Porous Immersed Boundaries

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  - Swimming worms, cells, other microorganisms **(Fauci, 1990–)**
  - Biofilms **(Dillon, 1995–)**, **(Alpkvist & Klapper, 2007)**
  - Suspensions of flexible fibers **(JS, 1997)**
  - Parachutes **(Kim & Peskin, 2006)**
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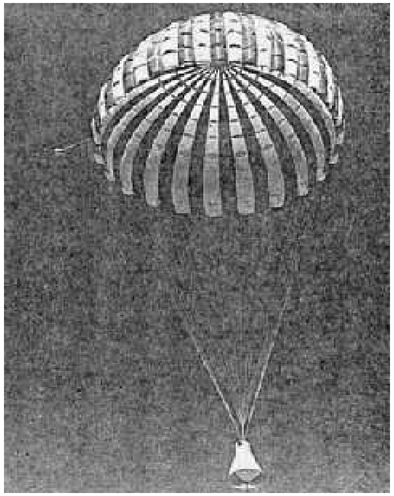
  - . . . and many other problems in biology and engineering
- Recent work involves extensions to:
  - Thick elastic solids **(Griffith & Peskin, 2004)**, **(Wang, 2006)**
  - Massive boundaries, not neutrally buoyant **(Zhu & Peskin, 2002)**
  - Variants specific to stationary walls **(Goldstein et al., 1993–)**
  - **Porous membranes** **(Kim & Peskin, 2006)**

- Related methods:
  - Immersed interface method **(LeVeque & Li, 1994–)** . . . including porous effects **(Layton, 2006)**
  - Immersed continuum method **(Wang, 2007)**
  - Blob projection method **(Cortez & Minion, 2000)**
  - IB finite element method **(Wang & Liu, 2004)**, **(Boffi & Gastaldi, 2007)**
  - Level set interface tracking **(Cottet & Maitre, 2004)**, **(Beale & Strain, 2007)**
- For an extensive review see:

  Mittal & Iaccarino, *Ann. Rev. Fluid Mech.*, <u>37</u>:239–261, 2005

Porous immersed boundaries

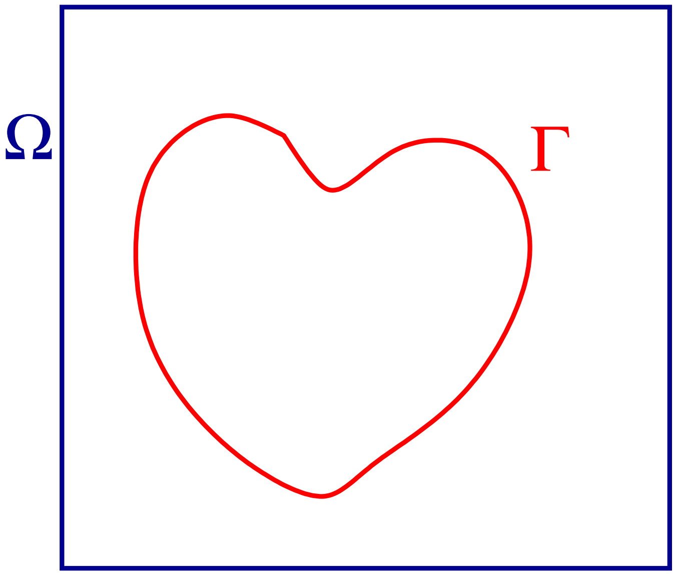
- Many applications involve a deformable **porous** membrane:
  - blood flow through porous capillary walls
  - vesicles and cells
  - cerebrospinal fluid in the brain (hydrocephalus)
  - ocean wave barriers and wave makers
  - biofilms
  - parachutes



- Most existing methods for simulating porous membranes assume the membrane is stationary or else deforms only slightly
- **This talk:** highly deformable porous boundaries

The mathematical formulation

- Peskin **(*Acta Numerica*, <u>11</u>, 2002)**: *"The IB method is both a mathematical formulation and a numerical method"*
- The IB formulation captures complex fluid-membrane interactions, under a few major assumptions:
  - Fluid lies both inside and outside the membrane
  - Membrane is neutrally buoyant and has zero mass and volume
  - Fluid + membrane form an incompressible composite material



$\Omega =$ fluid domain
$p(\vec{x}, t) =$ fluid pressure
$\vec{u}(\vec{x}, t) =$ fluid velocity

$\Gamma =$ immersed boundary/membrane
$s =$ arclength parameter
$\vec{X}(s, t) =$ IB position
$\vec{U}(s, t) =$ IB velocity
$\vec{F}(s, t) =$ IB force density

- The membrane is an elastic material obeying

$$\vec{F}(s,t) = \sigma \left[\vec{X}_s \left(1 - \frac{R_{eq}}{|\vec{X}_s|} \right) \right]_s$$

$R_{eq} =$ resting length, $\sigma =$ "spring constant" ($R_{eq} = 0 \implies \vec{F} = \sigma \vec{X}_{ss}$ linear)

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- The membrane exerts a **singular force** on adjacent fluid particles:

$$\rho \left(\vec{u}_t + \vec{u} \cdot \nabla \vec{u} \right) = \mu \Delta \vec{u} - \nabla p + \int_\Gamma \vec{F}(s,t)\, \delta(\vec{x} - \vec{X}(s,t))\, ds$$

$$\nabla \cdot \vec{u} = 0$$

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- The membrane in turn moves at the fluid velocity – **no slip condition**:

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- All interactions are mediated by **delta functions**

