if Mrs Henry Dashwood were to live considerably longer than 15 years. Mrs John Dashwood's error lies in comparing future and present sums of money. To make such a comparison valid, we must take into account the potential of money to earn interest. $100 in hand will always be worth more than $100 a year from now, by the amount of interest that the sum will earn over that year.

To make sensible comparisons between present and future sums, then, we must bring all the sums being compared into the same moment of time. This moment can be either present or future. The value of present money increases as we move it into the future, while the value of future income falls as we move it back to the present.

The central notion here is that of *equivalence*. A present and future sums of money are equivalent if a rational person would be indifferent as to which he or she received. Thus, for example, if the best available interest rate is 10%, I should be indifferent between receiving $100 now or $110 this time next year. (In this, and in the remainder of this lecture, we will assume an inflation rate of zero.)

The calculations needed to establish equivalence are relatively straightforward, but, since these calculations will often be steps in more complex calculations, it is useful to establish a standard notation for the quantities involved. The four most needed concepts are **P**, the present worth; **F**, future worth (or `compound amount'); **i**, the interest rate per period; and **N**, the number of time periods we are considering.

These quantities are related by the simple formula

P=F(P/F,i%,N)

and its inverse,

F=P(F/P,i%,N)

The conversion factor can be calculated from a simple formula, which you can readily deduce, but it is usually more convenient to look it up in compound interest tables, such as those found at the back of most engineering economics textbooks.

We also need to consider the common case in which a series of equal payments are made at regular intervals.

It is most convenient, and also most usual, for these intervals to be the same as those at which interest is calculated (for an exception, see question 5, below.) If the size of one payment is $A, then the present worth of the annuity can be calculated from

P=A(P/A,i%,N)

The inverse calculation, **A=P(A/P,i%,N)**, is known as `capital recovery', and would be used, for example, by a used-car salesperson to calculate the size of the regular payment needed to pay for a car worth $P.

It is sometimes desirable to set money aside on a regular basis, to be ready to meet an anticipated future expense. For example, when buying a new piece of equipment, a company will often set up a `sinking fund', into which they will make regular deposits, with the intention that when the current equipment must be replaced, the fund will have just enough in it to pay for the replacement. (Depending on the nature of the equipment, there may be some uncertainty as to what the replacement cost will be.) The `sinking fund' formula is

A=F(A/F,i%,N)

There are two final cases which are common enough to be worth discussing. First, there is the case where we have a regular series of payments, increasing by a fixed quantity each time (for example, a generous uncle might give you your age in dollars at every birthday). If the fixed increase is **G**, and the initial payment is zero, the present worth of such a series paid over **N** years is

P=G(P/G,i%,N)

Lastly, we consider a regular series of payments, where the quantity of the initial payment **A** changes by a fixed ratio **g** every time. (**g** may be greater or less than 1.) An example of this would be living on a fixed income in the presence of a steady rate of inflation. This case is not covered by the compound interest tables, but can be calculated from the following formula:

Some Loose Ends

To calculate the interest payable on a loan, we need to know the principal, the interest rate, and the interval at which interest is compounded. The typical case is where interest is compounded annually, that is, every year we calculate the interest accumulated over the past year, add it on to the amount already owed, and charge interest on the whole sum over the next year. But other compounding intervals are possible. For example, suppose you are quoted an interest rate of **i** per month, compounded monthly. What annual interest rate does this correspond to? We can answer this in two stages: firstly, **i** per month, compounded monthly, corresponds to **12i** per year, compounded monthly. This is the *nominal* interest rate. However, if we want to know what annual interest rate *compounded annually* corresponds to **i**, we need the *effective* annual interest rate, which is given by $\mathbf{j=(1+i)^{12}-1}$.

Consider two extreme cases:

Infinite Compounding Interval

In this case, we never add the interest to the principle. This corresponds to the case known as `simple interest', and, since this never occurs in practice, we shall not discuss it further.

Compounding Interval tends to Zero

Suppose we keep the nominal interest rate constant, at **r** per annum, say, and decrease the compounding interval towards zero. What happens to the effective interest rate?

If we divide the year into **M** intervals, the interest rate for each interval will be **r/M**. The effective annual interest rate will then be

$$j=(1+r/M)^{M}-1$$

$$=((1+r/M)^{M/r})^{r}-1$$

$$=e^{r}-1$$

(Or, you can just as easily convert the continuous interest rate to an equivalent effective annual rate, via the above formula.)

Cash Flow Diagrams

As we've seen, most of the conversion formulae are simple. The difficulty comes when we have to analyze a complex situation, keeping track of many things at once. It is essential to maintain maximum clarity at all times. One thing that can help is a cash flow diagram. A cash flow diagram is a device to show all the information available in a single format.

cf.eps

The cash-flow diagram shows receipts by an arrow above the horizontal axis, and payouts by an arrow below the axis. The horizontal axis is divided into time periods. The interest rate should be clearly shown. It's useful to keep the arrow heights roughly to scale -- this will give you an idea of which effects are negligible -- but there's no point in making them *exactly* to scale, since you're not going to be using the diagram for any kind of geometrical construction.

Bases for Comparison

We have established that revenues and expenditures over a period of time must be reduced to equivalent amounts at a single moment in time before they can be usefully compared. There are five methods in current use for making such a comparison: present worth, annual worth, future worth, rate of return, and benefit-cost ratio. (All of these methods should yield the same conclusion; the only reason we have to study all five is that all are in common use, and we have to be prepared to discuss analyses based on any one of them.)

