ARTIN GROUPS AND LOCAL INDICABILITY

by

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Abstract

This thesis consists of two parts. The first part (chapters 1 and 2) consists of an introduction to theory of Coxeter groups and Artin groups. This material, for the most part, has been known for over thirty years, however, we do mention some recent developments where appropriate. In the second part (chapters 3-5) we present some new results concerning Artin groups of finite-type. In particular, we compute presentations for the commutator subgroups of the irreducible finite-type Artin groups, generalizing the work of Gorin and Lin [GL69] on the braid groups. Using these presentations we determine the local indicability of the irreducible finite-type Artin groups (except for F_4 which at this time remains undetermined). We end with a discussion of the current state of the right-orderability of the finite-type Artin groups.

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To my Mother and Father

Chapter 0
Introduction and Statement of Results

0.1 Introduction

A number of recent discoveries regarding the Artin braid groups $\mathfrak{B}_n$ complete a rather interesting story about the orderability[1] of these groups. These discoveries were as follows.

In 1969, Gorin and Lin [GL69], by computing presentations for the commutator subgroups $\mathfrak{B}'_n$ of the braid groups $\mathfrak{B}_n$, showed that $\mathfrak{B}'_3$ is a free group of rank 2, $\mathfrak{B}'_4$ is the semidirect product of two free groups (each of rank 2), and $\mathfrak{B}'_n$ is finitely generated and perfect for $n \geq 5$. It follows from these results that $\mathfrak{B}_n$ is locally indicable[2] if and only if $n = 2, 3,$ and 4.

Neuwirth in 1974 [Neu74], observed $\mathfrak{B}_n$ is *not* bi-orderable if $n \geq 3$. However, Patrick Dehornoy [Deh94] showed the braid groups are in fact right-orderable for all n. Furthermore, Dale Rolfsen and Jun Zhu [RZ98] proved (non-constructively[3]) that the subgroups $\mathcal{P}_n$ of pure braids are *bi-orderable*.

So, by this point in time (1998), the orderability of the braid and pure braid groups were known. What remained unknown was the relationship between a right-ordering on $\mathfrak{B}_n$ and a bi-ordering on $\mathcal{P}_n$. That is, does a right-ordering on $\mathfrak{B}_n$ restrict to a bi-ordering on $\mathcal{P}_n$?

This question was recently answered by Rolfsen and Rhemtulla [RR02]

[1] A group G is *right-orderable* if there exists a strict total ordering $<$ of its elements which is right-invariant: $g < h$ implies $gk < hk$ for all $g, h, k \in G$. If in addition $g < h$ implies $kg < kh$, the group is said to be *orderable*, or for emphasis, *bi-orderable*.

[2] A group G is *locally indicable* if for every nontrivial, finitely generated subgroup there exists a *nontrivial* homomorphism into $\mathbb{Z}$ (called an *indexing function*).

[3] Rolfsen and Djun Kim construct a bi-ordering on $\mathcal{P}_n$ in [KR02].

by determining the connection between local indicability and orderability. In particular, they showed that since the braid groups $\mathfrak{B}_n$ are not locally indicable for $n \geq 5$ a right-ordering on $\mathfrak{B}_n$ could not restrict to a bi-ordering on $\mathcal{P}_n$.[4]

This thesis is concerned with investigating whether these results on the braid groups extend to all finite-type Artin groups. In particular, we are concerned with determining the local indicability of the finite-type Artin groups.

0.2 Outline and Statement of Results

In Chapter 1 we give a quick yet thorough introduction to the theory of Coxeter groups.

In Chapter 2 we introduce Artin groups and develop their basic theory. Most of these results have been known for over thirty years, however, we do mention recent developments where appropriate.

The remaining chapters consist of recent and new results.

In Chapter 3 we follow the direction of Gorin and Lin and compute presentations of the commutator subgroups of the finite-type Artin groups. The results here are new (aside from the particular case of the braid groups which were done, of course, by Gorin and Lin).

In Chapter 4 we use these presentations to extend the results of Gorin and Lin on the braid groups to the class of finite-type Artin groups as follows.

Theorem 0.1 *The following are finitely generated and perfect:*

1. $\mathcal{A}'_{A_n}$ *for* $n \geq 4$,
2. $\mathcal{A}'_{B_n}$ *for* $n \geq 5$,
3. $\mathcal{A}'_{D_n}$ *for* $n \geq 5$,
4. $\mathcal{A}'_{E_n}$ *for* $n = 6, 7, 8$,
5. $\mathcal{A}'_{H_n}$ *for* $n = 3, 4$.

Hence, the corresponding Artin groups are not locally indicable.

[4] see theorem 5.7.

On the other hand, we show the remaining finite-type Artin groups *are* locally indicable (excluding the type F_4 which at this time remains undetermined) .

In Chapter 5 we discuss the orderability of the finite-type Artin groups. We show that in order to determine the right-orderability (bi-orderability) of the finite-type Artin groups it is sufficient to determine whether the positive Artin monoid is right-orderable (bi-orderable). Furthermore, we show that in order to prove *all* finite-type Artin groups are right-orderable it suffices to show the Artin group of type E_8 is right-orderable.

Chapter 1
Basic Theory of Coxeter Groups

The first comprehensive treatment of finite reflection groups was given by H.S.M. Coxeter in 1934. In [Cox34] he completely classified the groups and derived several of their properties, using mainly geometrical methods. He later included a discussion of the groups in his book *Regular Polytopes* [Cox63]. Another discussion, somewhat more algebraic in nature, was given by E. Witt in 1941 [Wit41]. A more general class of groups; the Coxeter groups, to which finite reflection groups belong, has since been studied in N. Bourbaki's chapters on Lie Groups and Lie Algebras [Bou72], [Bou02]. Another discussion appears in Humphrey's book *Reflection Groups and Coxeter Groups* [Hum72].

In this chapter we develop the theory of Coxeter groups with emphasis on the "root system" (following Deodhar [Deo82]). The approach we take here is precisely that of Humphreys [Hum72]. All of the results found in this chapter may be found in some form or another in Humphreys book , however, its inclusion here has primarily two purposes: (1) to make this thesis self contained for the convience of the reader and (2) to draw a comparison with the theory of Artin groups developed in chapter 2. The material has been reorganized and emphasis has been put on the parts of the theory we wish to compare with the theory of Artin groups.

1.1 Definition

Let S be a finite set. A **Coxeter matrix** over S is a matrix $M = (m_{ss'})_{s,s' \in S}$ indexed by the elements of S and satisfying

(a) $m_{ss} = 1$ if $s \in S$,

(b) $m_{ss'} = m_{s's} \in \{2, \ldots, \infty\}$ if $s, s' \in S$ and $s \neq s'$.

A Coxeter matrix $M = (m_{ss'})_{s,s' \in S}$ is usually represented by its **Coxeter graph** Γ. This is defined by the following data.

(a) S is the set of vertices of Γ.
(b) Two vertices $s, s' \in S$ are joined by an edge if $m_{ss'} \geq 3$.
(c) The edge joining two vertices $s, s' \in S$ is labelled by $m_{ss'}$ if $m_{ss'} \geq 4$.

The **Coxeter system** of type Γ (or M) is the pair (W, S) where W is the group having the presentation

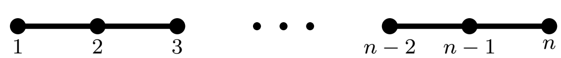
$$W = \langle s \in S : (ss')^{m_{ss'}} = 1 \text{ if } m_{ss'} < \infty \rangle.$$

The cardinality $|S|$ of S is called the **rank** of (W, S). The canonical image of S in W is a generating set which may conceivably be smaller than S, that is, under the above relations two generators in S may be equal in W. In 1.3 we show this does not happen. Furthermore, we show in theorem 1.14 that no proper subset of S generates W. In the meantime, we may allow ourselves to write $s \in W$ for the image of $s \in S$, whenever this creates no real ambiguity in the arguments. We refer to W itself as a **Coxeter group** of type Γ (or M), when the presentation is understood, and denote it by W_Γ. Although a good part of the theory goes through for arbitrary S, we shall always assume that S is finite. However, this does not mean that the Coxeter group W is finite.

Here are a couple of examples.

Example 1.1 *If $m_{ss'} = \infty$ when $s \neq s'$ then W is the free product of $|S|$ copies of $\mathbb{Z}/2\mathbb{Z}$. This group is sometimes referred to as a* **universal** *Coxeter group.*

Example 1.2 *It is well known that the symmetric group on $(n+1)$-letters is the Coxeter group associated with the Coxeter graph;*



where vertex i corresponds to the transposition $(i\; i+1)$.

When a group is given in terms of generators and relations it is quite difficult to say anything about the group – for example, is the group trivial or not? In our case it is quite easy to see that W has order at least 2. Consider the map from S into $\{\pm 1\}$, defined by taking each element of S to -1. Since this map takes each relation $(ss')^{m_{ss'}}$ to 1 it determines a homomorphism $\epsilon : W \longrightarrow \{\pm 1\}$ sending the image of each $s \in S$ to -1. The map ϵ is the generalization for an arbitrary Coxeter group of the sign character of the symmetric group.

Theorem 1.3 *There is a unique epimorphism $\epsilon : W \longrightarrow \{\pm 1\}$ sending each generator $s \in S$ to -1. In particular, each s has order 2 in W.*

Note that when $|S| = 1$, W is just a group of order 2, i.e. $\mathbb{Z}/2\mathbb{Z}$. When $|S| = 2$, say $S = \{s, s'\}$, W is the dihedral group of order $2m_{ss'} \leq \infty$.

1.2 Length Function

We saw that the generators $s \in S$ have order two in W, so each $w \neq 1$ in W can be written as a word in the generators with no negative exponents: $w = s_1 s_2 \cdots s_r$ for some s_i (not necessarily distinct) in S. If r is as small as possible we call it the **length** of w, written $l(w)$, and we call any expression of w as a product of r elements of S a **reduced expression**. By convention $l(1) = 0$. Note that if $s_1 s_2 \cdots s_r$ is a reduced expression then so are all initial segments, i.e. $s_1 s_2 \cdots s_i$, $i \leq r$. Some basic properties of the length function are included in the following lemma, whose proof is straightforward.

Lemma 1.4 *The length function l has the following properties:*

$(L1)$ $\quad l(w) = l(w^{-1})$,

$(L2)$ $\quad l(w) = 1$ *iff* $w \in S$,

$(L3)$ $\quad l(ww') \leq l(w) + l(w')$,

$(L4)$ $\quad l(ww') \geq l(w) - l(w')$,

$(L5)$ $\quad l(w) - 1 \leq l(ws) \leq l(w) + 1$, *for $s \in S$ and $w \in W$.*

Property $(L5)$ tells us that the difference in the lengths of ws and w is at most 1, the following theorem tells us that this difference is exactly 1.

Theorem 1.5 *The homomorphism $\epsilon : W \longrightarrow \{\pm 1\}$ of theorem 1.3 is given by $\epsilon(w) = (-1)^{l(w)}$. Thus, $l(ws) = l(w) \pm 1$, for all $s \in S$ and $w \in W$. Similarly for $l(sw)$.*

Proof. Let $w \in W$ have reduced expression $s_1 s_2 \cdots s_r$, then

$$\epsilon(w) = \epsilon(s_1)\epsilon(s_2)\cdots\epsilon(s_r) = (-1)^r = (-1)^{l(w)}.$$

Now $\epsilon(ws) = \epsilon(w)\epsilon(s) = -\epsilon(w)$ implies $l(ws) \neq l(w)$. □

In our study of Coxeter groups we will often use induction on $l(w)$ to prove theorems. It will therefore be essential to understand the precise relationship between $l(w)$ and $l(ws)$ (or $l(sw)$). It is clear that if $w \in W$ has a reduced expression ending in $s \in S$ then $l(ws) = l(w) - 1$, however it is not clear at this point whether the converse is true: for $w \in W$ and $s \in S$ if $l(ws) = l(w) - 1$ then w has a reduced expression ending in s. This turns out to be true, see section 1.5, but to prove this we need a way to represent W concretely.

1.3 Geometric Representation of W

Since Coxeter groups are generalizations of finite orthogonal reflection groups it should be no surprise that we wish to view W as a "reflection group" on some real vector-space V. It is too much to expect a faithful representation of W as a group generated by (orthogonal) reflections in a euclidean space. However, we can get a reasonable substitute if we redefine a **reflection** to be merely a linear transformation which fixes a hyperplane pointwise and sends some nonzero vector to its negative.

Define V to be the real vector space with basis $\{\alpha_s : s \in S\}$ in one-to-one correspondence with S. We impose a geometry on V in such a way that the "angle" between α_s and $\alpha_{s'}$ will be compatible with the given $m_{ss'}$. To do this, we define a symmetric bilinear form B on V by requiring

$$B(\alpha_s, \alpha_s') = -\cos\frac{\pi}{m_{ss'}}.$$

In the case of $m_{ss'} = \infty$ the expression is interpreted to be -1. From this definition we have $B(\alpha_s, \alpha_s) = 1$, while $B(\alpha_s, \alpha_s') \leq 0$ for $s \neq s'$. Note that B is not necessarily *positive definite*, i.e. there are Coxeter groups W for which some $v \in V$ does not satisfy $B(v, v) > 0$. Consider the following example.

Example 1.6 For the universal Coxeter group of rank two,

$$W = \langle s_1, s_2 : s_1^2, s_2^2 \rangle,$$

take $v = \alpha_{s_1} + \alpha_{s_2} \in V$. It is easy to check $B(\alpha_{s_1} + \alpha_{s_2}, \alpha_{s_1} + \alpha_{s_2}) = 0$.

Moreover, the following example shows that B may not even be *positive semidefinite*.