Jump formulation

- Delta function terms can be eliminated by integrating the momentum equations across the immersed boundary:

$$[p] = \frac{\vec{F} \cdot \vec{n}}{|\vec{X}_s|}$$

$$\mu \vec{\tau} \cdot \left[\frac{\partial \vec{u}}{\partial n} \right] = -\frac{\vec{F} \cdot \vec{\tau}}{|\vec{X}_s|}$$

  where $[\cdot] = (\cdot)|_{\Gamma^+} - (\cdot)|_{\Gamma^-}$

- The **immersed interface method** uses these jump conditions to derive high-order corrections to difference stencils within cells adjacent to Γ
- The jump formulation is more useful for deriving analytical results e.g. **(JS & Wetton, 1995)**, **(Cortez et al., 2004)**

The numerical method

Basic idea:

- Mixed, Eulerian–Lagrangian approach:
  - fixed, rectangular fluid grid
  - moving points define the membrane location
- Delta functions are replaced by discrete approximations with finite support

$$\delta(\vec{x}) \approx d_h(x) \cdot d_h(y) \quad \text{where} \quad d_h(x) = \frac{1}{4h}\left(1 + \cos\frac{\pi x}{2h}\right), \quad |x| \leq 2h$$

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Original implementation of the IB method **(Peskin, 1977)**:

- Split-step projection method for Navier-Stokes equations **(Chorin)**
- For periodic BC's, pressure solver is an FFT
- Boundary evolution equation is handled explicitly – can be **very stiff!**

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Disadvantages:

- Limited to low Reynolds number (e.g., bio-fluid applications)
- Only first order accurate, second order for “smooth” boundaries **(Griffith & Peskin, 2004)**
- Significant **volume conservation errors**

. . . all of these drawbacks have been addressed (at least partially) by IB method variants appearing in recent years

Adding porosity

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- **Advantage:** This is trivial to implement in the IB method!

Exact solution in 2D

Consider a 2D, radially–symmetric, porous membrane with radius $r(t)$

- Equilibrium configuration is a circle of radius R_{eq}

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$$U_p(s,t) = \alpha\sigma\left(1 - \frac{R_{eq}}{r(t)}\right) \quad \Longrightarrow \quad \frac{dr}{dt} = -\alpha\sigma\left(1 - \frac{R_{eq}}{r(t)}\right)$$

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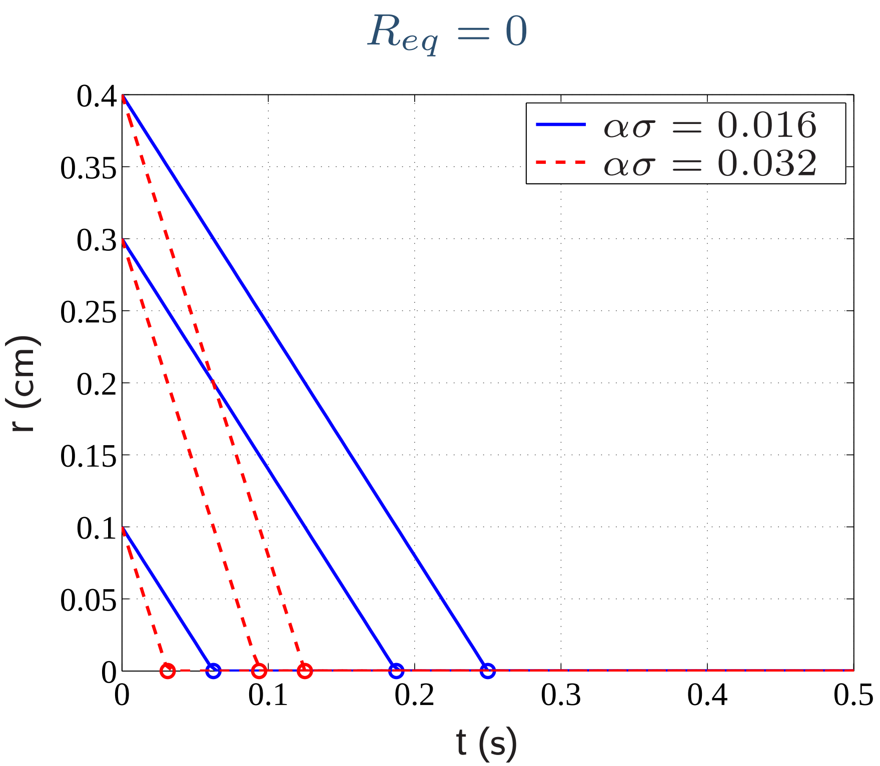
- Apply initial condition $r(0) = R_o$, and solve exactly:

