Chapter 24: Lights Out

Turning out the lights with linear algebra:

Sample gameplay:

Matrix Model for puzzle:

Represent the state of each light by an element of \( F_2 = \{ 0, 1 \} \):
- off \( \leftrightarrow 0 \)
- on \( \leftrightarrow 1 \)

Assign a matrix in \( M_{5 \times 5}(F_2) \) to a configuration of lights:

Assign a "toggle" matrix to each move.
Sample game play written in terms of matrices:

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A bit button configuration is solvable iff

\[(*) \sum_{i,j} x_{i,j} T_{i,j} = B \quad \text{(matrix equation)}\]

for some \( x_{i,j} \in \mathbb{F}_2 = \{0,1\} \)

\([x_{i,j}]\) is called the strategy matrix. (written as a vector it is called a strategy vector)

\[
\mathbf{x} = (x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, \ldots, x_{55})
\]

5x5 matrix | dimension 25 vector
---|---
\(T_{i,j}\) | \(\mathbf{t}_{i,j}\)
\(B\) | \(\mathbf{b}\)

Eqn \((*)\) can be written as a BIG linear system:

\[(***) A \mathbf{x} = \mathbf{b} \quad \text{where} \quad A = [\mathbf{t}_{11}, \mathbf{t}_{12}, \ldots, \mathbf{t}_{55}]\]

(25 equations in 25 unknowns)

Solving lights out \(\iff\) solving \((***)\) for a strategy vector \(\mathbf{x}\)

We now need to determine the matrix \(A\).
Lights Out Matrix: \[ A = [\vec{e}_{i,1}, \vec{e}_{i,2}, \ldots, \vec{e}_{i,r}] \] where \( \vec{e}_{i,j} \) is toggle vector for button \((i,j)\).
A little twist:

We can use techniques from linear algebra to solve (**), but arithmetic must be done in $\mathbb{F}_2$:

\[
\begin{array}{c|cc}
+ & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 0 \\
\end{array}
\quad\quad
\begin{array}{c|cc}
\cdot & 0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 0 & 1 \\
\end{array}
\]

**Definition 24.2.1** A set $F$ with two operations $+$ and $\cdot$ satisfying the following properties for every $a, b, c \in F$ is called a field:

(a) Addition is commutative, $a + b = b + a$.
(b) Addition is associative, $a + (b + c) = (a + b) + c$.
(c) There is a unique element 0 (zero) in $F$ such that $a + 0 = a$.
(d) For each $a \in F$ there is a unique element $-a \in F$ such that $a + (-a) = 0$.
(e) Multiplication is commutative, $ab = ba$.
(f) Multiplication is associative, $a(bc) = (ab)c$.
(g) There is a unique element 1 (one) in $F$ such that $a1 = a$.
(h) For each non-zero $a \in F$ there is a unique element $a^{-1} \in F$ such that $aa^{-1} = 1$.
(i) Multiplication distributes over addition, $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.

All results in linear algebra hold when $\mathbb{R}$ is replaced by a field $F$.

Solving linear systems over $\mathbb{F}_2$:

\[
\begin{align*}
x_1 + x_2 &= 1 \\
x_2 + x_3 &= 0 \\
x_1 + x_3 &= 1
\end{align*}
\]

Augmented

Matrix
Sagemath demo:

In Sagemath, $\mathbb{F}_2$ is denoted $GF(2)$, called the Galois field of size 2.

Example: Solve the following configuration:

$$ \vec{b} = \ldots $$

Solution:

Solvable Configurations:

$\vec{b}$ is a solvable configuration $\iff A\vec{x} = \vec{b}$ has a solution.

Moreover,

$A\vec{x} = \vec{b}$ is solvable for all $\vec{b} \in \mathbb{F}_2^{15}$ $\iff$ $A$ is invertible $\iff \det A \neq 0$

For $5 \times 5$ lights out $\det A =$

For example, $\ldots$ is
How many configurations are solvable?

\[ \bar{b} \text{ solvable } \iff \]

The dimension of \( \text{col}(A) \) is \( \text{rank}(A) = \) so there are \[ |\text{col}(A)| = \] solvable configurations out of \( 2^{25} \) configurations.

**Theorem:** The probability a random configuration is solvable is

\[ 2^{25/2^r} = \frac{1}{4} . \]

The nullspace of \( A \) (solutions to \( A\bar{x} = \bar{0} \)) has dimension

\[ \text{nullity}(A) = \text{(Rank-Nullity theorem)} \]

so

\[ \text{nul}(A) = \text{span}_F( ) = \]

\[ \delta \quad \delta_1 \quad \delta_2 \quad \delta_1 + \delta_2 \]

**Optimal solution:**

If \( \bar{x} \) and \( \bar{y} \) are two strategy vectors for configuration \( \bar{b} \) then

\[ A\bar{x} = \bar{b} = A\bar{y} \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ \therefore \bar{x} + \text{Null}(A) = \{ \} \quad \quad \{ \} \]

are four different ways to solve \( \bar{b} \). Pick the one that has the least number of 1's, that is the optimal solution.
**Light Chasing**: A way to solve with some memorization.

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<tr>
<th>Lights on bottom row</th>
<th>Press these on top row</th>
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<tbody>
<tr>
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