Example 1.7 For the Coxeter group

$$W = \langle s_1, s_2, s_3 : s_1^2, s_2^2, s_3^2, (s_1 s_2)^4, (s_1 s_3)^4, (s_2 s_3)^4 \rangle,$$

take $v = \alpha_{s_1} + \alpha_{s_2} + \alpha_{s_3} \in V$. Since $B(\alpha_{s_i}, \alpha_{s_j}) = -\cos \frac{\pi}{4} < -\frac{2}{3}$ for $i \neq j$, then $B(v, v) < -1$.

For each $s \in S$ we can now define a reflecton $\sigma_s : V \longrightarrow V$ by the rule:

$$\sigma_s(\lambda) = \lambda - 2B(\alpha_s, \lambda)\alpha_s.$$

Clearly $\sigma_s(\alpha_s) = -\alpha_s$, while σ_s fixes $H_s = \{\lambda \in V : B(\alpha_s, \lambda) = 0\}$ pointwise. In particular, we see that σ_s has order 2 in $GL(V)$.

Theorem 1.8 *There is a unique homomorphism $\sigma : W \longrightarrow GL(V)$ sending s to σ_s, and the group $\sigma(W)$ preserves the form B on V. Moreover, for each pair $s, s' \in S$, the order of ss' in W is precisely $m_{ss'}$.*

For a proof of this theorem see Humphreys [Hum72]. To avoid cumbersome notation, we usually write $w(\alpha_s)$ to denote $\sigma(w)(\alpha_s)$. The last statement in the theorem removes the possibility of $s = s'$ in W even though $s \neq s'$ in S, as promised in section 1.1. We will show next that this representation is indeed a faithful one. To do this we need to introduce the concept of a root system.

1.4 Root System

For a Coxeter system (W, S) a **root system** Φ of W is a set of vectors in V satisfying the conditions:

$$(R1) \qquad \Phi \cap \mathbb{R}\alpha = \{\pm\alpha\} \text{ for all } \alpha \in \Phi$$

$$(R2) \qquad s\Phi = \Phi \text{ for all } s \in S$$

The elements of Φ are called **roots**. We will only be concerned with the specific root system given by $\Phi = \{w(\alpha_s) : w \in W, s \in S\}$. It is clear that axiom $(R2)$ is satisfied for this choice of Φ, to check axiom $(R1)$ it suffices to note that since W (more precisely $\sigma(W)$) preserves the form B on V (theorem 1.8), Φ is a set of unit vectors. Note that $\Phi = -\Phi$ since if $\beta = w(\alpha_s) \in \Phi$ then $-\beta = ws(\alpha_s)$ is also in Φ. If α is any root then it can be expressed in the form

$$\alpha = \sum_{s \in S} c_s \alpha_s \quad (c_s \in \mathbb{R}).$$

If $c_s \geq 0$ for all $s \in S$ then we call α a **positive root** and write $\alpha > 0$. Similarly, if $c_s \leq 0$ for all $s \in S$ then we call α a **negative root** and write $\alpha < 0$. We write Φ^+ and Φ^- for the respective sets of positive and negative roots. It may come as some surprise that these two sets exhaust Φ, this follows from the following theorem. The proof of this theorem is nontrivial, we refer the reader to Humphreys [Hum72] for proof. The set of roots $\{\alpha_s : s \in S\}$ are called **simple roots** .

Theorem 1.9 *Let $w \in W$ and $s \in S$. Then*

$$l(ws) > l(w) \ \textit{iff} \ w(\alpha_s) > 0.$$

Equivalently,

$$l(ws) < l(w) \ \textit{iff} \ w(\alpha_s) < 0.$$

This tells us the precise criterion for $l(ws)$ to be greater than $l(w)$: w must take α_s to a positive root. This is the key to all further combinatorial properties of W relative to the generating set S.

Corollary 1.10 *The representation $\sigma : W \longrightarrow GL(V)$ is faithful.*

Proof. Let $w \in Ker(\sigma)$. If $w \neq 1$ then it has reduced expression $s_1 s_2 \cdots s_r$ where $r \geq 1$. Since $l(ws_r) = r - 1 < l(w)$ then $w(\alpha_{s_r}) < 0$ by theorem 1.9. But $w(\alpha_{s_r}) = \alpha_{s_r} > 0$, which is a contradiction. $\square$

Another consequence of Theorem 1.9 is that the length of $w \in W$ is completely determined by how it permutes Φ. For $w \in W$ let $\Pi(w)$ denote the set of positive roots sent to negative roots by w, i.e $\Pi(w) = \{\alpha \in \Phi^+ : w(\alpha) < 0\}$.

Theorem 1.11 *(a) If $s \in S$, then s sends α_s to its negative, but permutes the remaining positive roots. That is, $\Pi(s) = \{\alpha_s\}$.*
(b) For all $w \in W$, $l(w) = |\Pi(w)|$.

This theorem provides valuable information about the internal structure of W, see section 1.5. We refer the reader to Humphreys [Hum72] for the straightforward proof.

If W is infinite the length function takes on arbitrarily large values (recall we are assuming S is finite). It follows from theorem 1.11 that Φ is infinite. One the other hand, if W is finite (Φ is also finite by definition) it contains a unique element of maximal length . Indeed, clearly W must contain at least one element of maximal length, say w_0. For $s \in S$, $l(w_0 s) < l(w_0)$ so $w_0(\alpha_s) < 0$. Thus, w_0 sends all positive roots to negative roots, i.e. $\Pi(w_0) = \Phi^+$. Suppose that there is another element $w_1 \in W$ of maximal length, then w_1^{-1} is also of maximal length and so $\Pi(w_1^{-1}) = \Phi^+$. It follows that $w_0 w_1^{-1}(\Phi^+) = \Phi^+$, so $l(w_0 w_1^{-1}) = 0$. Therefore $w_0 = w_1$ so we have uniqueness. Since w_0 and w_0^{-1} have the same length uniqueness of the maximal element implies $w_0 = w_0^{-1}$, moreover it follows from theorem 1.11 that $l(w_0) = |\Phi^+|$.

1.5 Strong Exchange Condition

We are now in a position to prove some key facts about reduced expressions in W, which is at the heart of what it means to be a Coxeter group.

Theorem 1.12 *(**Exchange Condition**) Let $w = s_1 \cdots s_r$ ($s_i \in S$), not necessarily a reduced expression. Suppose a reflection $s \in S$ satisfies $l(ws) < l(w)$. Then there is an index i for which $ws = s_1 \cdots \widehat{s_i} \cdots s_r$ (omiting s_i). If the expression for w is reduced, then i is unique.*

There is a stronger version of this theorem, called the **Strong Exchange Condition** in which the simple reflection s can be replaced by any element $w \in W$ which acts on V as a reflection, in the sense that there exists a unit vector $\alpha \in V$ for which $w(\lambda) = \lambda - 2B(\lambda, \alpha)\alpha$. It turns out that the vector α must be a root for w to act on V in this way. On the other hand, to each positive root $\alpha \in \Phi^+$ there is a $w \in W$ which acts on V as a reflection along α. Indeed, take $w' \in W$, $s \in S$ such that $\alpha = w'(\alpha_s)$. Then $w = w's(w')^{-1}$ is such an element. Thus, there is a one-to-one correspondence between the set of positive roots Φ^+ and the set of reflections in W. For a complete discussion see Humphreys ([Hum72] sec. 5.7,5.8).

Before we prove theorem 1.12 we need to make the following observation. If $s, s' \in S$ and $w \in W$ satisfy $\alpha_{s'} = w(\alpha_s)$ then $wsw^{-1} = s'$. Indeed, $wsw^{-1}(\lambda) = w(w^{-1}(\lambda) - 2B(w^{-1}(\lambda), \alpha_s)\alpha_s)$ and since B is W-invariant the result follows.

Proof. Since $l(ws) < l(w)$ then $w(\alpha_s) < 0$. Because $\alpha_s > 0$ there exists an index $i \leq r$ for which $s_{i+1}\cdots s_r(\alpha_s) > 0$ but $s_i s_{i+1}\cdots s_r(\alpha_s) < 0$. From theorem 1.11 we have $s_{i+1}\cdots s_r(\alpha_s) = \alpha_{s_i}$, and by the above observation $s_{i+1}\cdots s_r s s_r \cdots s_{i+1} = s_i$, from which it follows $ws = s_1 \cdots \widehat{s_i} \cdots s_r$.

In case $l(w) = r$ consider what would happen if there were two distinct indices $i < j$ such that $ws = s_1 \cdots \widehat{s_i} \cdots s_r = s_1 \cdots \widehat{s_j} \cdots s_r$. After cancelling, this gives $s_{i+1}\cdots s_j = s_i \cdots s_{j-1}$, or $s_i \cdots s_j = s_{i+1}\cdots s_{j-1}$, allowing us to write $w = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_r$. This contradicts $l(w) = r$. $\square$

Corollary 1.13 *(a) (***Deletion Condition***) Suppose $w = s_1 \cdots s_r$ ($s_i \in S$), with $l(w) < r$. Then there exists $i < j$ such that $w = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_r$.*
(b) If $w = s_1 \cdots s_r$, ($s_i \in S$), then a reduced expression for w may be obtained by omitting on even number of s_i.

Proof. (a) There exists an index j such that $l(w's_j) < l(w')$ where $w' = s_1 \cdots s_{j-1}$. Applying the exchange condition gives $w's_j = s_1 \cdots \widehat{s_i} \cdots s_{j-1}$, allowing us to write $w = w's_j \cdots s_r = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_r$. $\square$

1.6 Parabolic Subgroups

In this section we show that for a Coxeter system (W, S) the subgroup of W generated by a subset of S is itself a Coxeter system with the obvious Coxeter graph.

Let (W, S) be a Coxeter system with values $m_{ss'}$ for $s, s' \in S$. For a subset $I \subset S$ we define W_I to be the subgroup of W generated by I. At the extremes, $W_\emptyset = 1$ and $W_S = W$. We call the subgroup W_I a **parabolic subgroup** . (More generally, we refer to any conjugate of such a subgroup as a parabolic subgroup.) Let l_I denote the length function on W_I in terms of the generators I.

Theorem 1.14 *(a) For each subset I of S, the pair (W_I, I) is a Coxeter system with the given values $m_{ss'}$.*
(b) Let $I \subset S$. If $w = s_1 \cdots s_r$ ($s_i \in S$) is a reduced expression, and $w \in W_I$, then all $s_i \in I$. In particular, the function l agrees with the length function l_I on W_I, and $W_I \cap S = I$.
(c) The assignment $I \longmapsto W_I$ defines a lattice isomorphism between the collection of subsets of S and the collection of subgroups W_I of W.
(d) S is a minimal generating set for W.

Proof. For (a). The set I and the corresponding values $m_{ss'}$ give rise to an abstractly defined Coxeter group $\overline{W}_I$, to which our previous results apply. In particular, $\overline{W}_I$ has a geometric representation of its own. This can obviously be identified with the action of the group generated by all σ_s ($s \in I$) on the subspace V_I of V spanned by all α_s ($s \in I$), since the bilinear form B restricted to V_I agrees with the form B_I defined by $\overline{W}_I$. The group generated by these σ_s is just the restriction to V_I of the group $\sigma(W_I)$. On the other hand, $\overline{W}_I$ maps canonically onto W_I, yielding a commutative triangle:

$$\begin{tikzcd} \overline{W}_I \arrow[rr] \arrow[dr] & & GL(V) \\ & W_I \arrow[ur] & \end{tikzcd}$$

Since the map $\overline{W}_I \longrightarrow GL(V_I)$ is injective by corollary 1.10, we conclude that W_I is isomorphic to $\overline{W}_I$ and is therefore itself a Coxeter group.

For (b), use induction on $l(w)$, noting that $l(1) = 0 = l_I(1)$. Suppose $w \neq 1$ and let $s = s_r$. Since $w \in W_I$ it also has a reduced expression $w = t_1 \cdots t_q$, where $t_i \in I$. Now,

$$w(\alpha_s) = \alpha_s + \sum_{i=1}^{q} c_i \alpha_{t_i} \quad (c_i \in \mathbb{R}).$$

According to theorem 1.9 $l(ws) < l(w)$ implies $w(\alpha_s) < 0$, so we must have $t_i = s$ for some i, forcing $s \in I$. Now, $ws = s_1 \cdots s_{r-1} \in W_I$, and the expression is reduced. The result follows by induction.

To prove (c), suppose $I, J \subset S$. If $W_I \subset W_J$, then, by (b), $I = W_I \cap S \subset W_J \cap S = J$ Thus $I \subset J$ (resp. $I = J$) if and only if $W_I \subset W_J$ (resp. $W_I = W_J$). It is clear that $W_{I \cup J}$ is the subgroup generated by W_I and W_J. On the other hand, (b) implies that $W_{I \cap J} = W_I \cap W_J$. This yields the desired lattice isomorphism. To prove (d), suppose that a subset I of S generates W then $W_I = W = W_S$, so by (c) $I = S$. □

If Γ is the Coxeter graph associated with the Coxeter system (W, S) then theorem 1.14 tells us that the Coxeter graph associated with (W_I, I) is precisely Γ_I: the subgraph induced by I, that is, the subgraph of Γ with vertex set I and all edges (from Γ) whose endpoints are in I. Another way to view this result is that every induced subgraph of Γ is a Coxeter graph for some (parabolic) subgroup of W.

We say that the Coxeter system (W, S) is **irreducible** if the Coxeter graph is connected. In general, let Γ_1, …, Γ_r be the connected components of Γ, and let I_i be the corresponding sets of generators from S, i.e. the verticies of Γ_i. Thus if $s \in I_i$ and $s' \in I_j$, we have $m_{ss'} = 2$ and therefore $ss' = s's$. The following theorem shows that the study of Coxeter groups can be largely reduced to the case when Γ is connected.

Theorem 1.15 *Let (W, S) have Coxeter graph Γ, with connected components Γ_1, $\ldots, \Gamma_r$, and let $I_1, \ldots, I_r$ be the corresponding subsets of S. Then*

$$W = W_{I_1} \oplus \cdots \oplus W_{I_r},$$

and each Coxeter system (W_{I_i}, I_i) is irreducible.