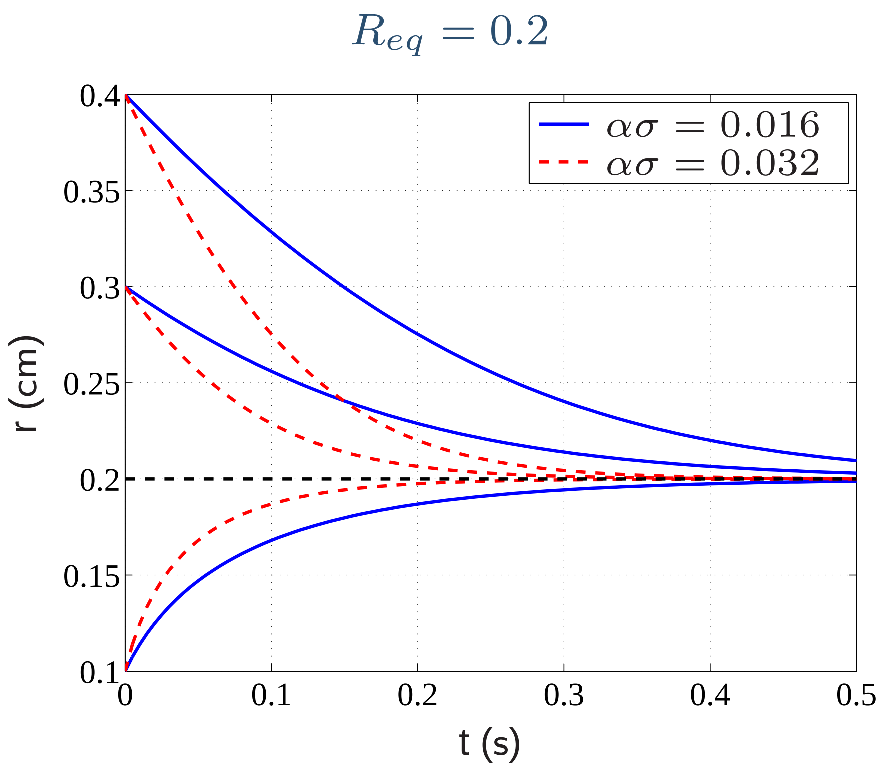
$$r(t) = \begin{cases} \max(0, R_o - \alpha\sigma t), & \text{if } R_{eq} = 0 \\ R_{eq}\left[1 + \boldsymbol{W}(m\exp[-\alpha\sigma t/R_{eq}])\right], & \text{if } R_{eq} > 0 \end{cases}$$

where $W(x)$ is the **Lambert W–function** satisfying $We^W = x$, and

$$m = \left(\frac{R_o}{R_{eq}} - 1\right)\exp\left(\frac{R_o}{R_{eq}} - 1\right)$$

Typical solutions





- Membrane velocity (rate of decay) increases for larger values of $\alpha\sigma$ (also for smaller R_{eq})
- The Lambert–W solution (for $R_{eq} > 0$) has an exponential character

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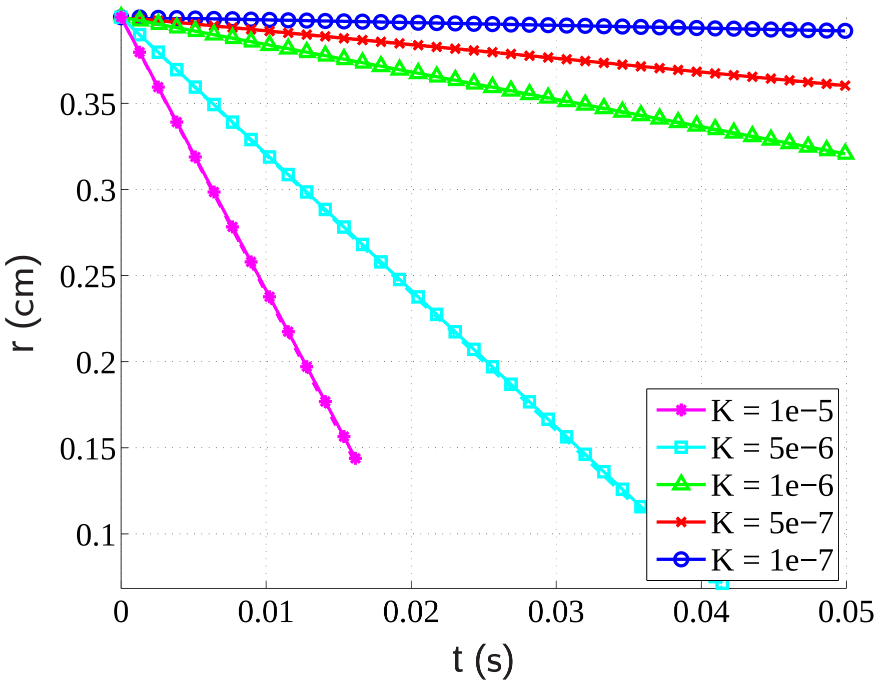
- This procedure can be automated!

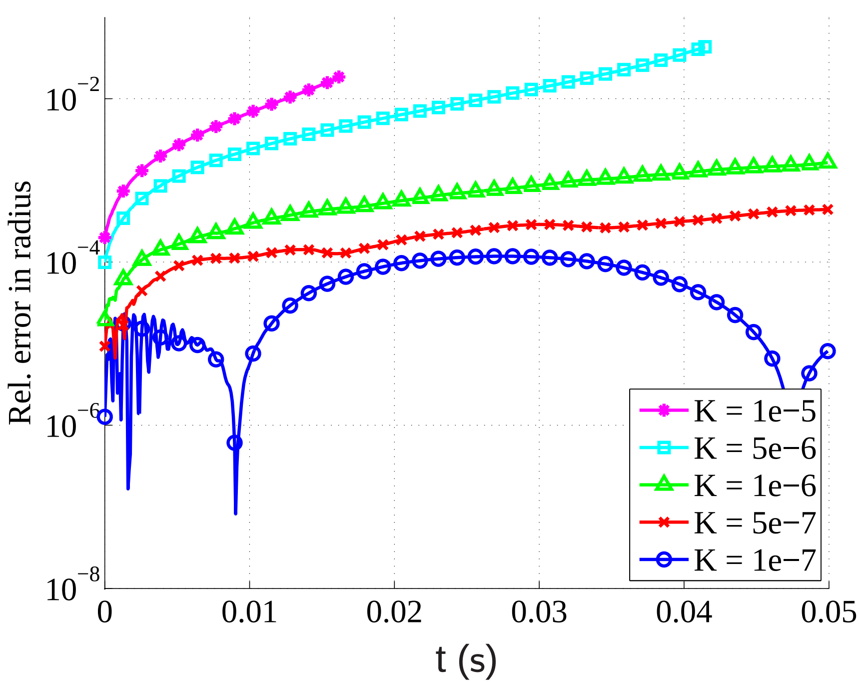
Numerical simulations: Circular membrane