All these methods compare alternative strategies available to you or your company. In addition to the financial consequences of each alternative strategy, there may be other relationships between the strategies. In particular, they may be *independent*, *exclusive*, or *contingent*. The most permissive of these three cases is where the strategies are independent; in this case, we can implement all of them, none of them, or any combination of them. The least permissive is the case where the strategies are exclusive -- we can implement at most one of them. In the third case, causal connections between the strategies may only permit us to implement them in certain combinations.

It is important to note that which of these categories we're dealing with must be determined *before* the economic analysis; it doesn't *emerge from* the economic analysis. Whichever case we have, we can re-define it as an example of the second case, by exhaustively listing all the legitimate combinations of strategies.

Present Worth

This is also known as the *discounted cash flow* or *net present value* method of comparison. Its basis is simply to reduce each of the mutually exclusive alternatives to its present value, then to choose the greatest.

Suppose we have a sum of money **P** to invest. If we don't choose any of the alternative strategies, we could just put the money in the bank. What is the present worth of this `do nothing' alternative?

Putting the money in the bank is worth **-P**, since we're paying **P** to the bank. If we leave the money in the bank for **N** years, it will then be worth **F=P(F/P,i,N)**. The present worth of **F** is then **F(P/F,i,N)=P(F/P,i,N)(P/F,i,N)=P**. So, adding the present worth of expenditures and receipts, we get 0. This shows that any alternative with a positive present worth is preferable to doing nothing.

Some strategies may guarantee us an infinite series of payments, or commit us to an infinite series of payouts. (For example, hiring Methuselah to a tenure-track position). What is the present worth of such a series?

The present worth of such a series is referred to as its *capitalized cost*, and can be calculated as follows. If **A** is one payment in the series, then

P = A (P/A,i,N) as **N** tends to infinity

= A((1+i)^N-1/(i(1+i)^N)

= A/i

An Example

Consider the following example:

A company has been using manual drafting methods for thirty years. It currently employs 10 drafters at $800/week each. The head of the drafting department is considering two alternatives:

(i) The department can buy 8 low-end workstations at $2,000 each. Two of the drafters can be given twelve months notice; at the end of the twelve months they will get $5,000 severance pay each. The remaining 8 can be trained in AutoCAD; the first training course is available in twelve months, and costs $2,000 for each participant. After completing this course, each drafter gets a $100/week raise.

(ii)The department can buy five high-end workstations at $5,000 each. All of the current drafters will be given a year's notice, and five new graduates hired at $1,200/week. These new graduates will be trained in Pro-Engineer; to keep current with this package, they will need a $5,000 retraining session every six months.

The department is currently able to perform its assigned drafting services for the rest of the company, and either of the two alternatives would allow it to continue that performance.

- Suppose the department decides to stay as it is. Following the discussion above, should this `do nothing' alternative be assigned a present worth of zero? If so, how will it compare with the other alternatives?
- What time frame would be appropriate for this analysis? Why? What effect would changing the time

frame have on the solution?
- Sketch the cash-flow diagrams for the alternatives. Can you see an easy solution? (It is helpful to draw the diagram to scale.)
- In reality, the economic analysis would be one of many factors relevant to the decision. What other factors might influence the department head?

Continuous and Discrete Cash Flows

This section deals with a small but potentially confusing detail which we skimmed over earlier in the semester.

We have discussed the difference between discrete and continuous compounding of interest. However, in both cases we have considered cash flows to occur at the end-points of discrete time intervals. There are many situations where this isn't true; for example, suppose an investment earns money continuously, and we put the weekly earnings in the bank every Friday, so over a year our bank balance grows from zero to $1,000. How does the bank calculate the interest it owes us?

We first introduce some notation: if an amount $F is paid in many regular, uniform installments over a year, we denote it by `F bar' [an F with a bar over the top, which I haven't yet figured out how to write in HTML.].

		Compounding	
		Discrete	**Continuous**
Cash Flow	**Discrete**	(Familiar formulae or tables in Appendix A)	(Convert $i=e^r$ OR use tables in Appendix B)
	Continuous	Up to the bank	`Funds Flow Conversion Factor' OR use tables in Appendix C

If the bank uses discrete compounding, it can choose any one of several conventions to determine the principal on which it calculates interest. For example, it can take the sum in the account at the end of the period; or the average sum in the account; or (this is what my bank does) the minimum amount in the account over the period.

One way of taking an average is to use the `mid-period convention', according to which all of the funds deposited over the year are considered to be deposited halfway through the year. If the interest rate is i, the interest on a principle P is then calculated from

$$F = P(1+i)^{0.5}$$

If the bank uses continuous compounding, there is only one formula that can be used for calculating interest, the *funds flow conversion* formula. (which you should *not* memorise.)

This formula gives the amount A you'll have in the bank at the end of the year, given that the bank continuously compounds at a nominal rate of r% per year, and that the total amount you pay into the bank over the year in regular installments is *A bar*.

$$A = A\ bar\ ((e^r-1)/r)$$

This is the formula used to generate the tables in Appendix C of the textbook.

(Neither the quizzes, mid-term nor final exam in this course will require you to know this formula.)

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