Proof. Since the elements of I_i commute with the elements of I_j, $i \neq j$, it is clear that the indicated parabolic subgroups centralize each other, hence that each is normal in W. Moreover, the product of these subgroups contains S and therefore must be all of W. According to theorem 1.14(c), for each $1 \leq i \leq r-1$, $(W_{I_1} W_{I_2} \ldots W_{I_i}) \cap W_{I_{i+1}} = \{1\}$. It follows that $W = W_{I_1} \oplus \cdots \oplus W_{I_r}$ (for example, see [Gal98]). □

1.7 The Word and Conjugacy Problem

Let a group G be given in terms of generators and relations.

(i) For an arbitrary word w in the generators, decide in a finite number of steps whether w defines the identity element of G, or not.

(ii) For two arbitrary words w_1, w_2 in the generators, decide in a finite number of steps, whether w_1 and w_2 define conjugate elements of G, or not.

The problems (i) and (ii) are called the **word problem** and the **conjugacy problem**, respectively, for the presentation defining G. It is shown in [Nov56], [Boo55] that there exist presentations of groups in which the word problem is not solvable, and there exist presentations of groups in which the conjugacy problem is not solvable [Nov54].

A very nice solution to the word problem for Coxeter groups was found by Tits [Tit69]. It allows one to transform an arbitrary product of generators from S into a reduced expression by making only the most obvious types of modifications coming from the defining relations. Here is a brief description.

Let F be a free group on a set Σ where Σ is in bijection with S, and let $\pi : F \longrightarrow W$ be the resulting epimorphism. The monoid F^+ generated by Σ already maps onto W. If $\omega \in F^+$ is a product of various elements $\sigma \in \Sigma$, we can define $l(\omega)$ to be the number of factors involved. If $m = m_{st}$ for $s, t \in S$, the product of m factors of σ and τ; $\sigma\tau\sigma\cdots$, maps to the same element of W as the product of m factors $\tau\sigma\tau\cdots$. Replacement of one of them by the other inside a given $\omega \in F^+$ is called an **elementary simplification of the first kind**; it leaves the length undisturbed. A **second kind** of elementary simplification reduces length, by omitting a consecutive pair $\sigma\sigma$. Write $\Sigma(\omega)$ for the set of all elements of F^+ obtainable from ω by a sequence of elementary simplifications. Since no new elements of Σ are introduced and length does not increase at each step, it is clear that $\Sigma(\omega)$ is finite. It is also effectively computable. Clearly the image of $\Sigma(\omega)$ under π is a single element of W.

Theorem 1.16 *Let $\omega, \omega' \in F^+$. Then $\pi(\omega) = \pi(\omega')$ iff $\Sigma(\omega) \cap \Sigma(\omega') \neq \emptyset$. In particular, $\pi(\omega) = 1$ iff $1 \in \Sigma(\omega)$.*

One direction is obvious. To go the other way, Tits assumes the contrary and analyses a minimal counterexample (in terms of lexicographic ordering of pairs (ω, ω')): both elements must have the same length and $\Sigma(\omega)$ consists of elements of equal length, etc., leading eventually to a contradiction.

Much less seems to be known about the conjugacy problem for Coxeter groups. Appel and Schupp [AS83] have given a solution for **extra large** Coxeter groups (those for which all $m_{ss'} \geq 4$ when $s \neq s'$.)

1.8 Finite Coxeter Groups

In this section we restrict our attention to finite Coxeter groups. We will classify all finite irreducible Coxeter groups in terms of their Coxeter graphs, in fact, we will give a complete list of all Coxeter graphs corresponding to finite irreducible Coxeter groups. According to theorem 1.15 every finite Coxeter group is isomorphic to a direct product of groups from this list.

Recall in 1.3 the bilinear form B was not necessarily positive definite, the next theorem tells us that it is precisly when W is finite.

Theorem 1.17 *The following conditions on the Coxeter group W are equivalent:*
(a) W is finite.
(b) The bilinear form B is positive definite.

The proof of this theorem is rather involved and so we refer the reader to Humphreys [Hum72].

If (W, S) is a Coxeter system with Coxeter graph Γ (resp. Coxeter matrix M) then we say that Γ (resp. M) is of **finite-type** if W is finite. Also, if the bilinear form B is positive definite then we call Γ **positive definite** as well. Theorem 1.17 tells us that Γ is positive definite if and only if it is of finite-type. Therefore, to classify the irreducible, finite Coxeter groups we just need to determine all connected, positive definite Coxeter graphs. Classification of all connected positive definite Coxeter graphs turns out to be relatively straightforward. For a wonderful discussion and solution of the problem see Humphreys ([Hum72] sec. $2.3 - 2.7$). It is shown in [Hum72] that the graphs in figure 1.1 are precisely all the connected positive definite Coxeter graphs.

The letter beside each of the graphs in figure 1.1 is called the **type** of the Coxeter graph, and the subscript denotes the number of vertices. Recall example 1.2 shows the symmetric group on (n+1)-letters is a Coxeter group of type A_n.

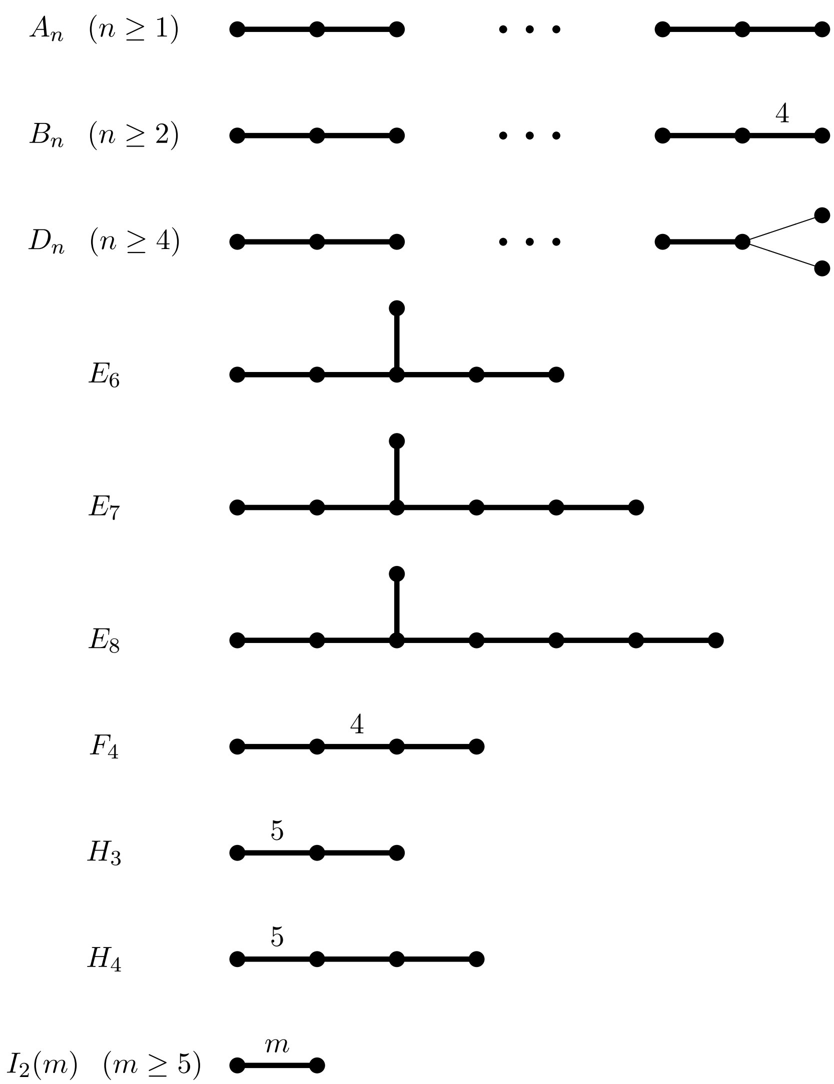


Figure 1.1: All the connected positive definite Coxeter graphs

W_Γ injects into $W_{\Gamma'}$	
Γ	Γ'
A_n	A_m (for $m \geq n$), B_m (for $m \geq n+1$), D_m (for $m \geq n+2$), E_8 (for $n \leq 7$), etc.
B_2	B_n (for $n \geq 2$), F_4, $I_2(4)$
B_3	B_n (for $n \geq 3$), F_4
E_6	E_7, E_8
E_7	E_8
H_3	H_4
$I_2(5)$	H_3, H_4

Table 1.1: Inclusions among Coxeter groups

The remarks after theorem 1.14 imply that if Γ is an induced subgraph of Γ' then the corresponding Coxeter group W_Γ injects into $W_{\Gamma'}$. Table 1.1 lists some such inclusions for the Coxeter graphs in figure 1.1.

Chapter 2
Basic Theory of Artin Groups

The braid groups, which are the Artin groups of type A_n, were first introduced by Artin in [Art25], he further developed the theory in [Art47a,b] and [Art50]. Since their introduction the braid groups have gone through a serious line of investigation. One of the most influential papers on the subject was that of Garside [Gar69], in which he solved the word and conjugation problems. Later, the connection of the braid groups with the fundamental group of a particular complex hyperplane arrangement lead to a natural generalization: the Artin groups. In this chapter we introduce the Artin groups and discuss some of their basic theory. We follow closely the work of Brieskorn and Saito [BS72], which is a generalization of the work of Garside.

2.1 Definition

Let M be a Coxeter matrix over S as described in section 1.1, and let Γ be the corresponding Coxeter graph. Fix a set Σ in one-to-one correspondence with S. In the following we will often consider words beginning with $a \in \Sigma$ and in which only letters a and b occur, such a word of length q is denoted $\langle ab\rangle^q$ so that

$$\langle ab\rangle^q = \underbrace{aba\ldots}_{\text{q factors}}$$

The **Artin system** of type Γ (or M) is the pair $(\mathcal{A}, \Sigma)$ where $\mathcal{A}$ is the group having presentation

$$\mathcal{A} = \langle a \in \Sigma : \langle ab\rangle^{m_{ab}} = \langle ba\rangle^{m_{ab}} \text{ if } m_{ab} < \infty\rangle.$$

The group $\mathcal{A}$ is called the **Artin group** of type Γ (or M), and is sometimes denoted by $\mathcal{A}_\Gamma$. So, similar to Coxeter systems, an Artin system is an Artin group with a prescribed set of generators.

There is a natural map $\nu : \mathcal{A}_\Gamma \longrightarrow W_\Gamma$ sending generator $a_i \in \Sigma$ to the corresponding generator $s_i \in \mathcal{S}$. This map is indeed a homomorphism since the equation $\langle s_i s_j \rangle^{m_{ij}} = \langle s_j s_i \rangle^{m_{ij}}$ follows from $s_i^2 = 1$, $s_j^2 = 1$ and $(s_i s_j)^{m_{ij}} = 1$. Since ν is clearly surjective it follows that the Coxeter group W_Γ is a quotient of the Artin group $\mathcal{A}_\Gamma$. The kernel of ν is called the **pure Artin group**, generalizing the definition of the *pure braid group*. From the observations in section 1.1 it follows that Σ is a minimal generating set for $\mathcal{A}_\Gamma$. The homomorphism ν has a natural set section $\tau : W_\Gamma \longrightarrow \mathcal{A}_\Gamma$ defined as follows. Let $w \in W$. We choose any reduced expression $w = s_1 \cdots s_r$ of w and we set

$$\tau(w) = a_1 \cdots a_r \in \mathcal{A}_\Gamma.$$

By Tits' solution to the word problem for Coxeter groups (sec. 1.7), the definition of $\tau(w)$ does not depend on the choice of the reduced expression of w. Note that τ is not a homomorphism.

The Artin group of a finite-type Coxeter graph is called an Artin group of **finite-type** . In other words, $\mathcal{A}_\Gamma$ is of finite-type if and only if the corresponding Coxeter group W_Γ is finite. An Artin group $\mathcal{A}_\Gamma$ is called **irreducible** if the Coxeter graph Γ is connected. In particular, the Artin groups corresponding to the graphs in figure 1.1 are irreducible and of finite-type. These Artin groups are our main interest in the remaining chapters.

2.2 Positive Artin Monoid

We now introduce the positive Artin monoid associated to the Artin system $(\mathcal{A}, \Sigma)$. All of the basic properties of Artin groups will follow from the study of the positive Artin monoid.

Let F_Σ be the free group generated by Σ and F_Σ^+ the free monoid generated by Σ inside F_Σ. We call the elements of F_Σ words and the elements of F_Σ^+ positive words. The positive words have unique representations as products of elements of Σ and the number of factors is the *length* l of a positive word. In the following we drop the subscript Σ when it is clear from the context. An

elementary transformation of positive words is a transformation of the form

$$U\langle ab\rangle^{m_{ab}}V \longrightarrow U\langle ba\rangle^{m_{ab}}V$$

where $U, V \in F^+$ and $a, b \in \Sigma$. A **positive transformation of length t** from a positive word U to a positive word V is a composition of t elementary transformations that begins with U and ends at V. Two words are **positive equivalent** if there is a positive transformation that takes one into the other. We indicate positive equivalence of U and V by $U =_p V$. Note, it follows from the definition that positive equivalent words have the same length. We use $=$ to denote equality in the group and $\equiv$ to express words which are equivalent letter by letter.

The monoid of positive equivalence classes of positive words relative to Γ (or M) is called the **positive Artin monoid** (or just the **Artin monoid**) and is denoted $\mathcal{A}_\Gamma^+$. The natural map $\mathcal{A}_\Gamma^+ \longrightarrow \mathcal{A}_\Gamma$ is a homomorphism. We will see that for Γ of finite-type this map is injective. Recently, Paris [Par01] has shown that for arbitrary Artin groups this map is injective.