- Initially circular membrane with $R_o = 0.4$, $R_{eq} = 0$, $\sigma = 10^5$
- Numerical parameters $N = 64$, $N_b = 200 \implies C \approx 0.794$

Average radius



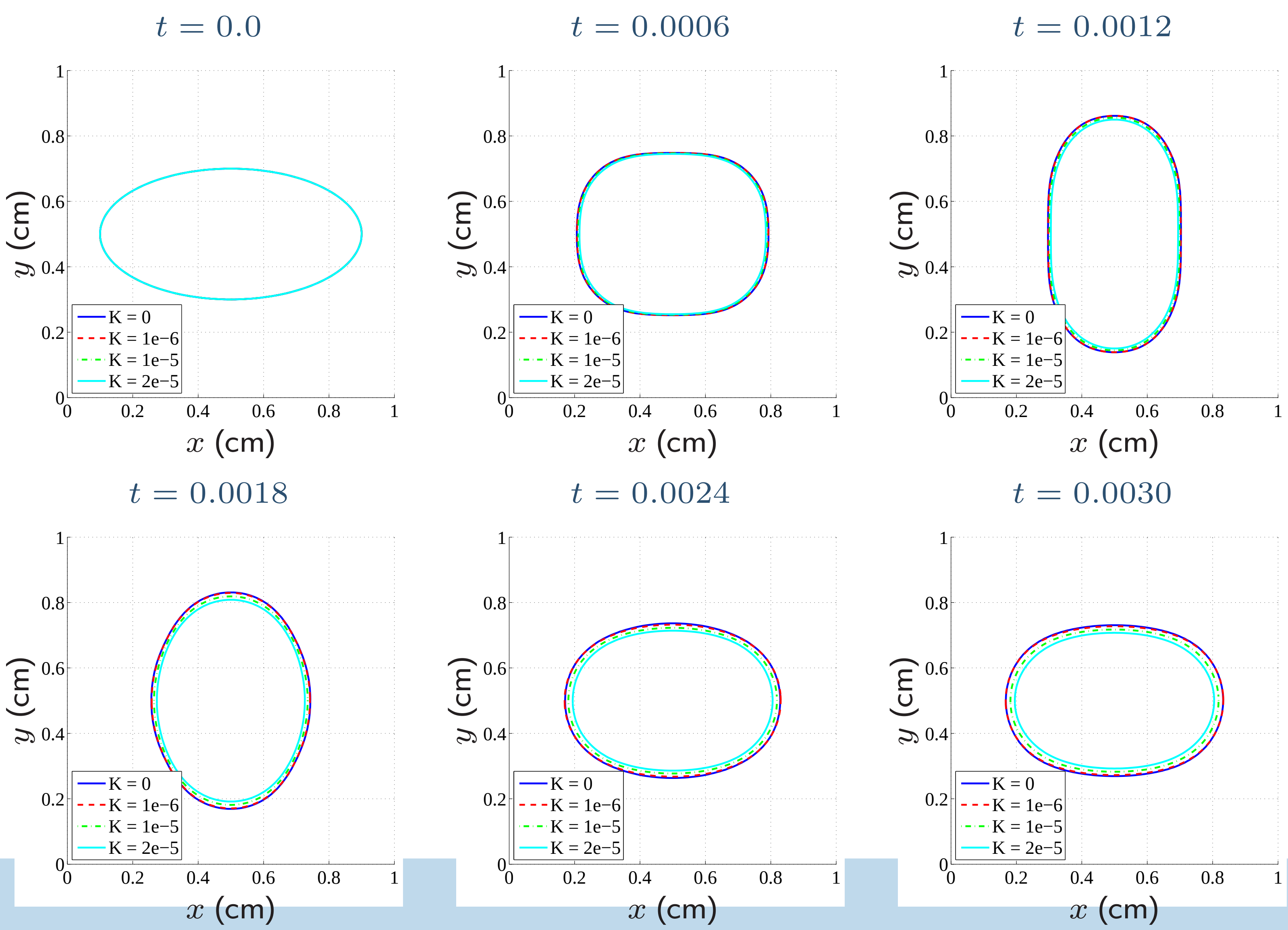
Relative error in radius



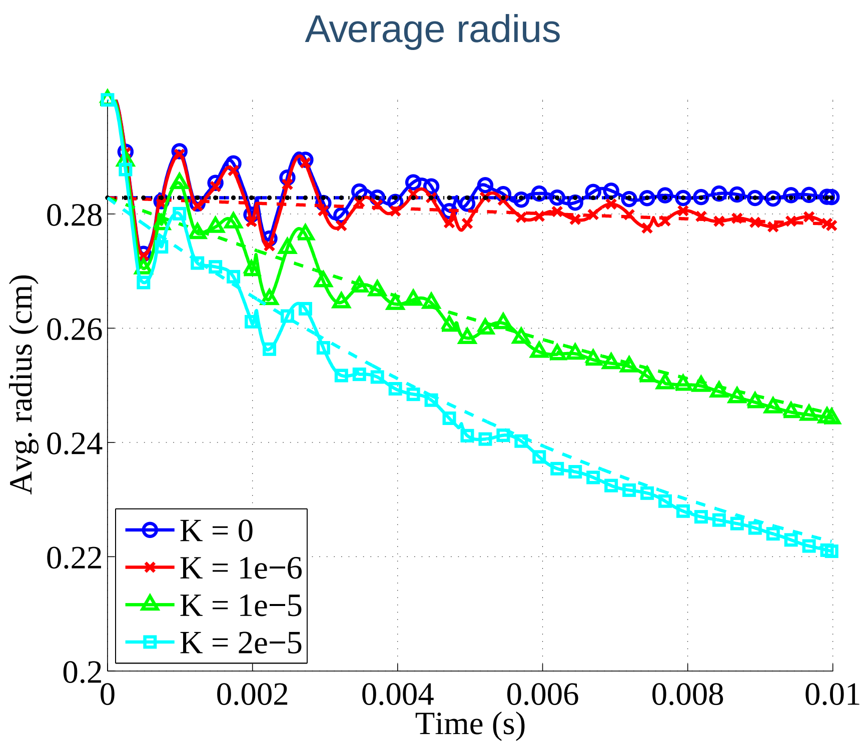
- Analytical solution (dashed lines, left) is indistinguishable from numerics!
- Error (right) increases with increased K, owing to larger velocities

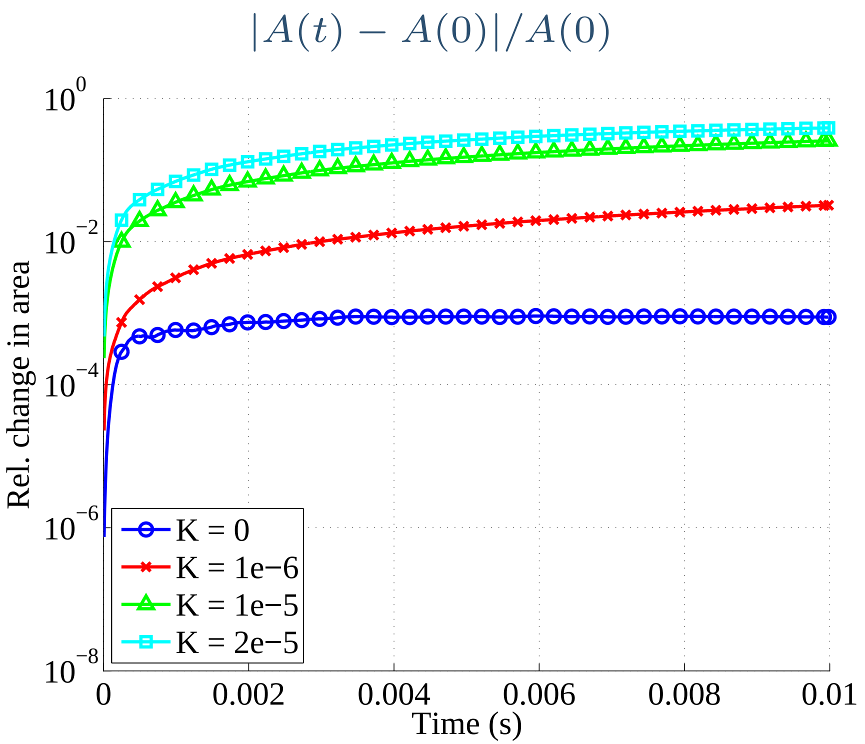
Numerical simulations: Elliptical membrane

- Initial membrane is an ellipse with $r_{max} = 0.4$ and $r_{min} = 0.2$
- Other parameters are $R_{eq} = 0.2$, $\sigma = 10^5$



- Ellipse simulations (solid lines) plotted alongside circular analytical solution with $R_{eq} = 0.2$ and $R_o = 0.2828$ (same area as initial ellipse)



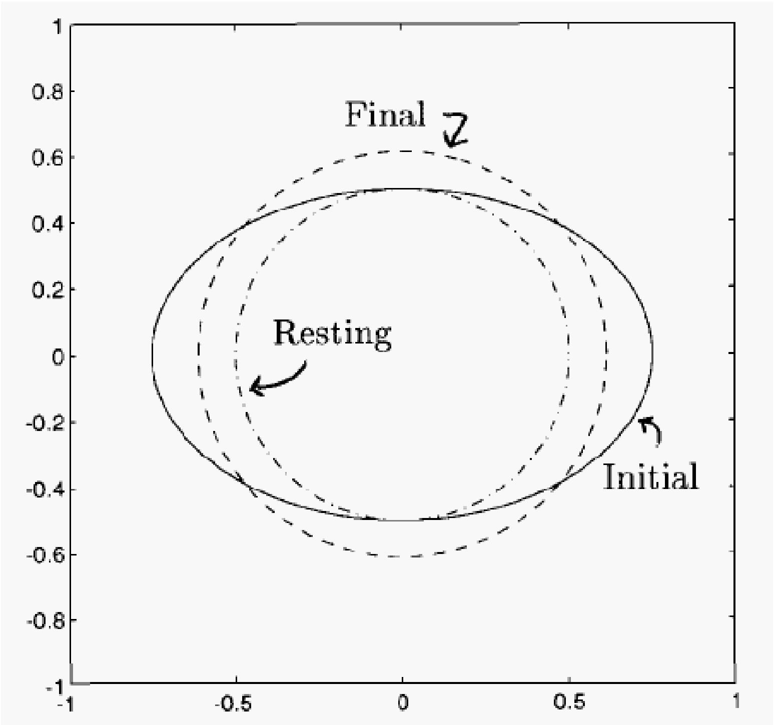


- As $K \to 0$, tends toward solution for non-porous membrane with circular equilibrium state $R_{eq} = \sqrt{r_{min} \cdot r_{max}} = 0.2828$