2.3 Reduction Property

The main result in this section concerns the positive Artin monoid and it accounts for most of the results we will encounter in this chapter. The statement is as follow.

Lemma 2.1 (Reduction Property) *For each Coxeter graph we have the following rule: If X and Y are positive words and a and b are letters such that $aX =_p bY$ then m_{ab} is finite and there exists a positive word U such that*

$$X =_p \langle ba\rangle^{m_{ab}-1}U \quad \textit{and} \quad Y =_p \langle ab\rangle^{m_{ab}-1}U.$$

In other words, if $aX =_p bY$ then there is a positive transformation of the form

$$aX \longrightarrow \cdots \longrightarrow \langle ab\rangle^{m_{ab}}U \xrightarrow{elem.} \langle ba\rangle^{m_{ab}}U \longrightarrow \cdots \longrightarrow bY$$

taking aX to bY.

The proof of this is long and tedious, we refer the reader to [BS72] for proof.

An analogous statement holds for reduction on the right side. We see this as follows. For each positive word

$$U \equiv a_{i_1} \cdots a_{i_k}$$

define the positive word $\mathrm{rev}\, U$ by

$$\mathrm{rev}\, U \equiv a_{i_k} \cdots a_{i_1},$$

called the **reverse** or **reversal** of U. Clearly $U =_p V$ implies $\mathrm{rev}\, U =_p \mathrm{rev}\, V$ by the symmetry in the relations and the definition of elementary transformation. It is clear that the application of rev to the words in lemma 2.1 gives the right-hand analog.

It follows from the reduction propery that the positive Artin monoid is left and right cancellative.

Theorem 2.2 *If U, V and X, Y are positive words with $UXV =_p UYV$ then $X =_p Y$.*

Proof. It suffices to show that left cancellativity holds since right cancellativity follows by applying the reversal map rev . For U a word of length 1, say a, the reduction property implies that if $aX =_p aY$ then a word Z exists such that

$$X =_p \langle aa \rangle^{m_{aa}-1} Z \equiv Z \quad \text{and} \quad Y =_p \langle aa \rangle^{m_{aa}-1} Z \equiv Z.$$

Thus $X =_p Y$. The result follows by induction on the length of U. □

Let X, Y and Z be positve words. We say X **divides** Z (**on the left**) if

$$Z \equiv XY \quad \text{(if working in } F^+\text{)},$$
$$Z =_p XY \quad \text{(if working in } \mathcal{A}^+\text{)},$$

and write $X|Z$ (interpreted in the context of F^+ or $\mathcal{A}^+$).

The term *reduction property*, which comes from [BS72], is appropriate as this property (in conjunction with left cancellativity) allows the problem of whether a letter divides a given word to be reduced to the same problem for a word of shorter length. In the following section we describe a method to determine when a given word is divisible by a given generator.

2.4 Divisibility Theory

In this section we present an algorithm used to decide whether a given letter divides a positive word (in $\mathcal{A}^+$), and to determine the smallest common multiple of a letter and a word if it exists.

2.4.1 Chains

Let $a \in \Sigma$ be a letter. The simplest positive words which are not multiples of a are clearly those in which a does not appear, since a letter appearing in a word must appear in all positive equivalent words by the definition of elementary transformation and the nature of the defining relations. Further, the words of the form $\langle ba \rangle^q$ with $q < m_{ab}$ are also not divisible by a. This follows from the reduction property. Of course many other quite simple words have this property, for example concatenations of the previous types of words in specific order, called a-chains, which we will now define.

Let C be a non-empty word and let a and b be letters. We say C is a **primitive** a**-chain** with **source** a and **target** a if $m_{ac} = 2$ for all letters c in C. We call C an **elementary** a**-chain** if $C \equiv \langle ba \rangle^q$ for some $q < m_{ab}$. The **source** is a and the **target** is b if m_{ab} even and a if m_{ab} odd. An a**-chain** is a product $C \equiv C_1 \cdots C_k$ where for each $i = 1, \ldots, k$, C_i is a primitive or elementary a_i-chain for some $a_i \in \Sigma$, such that $a_1 = a$ and the target of C_i is the source of C_{i+1}. This may be expressed as:

$$a = a_1 \xrightarrow{C_1} a_2 \xrightarrow{C_2} a_3 \cdots \xrightarrow{C_{k-1}} a_k \xrightarrow{C_k} a_{k+1} = b,$$

The **source** of C is a and the **target** of C is the target of C_k. If this target is b then we say: C is a **chain from** a **to** b.

Example 2.3 *Let $\Sigma = \{a, b, c, d\}$ and M be defined by $m_{ac} = m_{ad} = m_{bd} = 2$, $m_{ab} = m_{bc} = 3$, $m_{cd} = 4$.*

- *c, d, cd^2c^7 are primitive a-chains with target a,*
- *b, ba are elementary a-chains with targets a and b, respectively*
- *a, ab, c, cb are elementary b-chains with targets b, a, b, c, respectively,*

The word

$$\underbrace{ab}_{C_1}\underbrace{cd}_{C_2}\underbrace{bc}_{C_3}\underbrace{ab}_{C_4}\underbrace{dcc}_{C_5}\underbrace{ba}_{C_6}$$

is a d-chain with target b, since C_1 is a primitive d-chain with target d, C_2 is an elementary d-chain with target c, C_3 is an elementry c-chain with target b, C_4 is an elementary b-chain with target a, C_5 is a primitive a-chain with target a, and finally C_6 is a simple a-chain with target b. The chain diagram for this example is:

$$d \xrightarrow{C_1} d \xrightarrow{C_2} c \xrightarrow{C_3} b \xrightarrow{C_4} a \xrightarrow{C_5} a \xrightarrow{C_6} b.$$

As the example 2.3 indicates there is a unique decomposition of a given a-chain into primitive and elementary factors if one demands that the primitive factors are a large as possible. The number of elementary factors is the **length** of the chain.

Remark. If C is a chain from a to b then $\operatorname{rev} C$ is a chain from b to a.

We have already noted that primitive and elementary a-chains are not divisible by a, the next lemma shows that this is also the case for a-chains.

Lemma 2.4 *Let $C = C_1 \cdots C_k$ be a chain from a to b (where C_i is a primitive or elementary chain from a_i to a_{i+1} for $i = 1, \ldots, k$) and D is a positive word such that a divides CD. Then b divides D, and in particular a does not divide C.*

Proof. We prove this by induction on k. Suppose $k = 1$.

Suppose $C = x_1 \cdots x_m$ is primitive, so $m_{ax_i} = 2$ for all i. Then $x_1 \cdots x_m D =_p aV$ for some positive word V. By the reduction property there exists a word U such that $x_2 \cdots x_m D =_p \langle ax_1 \rangle^{m_{ax_1}-1} U = aU$. Continuing in this way we get that a divides D, where a is the target of C.

Supppose $C = \langle ba \rangle^q$ is elementary, where $m_{ab} > 2$ and $0 < q < m_{ab}$. Then

$$\langle ba \rangle^q D =_p aV$$

for some positive word V. By the reduction property, $\langle ab \rangle^{q-1} D {=_p} \langle ab \rangle^{m_{a,b}-1} U$ for some positive word U. So by cancellation, theorem 2.2,

$$D =_p \begin{cases} \langle ab \rangle^{m_{ab}-q} U & \text{if } q \text{ is odd,} \\ \langle ba \rangle^{m_{ab}-q} U & \text{if } q \text{ is even.} \end{cases}$$

so D is divisible by a if q is odd, and b if q is even, which in each case is the target of C.

This begins the induction. Suppose now $k > 1$. By the inductive hypothesis a_k divides $C_k D$, and by the base case, $b \equiv a_{k+1}$ divides D.

The last claim follows by taking D equal to the empty word. □

Corollary 2.5 *If C is an a-chain such that a divides Cb, then b is the target of C.*

2.4.2 Chain Operators K_a

An arbitrary word will in general not be an a-chain, for any particular a, and so we need to know firstly whether, given an arbitrary word U, there exists an a-chain C which is positive equivlent to U, and secondly how to calculate it and its target. We define operators K_a for each generator a which take as input a word U and output either

- a word beginning with a if U is divisible by a, or
- an a-chain equivalent to U if U is not divisible by a.

K_a is called a **chain operator** (the K stands for Kette, German for chain).

To state the precise definition of K_a, we need some preliminary definitions and notation. We call a primitive a-chain of length one or an elementary a-chain a **simple a-chain**, that is, a simple a-chain is a word of the form $\langle ba\rangle^q$ where $q < m_{ab}$ (where $m_{ab} = 2$ is allowed). For a simple a-chain of the form $C = \langle ba\rangle^{m_{ab}-1}$ we call C **imminent** and let C^+ denote $\langle ab\rangle^{m_{ab}}$, so $C^+ =_p Cc$ where c is the target of C. If D is any positive nonempty word denote by D^- the word obtained by deleting the first letter of D. For every letter $a \in \Sigma$, we define a function

$$K_a : F^+ \longrightarrow F^+$$

recursively. Let U be a word. If U is empty, begins with a or is a simple a-chain then

$$K_a(U) :\equiv U.$$

Otherwise, write $U \equiv C_a D_a$ where C_a and D_a are non-empty words, and C_a is the largest prefix of W which is a simple a-chain, with target b, say. The rest of the definition of $K_a(U)$ is recursive on the lengths of U and D_a:

$$K_a(U) :\equiv \begin{cases} C_a K_b(D_a) & \text{if } K_b(D_a) \text{ does not begin with } b\text{; or} \\ C_a^+ K_b(D_a)^- & \text{if } C_a \text{ imminent and } K_b(D_a) \text{ begins with } b\text{; or} \\ K_a(C_a b K_b(D_a)^-) & \text{otherwise} \end{cases}$$

Observe that $K_a(U)$ is calculable.

Example 2.6 Computing $K_a(U)$. *Let Σ and M be as defined in example 2.3. First we will compute K_a of the word $U = bcbabdc$ (notice U is not an a-chain). By the recursive nature of the definition of K_a we first need to decompose U as follows:*

$$U = \underbrace{b}_{C_1} \cdot \underbrace{c}_{C_2} \cdot \underbrace{ba}_{C_3} \cdot \underbrace{bdc}_{D}$$

where C_1 is an a-chain with target a, C_2 is an a-chain with target a, and C_3 is an a-chain with target b. Since D begins with the letter b then $K_b(D) \equiv D$. Since C_3 is imminent, $K_a(C_3 \cdot D) \equiv C_3^+ D^- \equiv abadc$. Since C_2 is imminent, and $K_a(C_3 \cdot D)$ begins with the letter a,

$$\begin{aligned} K_a(C_2 \cdot C_3 D) &\equiv C_2^+ \cdot K_a(C_3 \cdot D)^- \\ &\equiv ac \cdot badc. \end{aligned}$$

Now $K_a(C_2 C_3 D)$ begins with a but C_1 is not imminent, so

$$\begin{aligned} K_a(U) &\equiv K_a(C_1 \cdot C_2 C_3 D) \\ &\equiv K_a(C_1 \cdot acbadc) \quad \text{since } K_a(C_2 C_3 D) \equiv acbadc \\ &\equiv K_a(ba \cdot cbadc) \quad \text{by definition of } K_a. \end{aligned}$$

Applying the definition of K_a to the word $bacbadc$ just returns the same word (try it!). Therefore,

$$K_a(U) \equiv bacbadc,$$

which can be seen to be an a-chain positive equivalent to U, with target d.

For our second example we will compute K_a of the word $W \equiv bacbacab$. Again we need to decompose W as follows:

$$W \equiv \underbrace{ba}_{C_1} \cdot \underbrace{cb}_{C_2} \cdot \underbrace{a}_{C_3} \cdot \underbrace{cab}_{D},$$

where C_1 is an a-chain with target b, C_2 is an b-chain with target c, and C_3 is a c-chain with target c. Since D begins with the letter c then $K_c(D) \equiv D$, so $K_c(C_3 D) \equiv C_3^+ D^- \equiv ca \cdot ab$. Since C_2 is imminent, $K_c(C_2 \cdot C_3 D) \equiv bcb \cdot aab$. Finally, since C_1 is imminent, $K_a(W) \equiv aba \cdot cbaab$.

Lemma 2.7 *Let U be positive and $a \in \Sigma$. Then*
(a) $K_a(U) =_p U$ and $K_a(U)$ is either empty, begins with a or is an a-chain,
(b) $K_a(U) \equiv U$ if and only if U is empty, begins with a, or is an a-chain,
(c) a divides U if and only if $K_a(U)$ begins with a.

Proof. (a) If U is empty, begins with a or is a simple a-chain then $K_a(U) \equiv U$ and we are done. Otherwise, write $U \equiv C_a D_a$ where C_a and D_a are nonempty and C_a is the longest prefix of U which is a simple a-chain. Let c denote the target of C_a. Since $l(D_a) < l(U)$ then by induction on length, $K_c(D) =_p D_a$ and $K_c(D)$ is either a c-chain or begins with c. If $K_c(D)$ is a c-chain then it cannot begin with c (lemma 2.4), so $K_a(U) \equiv C_a K_c(D_s)$ which is an a-chain, and moreover $K_a(U) =_p C_a D_a \equiv U$. Otherwise $K_c(D)$ begins with c. Considering first when C_a is imminent, we have $K_a(U) \equiv C_a^+ K_b(D_a)^-$, which begins with a, and moreover,

$$K_a(U) =_p C_a c K_c(D_a)^- \equiv C_a K_c(D) =_p C_a D_a \equiv U.$$

Otherwise se have $K_c(D_a) =_p D_a$, $K_c(D_a)$ begins with c and C_a is not imminent; so

$$K_a(U) \equiv K_a(C_a c K_c(D_a)^-).$$

Now $C_a c$ is a simple a-chain of length greater than the length of C_a so by another induction, $K_a(C_a c K_b(D_a))$ begins with a or is an a-chain, and

$$K_a(C_a c K_c(D_a)^-) =_p C_a c K_c(D_a)^- \equiv C_a K_c(D_a) =_p C_a D_a \equiv U.$$

(b) The direction ($\Rightarrow$) follows from (a). To see the other direction notice the result is clear if U is empty, begins with a or is a simple a-chain. Suppose U is a nonempty a-chain, so $U \equiv C_a D_a$ where C_a is a simple a-chain with target c, say and D_a is a c-chain. By induction since $l(D_a) < l(U)$,

$$K_c(D_a) \equiv D_a.$$

Since D_a is a c-chain it does not begin with the letter c thus by definition of K_a,

$$K_a(U) \equiv C_a K_c(D_a) \equiv C_a D_a \equiv U.$$

(c) This follows from (a) and lemma 2.4 □

2.4.3 Division Algorithm

Let U and V be words. We present an algorithm to determine whether U divides V (in $\mathcal{A}_\Gamma^+$) and in the case U divides V it returns the cofactor, i.e. the word X such that $V =_p UX$. This can be done relatively easily using the chain operators K_a.

Write $U \equiv a_1 \cdots a_k$. If U is to divide V then certainly a_1 must divide V, this can be determined by calculating $K_{a_1}(V)$ and checking if a_1 is the first letter. If a_1 is not the first letter then a_1, and hence U, cannot divide V. Otherwise, we have $K_{a_1}(V) \equiv a_1 K_{a_1}(V)^-$. If $U \equiv a_1 \cdots a_k$ were to divide $V =_p K_{a_1}(V) \equiv a_1 K_{a_1}(V)^-$ then it is necessary for a_2 to divide $K_{a_1}(V)^-$. This can be determined by checking the first letter of $K_{a_2}(K_{a_1}(V)^-)$. Continuing this way we either get that some a_i does not divide

$$K_{a_i}(K_{a_{i-1}} \cdots K_{a_2}(K_{a_1}(V)^-)^- \cdots)^-)$$

in which case U does not divide V, or a_i divides the above word for each $1 \le i \le k$, in which case U divides V and the cofactor X is

$$K_{a_k}(K_{a_{k-1}} \cdots K_{a_2}(K_{a_1}(V)^-)^- \cdots)^-)^-.$$

We reformulate the above observations into the following definition. Let U and V be words. If U is empty then define $(V : U) :\equiv V$. Otherwise write $U \equiv Wa$ for some word W and some letter a. We make the recursive definition:

$$(V : U) \equiv \begin{cases} \infty & \text{if } (V : W) = \infty \text{, or if} \\ & \quad K_a(V : W) \text{ does not begin with } a\text{; or} \\ K_a(V : W)^- & \text{otherwise.} \end{cases}$$

Some remarks on the definition.

1. By induction of the length of U, if X is any word then $(UX : U) \equiv X$.

2. Since $K_a(X)$ is calculable for any word X, then $(V : U)$ is also calculable, for any pair of words V and U. Thus the following result gives a solution to the division problem in $\mathcal{A}_\Gamma^+$.

Lemma 2.8 *The word U divides V precisely when $(V : U) \neq \infty$, in which case*

$$V =_p U(V : U).$$

Proof. If U is empty then the result clearly holds. so we may write $U \equiv Wa$ for some word W and some letter a. Suppose U divides V, so there is a word X such that $UX \equiv WaX =_p V$. By induction $(V : W) \neq \infty$ and $V =_p W(V : W)$. By cancellation, $aX =_p (V : W)$, so a divides $(V : W)$. By lemma 2.7, $K_a(V : W)$ begins with a, so $(V : U) \neq \infty$ and $(V : U) =_p X$.

On the other hand, suppose $(V : U) \neq \infty$. Then $(V : W) \neq \infty$, and in fact $K_a(V : W)$ has to begin with a. By induction $V =_p W(V : W)$, so

$$V =_p W(V : W) =_p WK_a(V : W) =_p WaK_a(V : W)^- =_p U(V : U)$$

by the definition of $(V : U)$. □

Since we have a solution to the division problem in $\mathcal{A}_\Gamma^+$ we get a solution to the word problem in $\mathcal{A}_\Gamma^+$ for free.

Corollary 2.9 *Two positive words U and V are positive equivalent precisely when $(V : U)$ is the empty word.*

In section 2.6 we will show how to use this to solve the word problem in finite-type Artin groups A_Γ.

2.4.4 Common Multiples and Divisors

Given a set of words $V_i \in \mathcal{A}_\Gamma^+$ where i runs over some indexing set I, a **common multiple** of $\{V_i : i \in I\}$ is a word $U \in \mathcal{A}_\Gamma^+$ such that every V_i divides U (on the left). A **least common multiple** is a common multiple which divides all other common multiples. If U and U' are both least common multiples then they divide one another, it follows by cancellativity and the fact that equivalent words have the same length that $U =_p U'$. Thus, when a common multiple exists, it is unique. By a **common divisor** of $\{V_i : i \in I\}$ we mean a word W which divides every V_i. A **greatest common divisor** of $\{V_i : i \in I\}$ is a common divisor into which all other common divisors divide. Similarly, greatest common divisors, when they exist, are unique.

With the help of the chain operators K_a defined in 2.4.2 we get a simple algorithm for producing a common multiple of a letter a and a word U, if one exists.

The essence of the method lies in lemma 2.4 which can be rewritten to say: *If C is an a-chain to b, and U is a common multiple of a and C then U is a common*

multiple of a and Cb.
Given an arbitrary word X, to calculate a common multiple with a generator a, we begin by applying K_a to X. If $K_a(X)$ begins with a then we are done (X is divisible by a and so itself is a common multiple of a and X). Otherwise, $K_a(X)$ is an a-chain, we determine its target b, and then concatenate it to get $K_a(X)b \equiv X'$. If $K_a(X')$ begins with a then we may stop; otherwise we repeat the process. If a common multiple exists, then the process will hault, producing a word which is in fact the *least* common multiple of a and X.

Let a be a letter and W a word. The **a-sequence** of W is a sequence $W_0^a, W_1^a, \ldots$ over F^+ defined as follows. Set $W_0^a :\equiv K_a(W)$, so by lemma 2.7, either W_0^a is empty, an a-chain or begins with a. Then for $i \geq 1$, define recursively

$$W_i^a :\equiv \begin{cases} a & \text{if } W_{i-1}^a \text{ is empty;} \\ W_{i-1}^a & \text{if } W_{i-1}^a \text{ begins with } a; \\ K_a(W_{i-1}^a b) & \text{if } W_{i-1}^a \text{ is an } a\text{-chain to b.} \end{cases}$$

By lemma 2.7, W_i^a is either an a-chain of begins with a (or if $i = 0$, W_i^a may be empty). The a-sequence converges to a word W_k^a precisely when W_k^a begins with a. The following definition is intended to capture a notion of the limit of the a-sequence of W.

$$L(a, W) :\equiv \begin{cases} W_k^a & \text{if } W_k^a \equiv W_{k+1}^a \text{ ; or} \\ \infty & \text{otherwise.} \end{cases}$$

The following example illustrates the way in which $L(a, W) \equiv \infty$

Example 2.10 *Let $\Sigma = \{a, b, c\}$ and M, the Coxeter matrix, be defined by $m_{ab} = m_{ac} = m_{bc} = 3$. (Note, by the results in 1.8 $\mathcal{A}_\Gamma$ is not of finite type.) Consider the word $W \equiv bc$. Observe that for any $k \geq 1$, $U_k \equiv (bacbac)^k$ is an a-chain with target a. The first member of the a-sequence of W is $W_0^a \equiv bc \equiv U_0 bc$, and then for all $k \geq 0$,*

$$\begin{array}{lll} W_{6k}^a \equiv U_k bc, & W_{6k+1}^a \equiv U_k bca, & W_{6k+2}^a \equiv U_k baca, \\ W_{6k+3}^a \equiv U_k bacab, & W_{6k+4}^a \equiv U_k bacbab, & W_{6k+5}^a \equiv U_k bacbabc, \end{array}$$

and so $W_{6k+6}^a \equiv U_k bacbacbc \equiv U_{k+1} bc$ and so on. Thus, the a-sequence never converges to a word, and so $L(a, bc) \equiv \infty$.

The following result characterizes the situation when $L(a, W) \neq \infty$.

Lemma 2.11 $L(a, W) \neq \infty$ *precisely when* a *and* W *have a common multiple, in which case* $L(a, W)$ *is a least common multiple of* a *and* W *begins with* a.

Proof. If W is empty then $W_0^a \equiv W$ and $W_i^a \equiv a$ for all $i \geq 1$. Thus $L(a, W) \equiv a$, and so the result holds trivially. So we may that suppose W is nonempty.

Suppose that a and W have a common multiple M. By lemma 2.7, we know that $W_0^a \equiv K_a(W) =_p W$ and so divides M. Since W is nonempty, W_0^a either begins with a or is an a-chain, is a multiple of W and divides M. We will show that the same is true of all W_i^a, using induction on i. Suppose that, for a given $i \geq 0$, W_i^a is a multiple of W and divides M. If W_i^a begins with a, then $W_j^a \equiv W_i^a$ for all $j \geq i$, and so we are done. Otherwise, W_i^a is an a-chain to b and, by lemma 2.4, M is a common multiple of $W_i^a b =_p K_a(W_i^a b) \equiv W_{i+1}^a$ and a. Since W divides W_i^a then W must also divide W_{i+1}^a. Thus we have shown that when a and W have a common multiple, every element of the a-sequence of W is a multiple of W, and divides M. Since elements of the a-sequence increase in length until an element begins with a, and since divisors of M cannot exceed M in length, eventually there is a first W_k^a which begins with a. Hence $L(a, W) \equiv W_k^a$. Futhermore, we have shown that $L(a, W)$ divides every common multiple M of a and W, making it a least common multiple.

On the other hand, suppose $L(a, W) \neq \infty$. Then there is a first number $k \geq 0$ such that W_k^a begins with a. If $k = 0$, then $L(a, W) \equiv W_0^a =_p W$. If $k > 0$ then by definition of the a-sequence, there are letters $b_1, \ldots, b_k$ which are targets of the a-chains $W_0^a, \ldots, W_{k-1}^a$, respectively, and for each $i < k$, $W_i^a b_{i+1} =_p W_{i+1}^a$, so $L(a, W) \equiv W_k^a =_p W_0^a b_1 \cdots b_k =_p W b_1 \cdots b_k$. hence $L(a, W)$ is a common multiple of a and W. □

Thus we have in $L(a, W)$ a calculator of least common multiples of a generator and a word. By repeated application of this operation, we can obtain least common multiples of arbitrary pairs of words.

Let V and W be words. Define recursively:

$$\text{Ł}(V,W) :\equiv \begin{cases} W & \text{if } V \text{ is empty; or} \\ aL(U, L(a,W)^-) & \text{if } V \equiv aU,\ L(a,W) \neq \infty \text{ and} \\ & L(U, L(a,W)^-) \neq \infty\text{; or} \\ \infty & \text{otherwise.} \end{cases}$$

Similar to lemma 2.11 we get the following lemma.

Lemma 2.12 *$L(V,W) \neq \infty$ precisely when V and W have a common multiple, in which case $L(V,W)$ begins with V and is a least common multiple of V and W. Moreover, $L(V,W) \neq \infty$ precisely when $L(W,V) \neq \infty$, in which case $L(V,W) =_p L(W,V)$.*

We can also compute the least common multiple of any finite collection of words by induction on the number of words. In particular, let $V_1, \ldots, V_m$ be words and let 1 denote the empty word. Define recursively:

$$\text{Ł}(V_1,\ldots,V_m) :\equiv \begin{cases} 1 & m = 0\text{; or} \\ V_1 & \text{if } m = 1\text{; or} \\ \infty & m \geq 2 \text{ and } L(V_2,\ldots,V_m) = \infty\text{; or} \\ L(V_1, L(V_2,\ldots,V_m)) & \text{if } m \geq 2 \text{ and } L(V_2,\ldots,V_m) \neq \infty. \end{cases}$$

The next result follows by induction on m using lemma 2.12.

Lemma 2.13 *$L(V_1,\ldots,V_m) \neq \infty$ precisely when $V_1,\ldots,V_m$ have a common multiple, in which case $L(V_1,\ldots,V_m)$ begins with V_1 and is a least common multiple of $V_1,\ldots,V_m$. Moreover, for any permutation σ of $\{1,\ldots m\}$, $L(V_1,\ldots,V_m) \neq \infty$ if and only if $L(V_{\sigma(1)},\ldots,V_{\sigma(m)}) \neq \infty$, in which case $L(V_1,\ldots,V_m) =_p L(V_{\sigma(1)},\ldots,V_{\sigma(m)})$.*

Corollary 2.14 *Let Ω be a finite set of words. Then Ω has a common multiple if and only if it has a least common multiple.* $\square$

Since Σ is finite then an infinite set of words in F^+ must have elements of arbitrary length. Since positive equivalent words have the same length it follows that a common multiple must be at least as long as any of the factors. So an infinite set of words can have no common multiples. On the other hand, the empty word divides every other word, so an arbitrary nonempty set Ω of words has a common divisor. If D denotes the set of all common divisors of Ω, then D is finite by the preceding discussion. Since every element of Ω is a comon multiple of D, then by corollary 2.14, D has a least common multiple, which is a greastest common divisor of Ω. Thus, greatest common divisors for nonempty sets of words always exist.

Remark. The only letters arising in the greatest common divisor and the least common multiple of a set of words are those occurring in the words themselves.