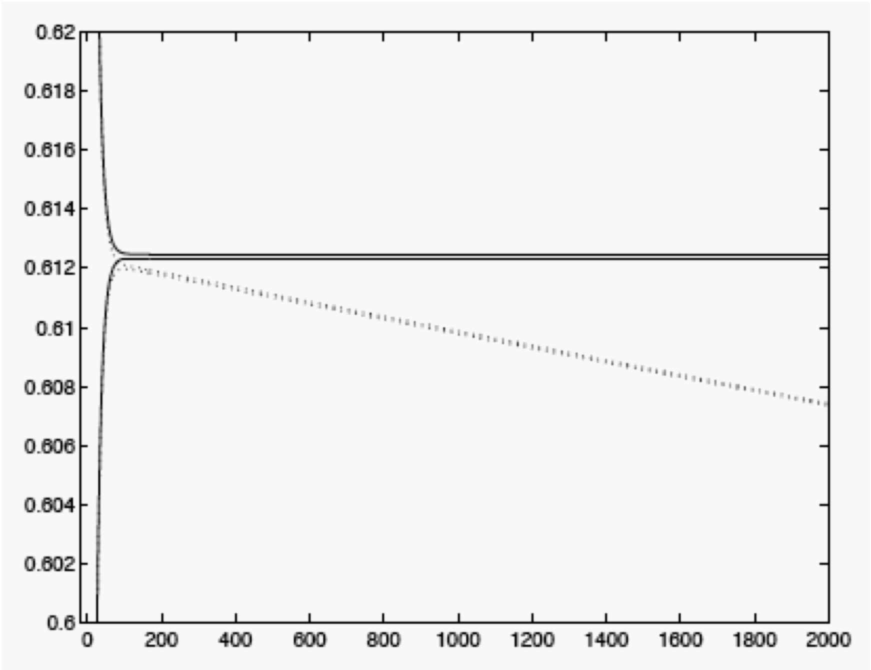
Volume conservation errors

- A closed, elastic membrane should conserve volume owing to incompressibility condition, **BUT** . . .
- A big drawback of the IB method is an inherent loss of volume

Example: LeVeque & Li, 1997



IB (dotted) vs. Immersed Interface (solid)



Time (s)

- Various attempts have been made to improve volume conservation:
  - Modified discrete divergence stencil **(Peskin & Printz, 1993)**
  - Immersed interface method **(LeVeque & Li, 1994)**
  - Blob projection method **(Cortez & Minion, 2000)**
  - Hybrid IB/immersed interface method **(Lee & LeVeque, 2003)**
  - Modified velocity interpolation **(Newren, 2007)**
- Volume loss has been attributed to various sources, but was **finally** correctly associated with errors in incompressibility from the velocity interpolation step **(Newren, 2007)**

Correction for volume loss

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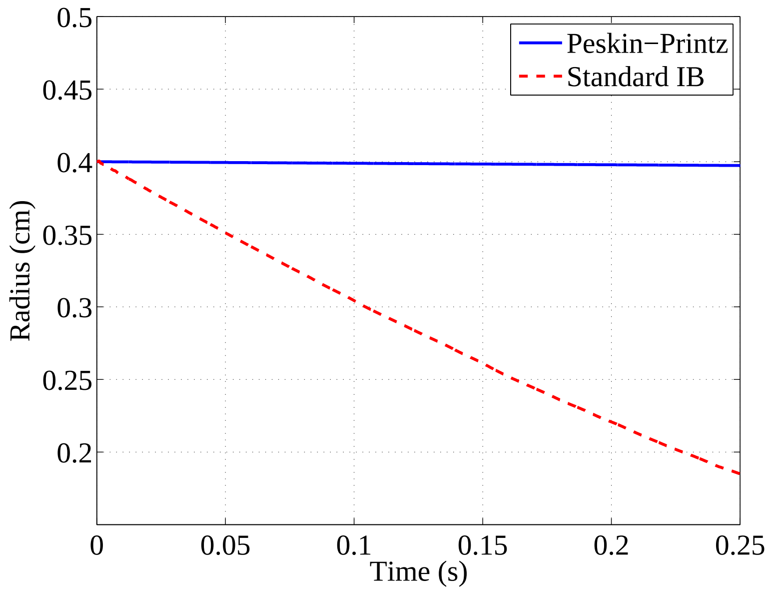
- One interpretation of these results is that discretization errors introduce an **intrinsic permeability** K_{int} into immersed boundaries
- **Idea:** An extra Darcy-like correction term might be used to minimize (or eliminate?) errors in volume conservation

Intrinsic permeability

A method for estimating intrinsic permeability K_{int} ...

- Simulate a non-porous circular membrane ($K = 0$)



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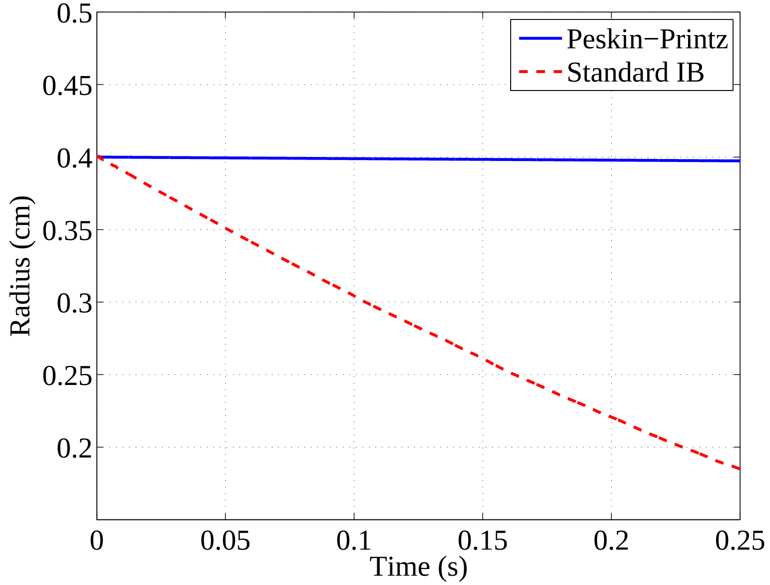