Proof. For the greatest common divisor it is clear, because in any pair of positive words exactly the same letters occur. For the least common multiple, recall how we found $L(a, W)$: $W_0^a \equiv K_a(W)$, and $W_{i+1}^a \equiv W_i^a$ if W_i^a starts with a, or $W_{i+1}^a \equiv K_a(W_i^a b)$ if W_i^a is an a-chain from a to b. But if $b \neq a$, then the only way we can have an a-chain from a to b is if there is an elementary subchain somewhere in the a-chain containing b. So W_{i+1}^a only contain letters which are already in W_i^a. □

2.4.5 Square-Free Positive Words

When a positive word U is of the form $U \equiv XaaY$ where X and Y are positive words and a is a letter then we say U has a **quadratic factor**. A word is **square-free** relative to a Coxeter graph Γ when U is not positive equivalent to a word with a quadratic factor. The image of a square-free word in A_Γ^+ is called **square-free**.

Lemma 2.15 *Let V be a positive word which is divisible by a and contains a square. Then there is a positive word $\widetilde{V}$ with $\widetilde{V} =_p V$ which contains a square and which begins with a. Thus, if W is a square-free positive word and a is a letter such that aW is not square free then a divides W.*

Proof. The proof is by induction on the length of V. Decompose V, as

$$V \equiv C_a(V) D_a(V)$$

where $C_a(V)$ and $D_a(V)$ are non-empty words, and $C_a(V)$ is the largest prefix of V which is a simple a-chain. Without loss of generality we may assume that V is a representative of its positive equivalence class which contains a square and is such that $l(C_a(V))$ is maximal.

When $C_a(V)$ is the empty word it follows naturally that $\widetilde{V} \equiv V$ satisfies the conditions for $\widetilde{V}$. For nonempty $C_a(V)$ we have two cases:

(i) $D_a(V)$ contains a square. By the induction assumption, one can assume, without loss of generality that $D_a(V)$ begins with the target of the simple a-chain $C_a(V)$. Thus, since the length of $C_a(V)$ is maximal, $C_a(V)$ is of the form $\langle ba\rangle^{m_{ab}-1}$. From this it follows that when $D_a(V)^-$ contains a square then $\widetilde{V} \equiv aC_a(V)D_a(V)^-$ satisfies the conditions for $\widetilde{V}$, and otherwise $\widetilde{V} \equiv a^2C_a(V)D_a(V)^{--}$ does.

(ii) Neither $C_a(V)$ nor $D_a(V)$ contains a square. Then V is of the form $V \equiv \langle ba\rangle^q D_a(V)$ where $q \geq 1$, and $D_a(V)$ begins with a if q is even, and b if q is odd. If q even then $\langle ba\rangle^q$ is a simple a-chain with target b so, by lemma 2.4, since a divides $\langle ba\rangle^q D_a(V)$, b divide $D_a(V)$. But $D_a(V)$ begins with a so by an application of the reduction property there exists E such that

$$D_a(V) =_p \langle ba\rangle^{m_{ab}} E.$$

Similarly, for q odd. Then

$$\begin{aligned} \widetilde{V} &\equiv a\langle ba\rangle^{m_{ab}-1}\langle ba\rangle^q E \quad \text{if } m_ab \text{ is even,} \\ \widetilde{V} &\equiv a\langle ba\rangle^{m_{ab}-1}\langle ab\rangle^q E \quad \text{if } m_ab \text{ is odd.} \end{aligned}$$

satisfies the conditions.

To prove the second statement, we have that there exists a positive word U, such that aU contains a square and $aW =_p aU$ from the first statement. It follow from cancellativity that $U =_p W$ and, since W is square free, that U does not contain a square. So U begins with a and W is divisible by a. □

From this lemma we get the following result concerning the a-sequence of a square-free word W, which will be needed in the next section.

Lemma 2.16 *If W is a square-free positive word and a is a letter then each word W_i^a in the a-sequence of W is also square-free.*

Proof. W_0^a is square-free since $W_0^a =_p W$. Assume W_i^a is square-free. Then either $W_{i+1}^a \equiv W_i^a$ or $W_{i+1}^a =_p W_i^a b_i$ where b_i is the target of the chain W_i^a. If $W_i^a b_i$ is not square-free then $b_i \mathrm{rev}\, W_i^a$ is not square-free and by lemma 2.15, the b_i-chain $\mathrm{rev}\, W_i^a$ is divisible by b_i, in contradiction to lemma 2.4. □

Let $QF\mathcal{A}_\Gamma^+$ be the set of square-free elements of $\mathcal{A}_\Gamma^+$. Consider the canonical map of $QF\mathcal{A}_\Gamma^+$ into the Coxeter group W_Γ defined by the composition of the canonical maps $\mathcal{A}_\Gamma^+ \longrightarrow \mathcal{A}_\Gamma \longrightarrow W_\Gamma$. It follows from theorem 3 of Tits [Tit69] that

$$QF\mathcal{A}_\Gamma^+ \longrightarrow W_\Gamma \quad \text{is bijective.}$$

Thus, $QF\mathcal{A}_\Gamma^+$ is finite precisely when $\mathcal{A}_\Gamma$ is of finite type (i.e. W_Γ is finite). This result is needed in the next section.

2.5 The Fundamental Element

Let M be a Coxeter matrix over Σ, and let $I \subset \Sigma$ such that the letters of I in $\mathcal{A}_\Gamma^+$ have a common multiple. Then the uniquely determined least common multiple (which exists by lemma 2.13) of the letters of I in $\mathcal{A}_\Gamma^+$ is called the **fundamental element** Δ_I for $I \in \mathcal{A}_\Gamma^+$.

The word "fundamental", introduced by Garside [Gar69], refers to the fundamental role which these elements play. It is shown in [BS72] that when $\mathcal{A}_\Gamma$ is irreducible (i.e. Γ connected) and there exists a fundamental element Δ_Σ, then Δ_Σ or Δ_Σ^2 generates the center of $\mathcal{A}_\Gamma$. The conditions for the existence of Δ_Σ are very strong and are outlined in the following two theorems, which appear in [BS72].

Theorem 2.17 *For a Coxeter graph Γ the following statements are equivalent:*
(i) There is a fundamental element Δ_Σ in $\mathcal{A}_\Gamma^+$.
(ii) Every finite subset of $\mathcal{A}_\Gamma^+$ has a least common multiple.
(iii) The canonical map $\mathcal{A}_\Gamma^+ \longrightarrow \mathcal{A}_\Gamma$ is injective, and for each $Z \in \mathcal{A}_\Gamma$ there exist $X, Y \in A_\Gamma^+$ with $Z = XY^{-1}$.
(iv) The canonical map $\mathcal{A}_\Gamma^+ \longrightarrow \mathcal{A}_\Gamma$ is injective, and for each $Z \in \mathcal{A}_\Gamma$ there exist $X, Y \in A_\Gamma^+$ with $Z = XY^{-1}$, where the image of Y lies in the center of $\mathcal{A}_\Gamma$.

Theorem 2.18 *Let Γ be a Coxeter graph. There exists a fundamental element Δ_Σ in $\mathcal{A}_\Gamma^+$ if and only if Γ is of finite-type (i.e. W_Γ is finite).*

To prove theorem 2.18 we need to recall the theorem of Tits we discussed at the end of section 2.4.5 on page 34: Γ is of finite-type if and only if $QF\mathcal{A}_\Gamma^+$ is finite. It is shown in [BS72] that every element of $QF\mathcal{A}_\Gamma^+$ divides Δ_Σ thus if Δ_Σ exists then $QF\mathcal{A}_\Gamma^+$ must be finite. To prove the converse suppose that Δ_Σ does not exist in $\mathcal{A}_\Gamma^+$. Let $J = \{a_1, \ldots, a_k\} \subset \Sigma$ be such that Δ_J exists but $\Delta_{J\cup\{a_{k+1}\}}$ does not exist (here we have assumed Σ has been ordered). Then the a_{k+1}-sequence of Δ_J does not terminate. Since Δ_J is square-free (see [BS72]) then by lemma 2.16 every element of the a_{k+1}-sequence of Δ_J is square free (and distinct). Thus $QF\mathcal{A}_\Gamma^+$ is infinite.

It is important to note that in theorem 2.17 the positive words X and Y such that $Z = XY^{-1}$ are calculable. This can be seen from the proof given in [BS72]. We use this fact in 2.6 to solve the word problem for finite-type Artin groups.

For a complete discussion on properties of the fundamental element see [BS72]. There it is shown that the image of the fundamental element of $\mathcal{A}_\Gamma^+$ in the Coxeter group W_Γ is precisely the longest element. Also they give formulae for the fundamental elements of irreducible finite-type Artin groups, i.e. the Artin groups corresponding to the Coxeter graphs in figure 1.1.

2.6 The Word and Conjugacy Problem

In this section we use the machinary developed thus far to give a quick solution to the word problem for finite-type Artin groups. The conjugacy problem is also discussed.

Let $U, V \in \mathcal{A}_\Gamma$, where Γ is of finite-type. We want to decide if $U = V$. By theorem 2.17 we know there exists (calculable) positive words $X_1, X_2, Y_1, Y_2 \in \mathcal{A}_\Gamma^+$ such that

$$U = X_1 Y_1^{-1} \quad \text{and} \quad V = X_2 Y_2^{-1}$$

where the images of Y_1 and Y_2 are central in $\mathcal{A}_\Gamma$. To decide if $U = V$ it is equivalent to decide if $X_1Y_2 = X_2Y_1$, but since the canonical map $\mathcal{A}_\Gamma^+ \longrightarrow \mathcal{A}_\Gamma$ is injective this is equivalent to deciding if $X_1Y_2 =_p X_2Y_1$. In 2.4.3 we gave a solution to the word problem for $\mathcal{A}_\Gamma^+$, thus a solution to the word problem for $\mathcal{A}_\Gamma$ follows.

In [BS72] it is shown elements of $\mathcal{A}_\Gamma^+$ and $\mathcal{A}_\Gamma$ can be put into a normal form using the fundamental element. This also gives a solution to the word problem in both $\mathcal{A}_\Gamma^+$ and $\mathcal{A}_\Gamma$. Brieskorn and Saito also give a solution to the conjugacy problem in finite type Artin groups.

Another solution to the word and conjugacy problems appears in [Cha92]. It is shown that finite-type Artin groups are *biautomatic* in which case they are known to have solvable word and conjugacy problems.

Some infinite-type Artin groups have been shown to have solvable word and conjugacy problems. Appel and Schupp [AS83] solve these problems for Artin groups of extra-large type (i.e. $m_{ab} \geq 4$ for all $a, b \in \Sigma$).

2.7 Parabolic Subgroups

Let $(\mathcal{A}_\Gamma, \Sigma)$ be an Artin system with values m_{ab} for $a, b \in \Sigma$. For a subset $I \subset \Sigma$ we define $\mathcal{A}_{\Gamma_I}$ to be the subgroup of $\mathcal{A}_\Gamma$ generated by I. We call the subgroup $\mathcal{A}_{\Gamma_I}$ a **parabolic subgroup**. (More generally, we refer to any conjugate of such a subgroup as a parabolic subgroup.)

Van der Lek [Lek83] has shown that for each $I \subset \Sigma$ the pair $(\mathcal{A}_{\Gamma_I}, I)$ is an Artin system associated with Γ_I. That is, parabolic subgroups of Artin groups are indeed Artin groups. A proof of this fact also appears in [Pa97]. Thus the inclusions among Coxeter groups in table 1.1 also hold for the associated Artin groups. Crisp [Cri99] shows quite a few more inclusions hold among the irreducible finite-type Artin groups. Table 2.1 lists these inclusions. Notice that every irreducible finite-type Artin group embeds into an Artin group of type A, D or E.

Similar to that of Coxeter groups we have that the study of Artin groups can be largely reduced to the case when Γ is connected.

Theorem 2.19 *Let $(\mathcal{A}_\Gamma, \Sigma)$ have Coxeter graph Γ, with connected components Γ_1, …, Γ_r, and let $I_1, \ldots, I_r$ be the corresponding subsets of Σ. Then*

$$\mathcal{A}_\Gamma = \mathcal{A}_{\Gamma_{I_1}} \oplus \cdots \oplus \mathcal{A}_{\Gamma_{I_r}},$$

and each Artin system $(\mathcal{A}_{\Gamma_{I_i}}, I_i)$ is irreducible.

Cohen and Wales [CW01] use this fact and the fact that irreducible finite type Artin groups embed into an Artin group of type A, D or E to show all Artin groups of finite-type are *linear* (have a faithful linear representation) by showing Artin groups of type D, and E are linear, thus generalizing the recent result that the braid groups (Artin groups of type A) are linear [Bi01], [Kr02].

$\mathcal{A}_\Gamma$ injects into $A_{\Gamma'}$	
Γ	Γ'
A_n	A_m $(m \geq n)$, B_{n+1} $(n \geq 2)$, D_{n+2}, E_6 $(1 \leq n \leq 5)$, E_7 $(1 \leq n \leq 6)$, E_8 $(1 \leq n \leq 7)$, F_4, H_3 $(1 \leq n \leq 2)$, H_4 $(1 \leq n \leq 3)$ $I_2(3)$ $(1 \leq n \leq 2)$
B_n	A_n, A_{2n-1}, A_{2n}, D_{n+1}
E_6	E_7, E_8
E_7	E_8
F_4	E_6, E_7, E_8
H_3	D_6
H_4	E_8
$I_2(m)$	A_{m-1}

Table 2.1: Inclusions among Artin groups

2.8 Geometric Realization of Artin Groups

In this section we discuss how finite-type Artin groups appear as fundamental groups of complex hyperplane arrangements. From this point of view we can see that finite-type Artin groups are torsion free.

Let (W_Γ, S) be a Coxeter system where W_Γ is finite and $|S| = n$. Let V be the associated (real) n-dimensional vector space, and B the bilinear form on V introduced in section 1.3. We know from theorem 1.17 that V is a Euclidean space. Let T denote the set of reflections in W. For each $t \in T$ let H_t denote the hyperplane in V (pointwise) fixed by t. Let $\mathcal{H} = \{H_t\}_{t\in T}$ be the collection of such hyperplanes. The complement of $\mathcal{H}$ in V is defined by

$$M(\mathcal{H}) = V \setminus \bigcup_{H\in\mathcal{H}} H.$$

Note that since V is a real vector space $M(\mathcal{H})$ is not connected. However, if we "complexify" V and the arrangement of hyperplanes $\mathcal{H}$ we get a connected space. This is done as follows. The *complexification* of V is $V_\mathbb{C} = \mathbb{C}^n$. The *complexification* of a hyperplane H is the hyperplane $H_\mathbb{C}$ of $V_\mathbb{C}$ having the same equation as H. The *complexification* of $\mathcal{H}$ is the arrangement $\mathcal{H}_\mathbb{C} = \{H_\mathbb{C} : H \in \mathcal{H}\}$ in $V_\mathbb{C}$. The topological space

$$M(\mathcal{H}_\mathbb{C}) = V_\mathbb{C} \setminus \bigcup_{H\in\mathcal{H}_\mathbb{C}} H.$$

is our primary interest.

Before we proceed any futher we need to make some definitions. A collection of hyperplanes $\mathcal{H}$ in a (real) vector space is called a (real) **arrangement of hyperplanes**. We say $\mathcal{H}$ is **central** if all the hyperplanes of $\mathcal{H}$ contain the origin. We say further that $\mathcal{H}$ is **essential** if the intersection of all the elements of $\mathcal{H}$ is $\{0\}$. Call $\mathcal{H}$ **simplicial** if it is central and essential, and if all the chambers of $\mathcal{H}$ (i.e. connected components of $V \setminus \bigcup_{H\in\mathcal{H}} H$) are cones over simplices. The following theorem indicates the importance of knowing an arrangement is simplicial.

Theorem 2.20 *(Deligne [Del72]). Let $\mathcal{H}$ be a simplicial arrangement of hyperplanes. Then $M(\mathcal{H}_\mathbb{C})$ is an Eilenberg-Maclane space (i.e. its universal cover is contractible).*

The importance of this theorem lies in the fact that if $M(G)$ is a finite dimensional Eilenberg-Maclean space for a group G then G has finite cohomological dimension and so, from a result in homological algebra, G is torsion-free.

Let us return now to our particular hyperplane arrangement $\mathcal{H}$ defined above. It follows from our work in chapter 1 that the arrangement of hyperplanes $\mathcal{H} = \{H_t\}_{t \in T}$ is central and essential. Futhermore, Deligne [Del72] showed that $\mathcal{H}$ is simplicial. Thus, it follows from theorem 2.20 that $M(\mathcal{H}_{\mathbb{C}})$ is an Eilenberg-Maclean space. Deligne has shown that the fundamental group of $M(\mathcal{H}_{\mathbb{C}})$ is precisely the pure Artin group associated with Γ. Moreover, Deligne showed that W_Γ acts freely on $M(\mathcal{H}_{\mathbb{C}})$ so that $M(\mathcal{H}_{\mathbb{C}})/W_\Gamma$ is also an Eilenberg-Maclean space and $\pi_1(M(\mathcal{H}_{\mathbb{C}})/W_\Gamma)$ is the Artin group $\mathcal{A}_\Gamma$. Thus, $\mathcal{A}_\Gamma$ is torsion-free.

For arbitrary Artin groups $\mathcal{A}_\Gamma$ (not necessarily of finite-type) more general constructions of $K(\mathcal{A}_\Gamma, 1)$-spaces have been done, for example see [CD95].

An algebraic argument showing finite-type Artin groups are torsion free was discovered by Dehornoy [Deh98]. The proof uses the divisibility theory we developed in this chapter.

Chapter 3

Commutator Subgroups of Finite-Type Artin Groups

Gorin and Lin [GL69] gave a presentation for the commutator subgroup $\mathfrak{B}'_n$ of the braid group $\mathfrak{B}_n$, $n \geq 3$, which showed $\mathfrak{B}'_n$ is finitely generated and perfect for $n \geq 5$. This has some interesting consequences concerning $\mathfrak{B}_n$ and "orderability", which we discuss in chapter 5. In this chapter we extend the work of Gorin and Lin and compute presentations for the commutator subgroups of all the other irreducible finite-type Artin groups; those corresponding to the Coxeter graphs in figure 1.1. This will be applied in chapter 4 to "local indicability" of finite-type Artin groups.

3.1 Reidemeister-Schreier Method

We will use the *Reidemeister-Schreier method* to compute the presentation for the commutator subgroups so we give a brief overview of this method in this section. For a complete discussion of the Reidemeister-Schreier method see [MKS76].

Let G be an arbitrary group with presentation $\langle a_1, \ldots, a_n : R_\mu(a_\nu), \ldots \rangle$ and H a subgroup of G. A system of words $\mathcal{R}$ in the generators $a_1, \ldots, a_n$ is called a **Schreier system** for G modulo H if (i) every right coset of H in G contains exactly one word of $\mathcal{R}$ (i.e. $\mathcal{R}$ forms a system of right coset representatives), (ii) for each word in $\mathcal{R}$ any initial segment is also in $\mathcal{R}$ (i.e. initial segments of right coset representatives are again right coset representatives). Such a Schreier system always exists, see for example [MKS76]. Suppose now that we have fixed a Schreier system $\mathcal{R}$. For each word W in the generators

$a_1, \ldots, a_n$ we let $\overline{W}$ denote the unique representative in $\mathcal{R}$ of the right coset HW. Denote

$$s_{K,a_v} = Ka_v \cdot \overline{Ka_v}^{-1}, \tag{3.1}$$

for each $K \in \mathcal{R}$ and generator a_v. A theorem of Reidemeister-Schreier (theorem 2.9 in [MKS76]) states that H has presentation

$$\langle s_{K,a_\nu}, \ldots : s_{M,a_\lambda}, \ldots, \tau(KR_\mu K^{-1}), \ldots \rangle \tag{3.2}$$

where K is an arbitrary Schreier representative, a_v is an arbitrary generator and R_μ is an arbitrary defining relator in the presentation of G, and M is a Schreier representative and a_λ a generator such that

$$Ma_\lambda \approx \overline{Ma_\lambda},$$

where $\approx$ means "freely equal", i.e. equal in the free group generated by $\{a_1, \ldots, a_n\}$. The function τ is a **Reidemeister rewriting function** and is defined according to the rule

$$\tau(a_{i_1}^{\epsilon_1} \cdots a_{i_p}^{\epsilon_p}) = s_{K_{i_1},a_{i_1}}^{\epsilon_1} \cdots s_{K_{i_p},a_{i_p}}^{\epsilon_1} \tag{3.3}$$

where $K_{i_j} = \overline{a_{i_1}^{\epsilon_1} \cdots a_{i_{j-1}}^{\epsilon_{j-1}}}$, if $\epsilon_j = 1$, and $K_{i_j} = \overline{a_{i_1}^{\epsilon_1} \cdots a_{i_j}^{\epsilon_j}}$, if $\epsilon_j = -1$. It should be noted that computation of $\tau(U)$ can be carried out by replacing a symbol a_v^ϵ of U by the appropriate s-symbol s_{K,a_ν}^ϵ. The main property of a Reidemeister rewriting function is that for an element $U \in H$ given in terms of the generators a_ν the word $\tau(U)$ is the same element of H rewritten in terms of the generators s_{K,a_ν}.

3.2 A Characterization of the Commutator Subgroups

The **commutator subgroup** G' of a group G is the subgroup generated by the elements $[g_1, g_2] := g_1 g_2 g_1^{-1} g_2^{-1}$ for all $g_1, g_2 \in G$. Such elements are called **commutators**. It is an elementary fact in group theory that G' is a normal subgroup in G and the quotient group G/G' is abelian. In fact, for any normal subgroup $N \lhd G$ the quotient group G/N is abelian if and only if $G' < N$. If G

is given in terms of a presentation $\langle \mathcal{G} : \mathcal{R} \rangle$ where $\mathcal{G}$ is a set of generators and $\mathcal{R}$ is a set of relations, then a presentation for G/G' is obtained by *abelianizing* the presentation for G, that is, by adding relations $gh = hg$ for all $g, h \in \mathcal{G}$. This is denoted by $\langle \mathcal{G} : \mathcal{R} \rangle_{\mathrm{Ab}}$.

Let $U \in \mathcal{A}_\Gamma$, and write $U = a_{i_1}^{\epsilon_1} \cdots a_{i_r}^{\epsilon_r}$, where $\epsilon_i = \pm 1$. The **degree of** U is defined to be

$$\deg(U) := \textstyle\sum_{j=1}^{r} \epsilon_j.$$

Since each defining relator in the presentation for $\mathcal{A}_\Gamma$ has degree equal to zero the map $\deg$ is a well defined homomorphism from $\mathcal{A}_\Gamma$ into $\mathbb{Z}$. Let $\mathcal{Z}_\Gamma$ denote the kernel of $\deg$; $\mathcal{Z}_\Gamma = \{U \in \mathcal{A}_\Gamma : \deg(U) = 0\}$. It is a well known fact that for the braid group (i.e. $\Gamma = A_n$) $\mathcal{Z}_{A_n}$ is precisely the commutator subgroup. In this section we generalize this fact for all Artin groups.

Let Γ_{odd} denote the graph obtained from Γ by removing all the even-labelled edges and the edges labelled ∞. The following theorem tells us exactly when the commutator subgroup $\mathcal{A}'_\Gamma$ is equal to $\mathcal{Z}_\Gamma$.

Theorem 3.1 *For an Artin group $\mathcal{A}_\Gamma$, Γ_{odd} is connected if and only if the commutator subgroup $\mathcal{A}'_\Gamma$ is equal to $\mathcal{Z}_\Gamma$.*

Proof. For the direction ($\Longrightarrow$) the hypothesis implies

$$\mathcal{A}_\Gamma/\mathcal{A}'_\Gamma \simeq \mathbb{Z}.$$

Indeed, start with any generator a_i, for any other generator a_j there is a path from a_i to a_j in Γ_{odd}:

$$a_i = a_{i_i} \longrightarrow a_{i_2} \longrightarrow \cdots \longrightarrow a_{i_m} = a_j.$$

Since $m_{i_k i_{k+1}}$ is odd the relation

$$\langle a_{i_k} a_{i_{k+1}} \rangle^{m_{i_k i_{k+1}}} = \langle a_{i_{k+1}} a_{i_k} \rangle^{m_{i_k i_{k+1}}}$$

becomes $a_{i_k} = a_{i_{k+1}}$ in $\mathcal{A}_\Gamma/\mathcal{A}'_\Gamma$. Hence, $a_i = a_j$ in $\mathcal{A}_\Gamma/\mathcal{A}'_\Gamma$. It follows that,

$$\begin{aligned} \mathcal{A}_\Gamma/\mathcal{A}'_\Gamma &\simeq \langle a_1, \ldots, a_n : a_1 = \cdots = a_n \rangle \\ &\simeq \mathbb{Z}, \end{aligned}$$

where the isomorphism $\phi : \mathcal{A}_\Gamma/\mathcal{A}'_\Gamma \longrightarrow \mathbb{Z}$ is given by

$$U\mathcal{A}'_\Gamma \longmapsto \deg(U).$$

Therefore, $\mathcal{A}'_\Gamma = \ker\phi = \mathcal{Z}_\Gamma$.

We leave the proof of the other direction to theorem 3.2, where a more general result is stated. □

For the case when Γ_{odd} is not connected we can get a more general description of $\mathcal{A}'_\Gamma$ as follows. Let Γ_{odd} have m connected components; $\Gamma_{odd} = \Gamma_1 \sqcup \ldots \sqcup \Gamma_m$. Let $\Sigma_i \subset \Sigma$ be the corresponding sets of vertices. For each $1 \leq k \leq m$ define the map

$$\deg_k : \mathcal{A}_\Gamma \longrightarrow \mathbb{Z}$$

as follows: If $U = a_{i_1}^{\epsilon_1} \cdots a_{i_r}^{\epsilon_r} \in \mathcal{A}_\Gamma$ take

$$\deg_k(U) = \sum_{1 \leq j \leq r \text{ where } a_{i_j} \in \Sigma_k} \epsilon_j.$$

It is straight forward to check that for each $1 \leq k \leq m$ the map $\deg_k$ agrees on $\langle ab \rangle^{m_{ab}}$ and $\langle ba \rangle^{m_{ab}}$ for all $a, b \in \Sigma$. Hence, $\deg_k : \mathcal{A}_\Gamma \longrightarrow \mathbb{Z}$ is a homomorphism for each $1 \leq k \leq m$. Let

$$\mathcal{Z}_\Gamma{}^{(m)} := \bigcap_{1 \leq k \leq m} \ker(\mathrm{deg_k}).$$

The following theorem tells us that this is precisely the commutator subgroup of $\mathcal{A}_\Gamma$.

Theorem 3.2 *Let Γ be a Coxeter graph such that Γ_{odd} has m connected components. Then $\mathcal{A}'_\Gamma = \mathcal{Z}_\Gamma{}^{(m)}$.*

Proof. Clearly $\mathcal{A}'_\Gamma \subset \mathcal{Z}_\Gamma{}^{(m)}$ since commutators certainly lie in the kernel of $\deg_k$ for each k. To show the opposite inclusion let $W \in \mathcal{Z}_\Gamma{}^{(m)}$, i.e. $\deg_k(W) = 0$ for all $1 \leq k \leq m$. Since

$$\begin{aligned} \mathcal{A}_\Gamma / \mathcal{A}'_\Gamma &\simeq \langle a_1, \ldots, a_n : \langle a_i a_j \rangle^{m_{a_i a_j}} = \langle a_j a_i \rangle^{m_{a_i a_j}} \rangle_{\mathrm{Ab}} \\ &\simeq \langle a_1, \ldots, a_n : a_i = a_j \text{ iff } i \text{ and j lie in the same connected} \\ &\qquad\qquad \text{component of } \Gamma_{odd} \rangle, \\ &\simeq \mathbb{Z}^m, \end{aligned}$$

with the isomorphism given by

$$U\mathcal{A}'_\Gamma \longmapsto (\deg_1(U), \dots, \deg_m(U)),$$

then $W\mathcal{A}'_\Gamma$ must be the identity in $\mathcal{A}_\Gamma/\mathcal{A}'_\Gamma$ (since it is in the kernel). In which case $W \in \mathcal{A}'_\Gamma$. □

It is this characterization of the commutator subgroup which allows us to use the Reidemeister-Schreier method to compute its presentation. In particular, we can find a relatively simple set of Schreier right coset representatives.