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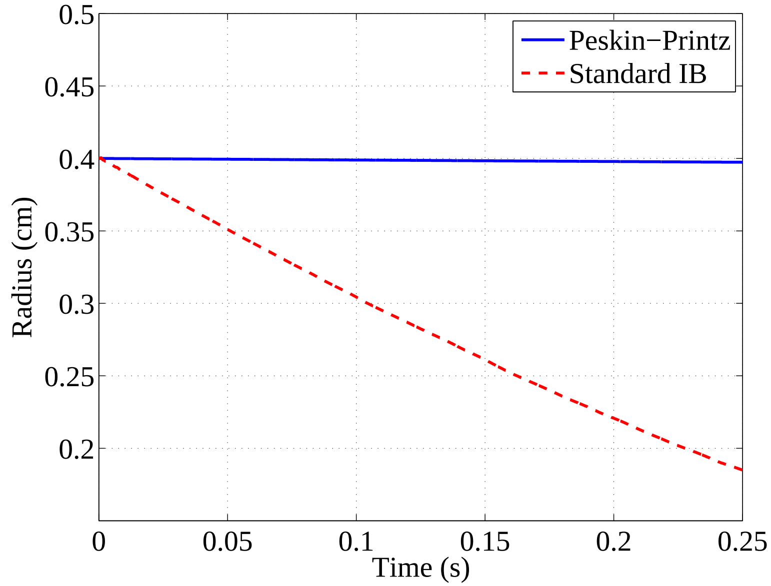

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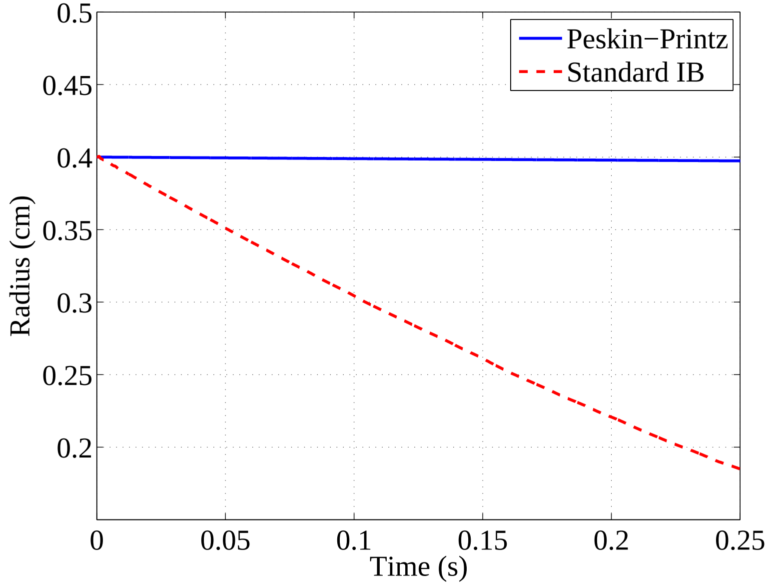

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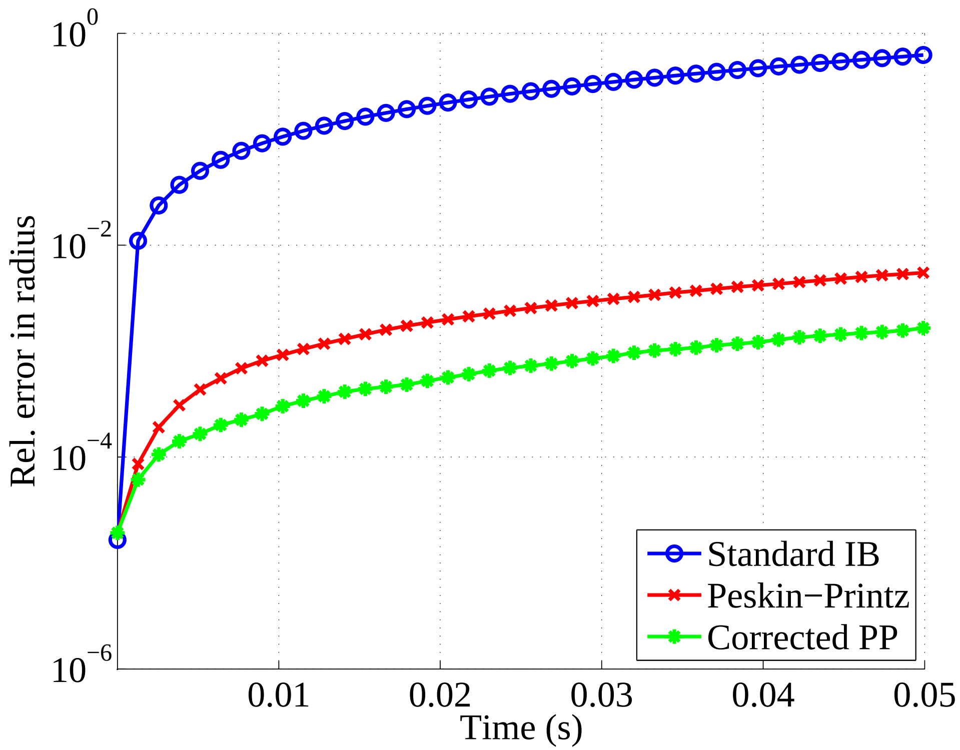
- Volume loss is only noticeable if $K \sim K_{int}$ or t is large enough

Numerical simulations: Circle revisited

- Consider a non-porous membrane ($K = 0$)
- Initial state is circular with $R_o = 0.4$, $R_{eq} = 0$, $\sigma = 10^5$
- Correct for volume loss using a porous slip velocity term with $K = -K_{int}$



- Similar results for porous membranes with corrected permeability $K - K_{int}$

Summary

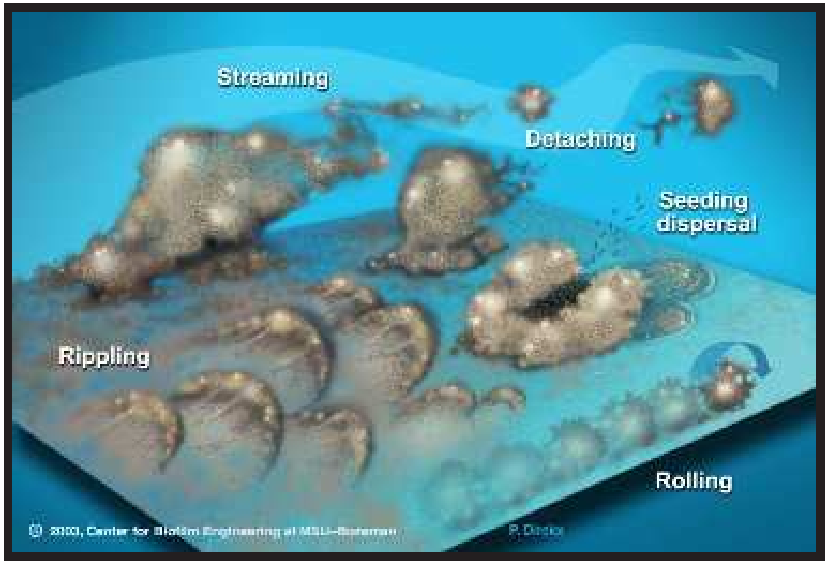
- Generalized the IB formulation to handle porous immersed boundaries, characterized by permeability K and membrane thickness a
- Trivial to implement in the IB method
- Derived a radially-symmetric analytical solution
- Numerical simulations match very well
- Demonstrated that volume loss appears as an intrinsic permeability K_{int}, and can be corrected by replacing K by $K - K_{int}$

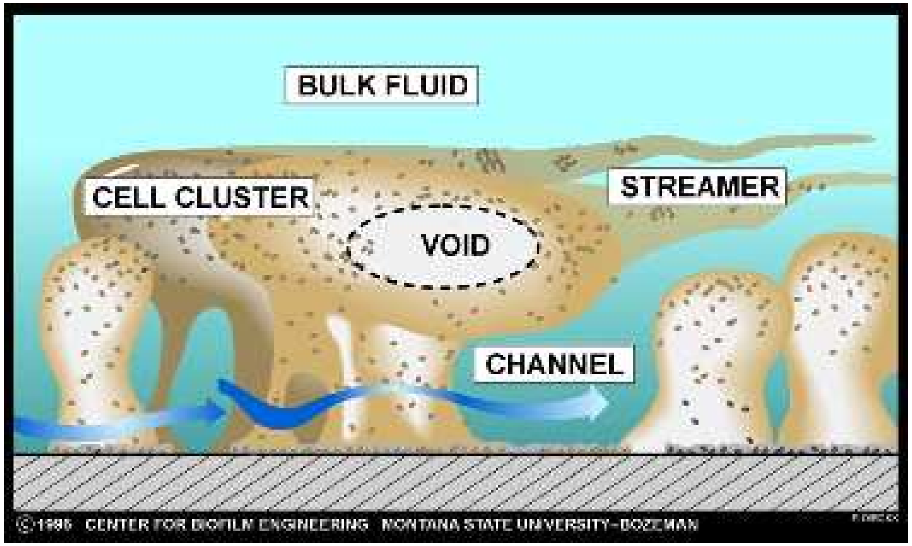
Future work

- A comprehensive study of volume loss and the proposed porous slip correction
- Investigate sensitivity of K_{int} and C to discretization and choice of δ_h
- Apply the method to deformable porous boundaries, e.g. porous wavemakers, biofilms ...

Future work: Biofilms

- Develop an IB model for deformation and detachment in food processing and wastewater treatment





- Porous flow within the biofilm is important for nutrient transport
- Two main differences from IB model considered earlier:
  - Biofilm is no longer an interface but rather a solid matrix, with a definite thickness
  - Biofilm is a non-Newtonian, visco/elasto-plastic material

Biofilms (cont'd)

Main differences:

- Immersed boundary parametrization: $\vec{F}(s,t) \;\rightarrow\; \vec{F}(\vec{q},t)$
- **Brinkman** extension to Navier-Stokes equations

$$\rho\left(\vec{u}_t + \vec{u}\cdot\nabla\vec{u}\right) = \mu\Delta\vec{u} - \nabla p + \underbrace{\iint_B \vec{F}_{IB}(\vec{q},t)\,\delta(\vec{x} - \vec{X}(\vec{q},t))\,d\vec{q}}_{\text{not singular}} - \frac{\mu}{K}\,\vec{u}$$

- Viscosity is solution-dependent, $\mu(\vec{u})$