3.3 Computing the Presentations

In this section we compute presentations for the commutator subgroups of the irreducible finite-type Artin groups. We will show that, for the most part, the commutator subgroups are finitely generated and *perfect* (equal to its commutator subgroup).

Figure 3.1 shows that each irreducible finite-type Artin group falls into one of two classes; (i) those in which Γ_{odd} is connected and (ii) those in which Γ_{odd} has two components. Within a given class the arguments are quite similar. Thus, we will only show the complete details of the computations for types A_n and B_n. The rest of the types have similar computations.

3.3.1 Two Lemmas

We will encounter two sets of relations quite often in our computations and it will be necessary to replace them with sets of simpler but equivalent relations. In this section we give two lemmas which allow us to make these replacements.

Let $\{p_k\}_{k\in\mathbb{Z}}$, a, b, and q be letters. In the following keep in mind that the relators $p_{k+1}p_{k+2}^{-1}p_k^{-1}$ split up into the two types of relations $p_{k+2} = p_k^{-1}p_{k+1}$ (for $k \geq 0$), and $p_k = p_{k+1}p_{k+2}^{-1}$ (for $k < 0$). The two lemmas are:

Lemma 3.3 *The set of relations*

$$p_{k+1}p_{k+2}^{-1}p_k^{-1} = 1, \quad p_k a p_{k+2} a^{-1} p_{k+1}^{-1} a^{-1} = 1, \quad b = p_0 a p_0^{-1}, \tag{3.4}$$

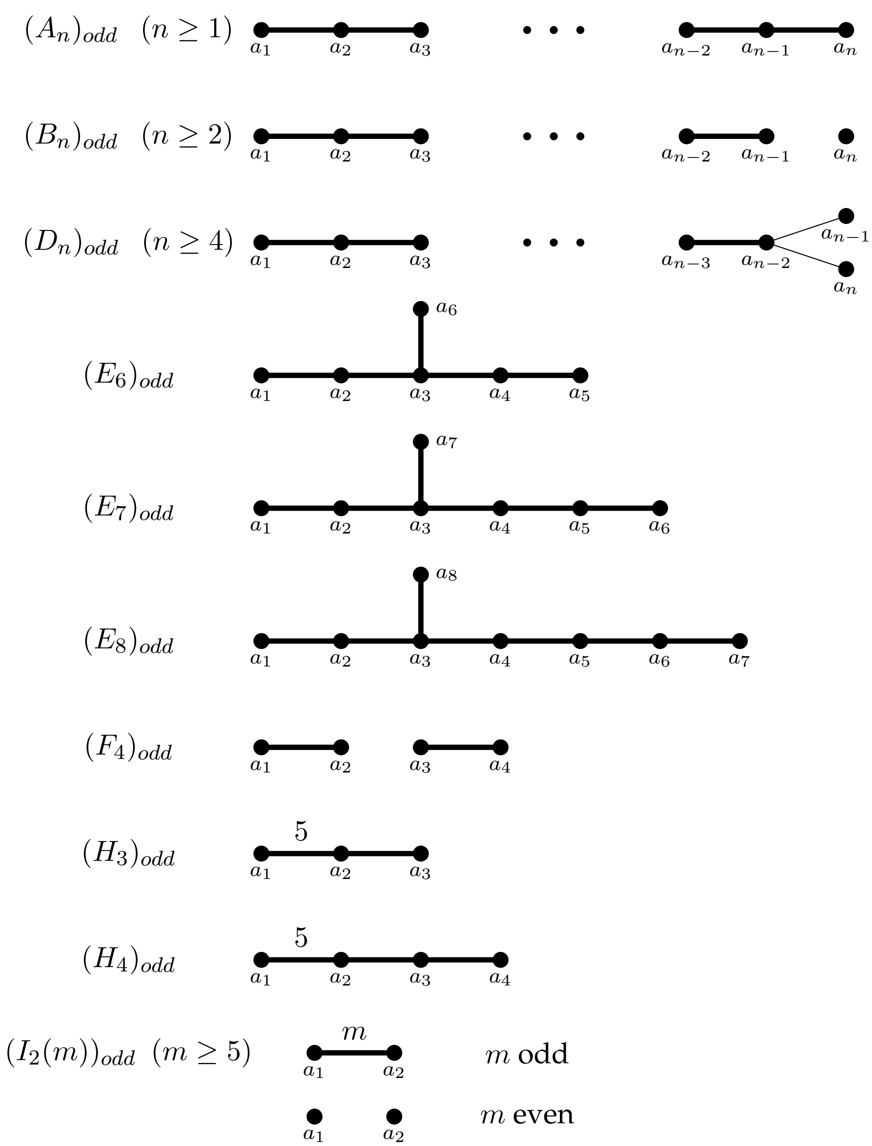


Figure 3.1: Γ_{odd} for the irreducible finite-type Coxeter graphs Γ

is equivalent to the set

$$\begin{aligned} p_{k+1}p_{k+2}^{-1}p_k^{-1} &= 1, & (3.5)\\ p_0 a p_0^{-1} &= b, & (3.6)\\ p_0 b p_0^{-1} &= b^2a^{-1}b & (3.7)\\ p_1 a p_1^{-1} &= a^{-1}b, & (3.8)\\ p_1 b p_1^{-1} &= (a^{-1}b)^3a^{-2}b. & (3.9) \end{aligned}$$

Lemma 3.4 *The set of relations:*

$$p_{k+1}p_{k+2}^{-1}p_k^{-1} = 1, \quad p_k q = q p_{k+1},$$

is equivalent to the set

$$p_{k+1}p_{k+2}^{-1}p_k^{-1} = 1, \quad p_0 q = q p_1, \quad p_1 q = q p_0^{-1} p_1.$$

The proof of lemma 3.4 is straightforward. On the other hand, the proof of the lemma 3.3 is somewhat long and tedious.

Proof. [Lemma 3.4] Clearly the second set of relations follows from the first set of relations since $p_2 = p_0^{-1}p_1$. To prove the converse we first prove that $p_k q = q p_{k+1}$ $(k \geq 0)$ follows from the second set of relations by induction on k. It is easy to see then that the same is true for $k < 0$. For $k = 0, 1$ the result clearly holds. Now, for $k = m + 2$;

$$\begin{aligned} p_{m+2} q p_{m+3}^{-1} q^{-1} &= p_{m+2} q p_{m+2}^{-1} p_{m+1} q^{-1}, \\ &= p_{m+2}(p_{m+1}^{-1} q) p_{m+1} q^{-1} \quad \text{by IH } (k = m+1), \\ &= p_{m+2} p_{m+1}^{-1} (q p_{m+1}) q^{-1}, \\ &= p_{m+2} p_{m+1}^{-1} (p_m q) q^{-1} \quad \text{by IH } (k = m), \\ &= p_{m+2} p_{m+1}^{-1} p_m, \\ &= 1. \end{aligned}$$

□

Proof. [Lemma 3.3] First we show the second set of relations follows from the first set. Taking $k = 0$ in the second relation in (3.4) we get the relation

$$p_0 a p_2 a^{-1} p_1^{-1} a^{-1} = 1,$$

and, using the relations $p_2 = p_0^{-1}p_1$ and $b = p_0ap_0^{-1}$, (3.8) easily follows. Taking $k = 1$ in the second relation in (3.4) we get the relation

$$p_1ap_3a^{-1}p_2^{-1}a^{-1} = 1.$$

Using the relations $p_3 = p_1^{-1}p_2$ and $p_2 = p_0^{-1}p_1$ this becomes

$$p_1ap_1^{-1}p_0^{-1}p_1a^{-1}p_1^{-1}p_0a^{-1} = 1.$$

But $p_1ap_1^{-1} = a^{-1}b$ (by (3.8)) so this reduces to

$$a^{-1}bp_0^{-1}b^{-1}ap_0a^{-1} = 1.$$

Isolating bp_0^{-1} on one side of the equation gives

$$bp_0^{-1} = a^2p_0^{-1}a^{-1}b.$$

Multiplying both sides on the left by p_0 and using the relation $p_0ap_0^{-1} = b$ it easily follows $p_0bp_0^{-1} = b^2a^{-1}b$, which is (3.7). Finally, taking $k = 2$ in the second relation in (3.4) we get the relation

$$p_2ap_4a^{-1}p_3^{-1}a^{-1} = 1.$$

Using the relation $p_4 = p_2^{-1}p_3$ this becomes

$$p_2ap_2^{-1}p_3a^{-1}p_3^{-1}a^{-1} = 1. \tag{3.10}$$

Note that

$$\begin{aligned} p_2ap_2^{-1} &= p_0^{-1}p_1ap_1^{-1}p_0 \quad \text{by } p_2 = p_0^{-1}p_1 \\ &= p_0^{-1}a^{-1}bp_0 \quad \text{by (3.8)} \\ &= a^{-2}ba^{-1}a \quad \text{using (3.4) and (3.7)} \\ &= a^{-2}b \end{aligned}$$

and

$$\begin{aligned} p_3ap_3^{-1} &= p_1^{-1}p_2ap_2^{-1}p_1 \quad \text{by } \mathrm{p}_3 = \mathrm{p}_1^{-1}\mathrm{p}_2 \\ &= p_1^{-1}a^{-2}bp_1, \end{aligned}$$

where the second equality follows from the previous statement. Thus, (3.10) becomes

$$a^{-2}bp_1^{-1}b^{-1}a^2p_1a^{-1} = 1$$

Isolating the factor bp_1^{-1} on one side of the equation, multiplying both sides by p_1, and using the relation (3.8) we easily get the relation (3.9). Therefore we have that the second set of relations (3.5)-(3.9) follows from the first set of relations (3.4).

In order to show the relations in (3.4) follow from the relations in (3.5)-(3.9) it suffices to just show that the second relation in (3.4) follows from the relations in (3.5)-(3.9). To do this we need the following fact: The relations

$$\begin{aligned} p_kap_k^{-1} &= a^kb, && (3.11)\\ p_kbp_k^{-1} &= (a^{-k}b)^{k+2}a^{-(k+1)}b, && (3.12)\\ p_k^{-1}ap_k &= ab^{-1}a^{k+2}, && (3.13)\\ p_k^{-1}bp_k &= (ab^{-1}a^{k+2})^ka, && (3.14) \end{aligned}$$

follow from the relations in (3.5)-(3.9). The proof of this fact is left to lemma 3.5 below. From the relations (3.11)-(3.14) we obtain

$$p_{k+1}ap_{k+1}^{-1} = a^{-(k+1)}b = a^{-1}\cdot a^{-k}b = a^{-1}p_kap_k^{-1}, \tag{3.15}$$

and

$$p_{k+1}^{-1}ap_{k+1} = ab^{-1}a^{k+3} = ab^{-1}a^{k+2}a = p_k^{-1}ap_ka. \tag{3.16}$$

Now we are in a position to show that that the second relation in (3.4) follows from the relations in (3.5)-(3.9). For $k \geq 0$

$$\begin{aligned} p_kap_{k+2}a^{-1}p_{k+1}^{-1}a^{-1} &= p_kap_k^{-1}\underbrace{p_{k+1}a^{-1}p_{k+1}^{-1}}a^{-1} \quad \text{by (3.5)}\\ &= p_kap_k^{-1}(a^{-1}p_kap_k^{-1})^{-1}a^{-1} \quad \text{by (3.15)}\\ &= 1. \end{aligned}$$

and for $k < 0$

$$\begin{aligned} p_kap_{k+2}a^{-1}p_{k+1}^{-1}a^{-1} &= p_{k+1}\underbrace{p_{k+2}^{-1}ap_{k+2}}a^{-1}p_{k+1}^{-1}a^{-1} \quad \text{by (3.5)}\\ &= p_{k+1}(p_{k+1}^{-1}ap_{k+1}a)a^{-1}p_{k+1}^{-1}a^{-1} \quad \text{by (3.16)}\\ &= 1. \end{aligned}$$

Therefore, the relations

$$p_k a p_{k+2} a^{-1} p_{k+1}^{-1} a^{-1} = 1, \quad k \in \mathbb{Z}$$

follow from the relations in (3.5)-(3.9). $\square$

To complete the proof of lemma 3.3 we need to prove the following.

Lemma 3.5 *The relations*

$$\begin{aligned} p_k a p_k^{-1} &= a^k b \\ p_k b p_k^{-1} &= (a^{-k} b)^{k+2} a^{-(k+1)} b \\ p_k^{-1} a p_k &= a b^{-1} a^{k+2} \\ p_k^{-1} b p_k &= (a b^{-1} a^{k+2})^k a \end{aligned}$$

follow from the relations in (3.5)-(3.9).

Proof. We will use induction to prove the result for nonnegative indices k, the result for negative indices k is similar. Clearly this holds for $k = 0, 1$. For $k = m + 2$ we have

$$\begin{aligned} p_{m+2} a p_{m+2}^{-1} &= p_m^{-1} p_{m+1} a p_{m+1}^{-1} p_m \quad \text{by (3.5),} \\ &= p_m^{-1} a^{-(m+1)} b p_m \quad \text{by induction hypothesis (IH),} \\ &= (p_m^{-1} a^{-(m+1)} p_m)(p_m^{-1} b p_m), \\ &= (p_m^{-1} a p_m)^{-(m+1)} (p_m^{-1} b p_m), \\ &= (a b^{-1} a^{m+2})^{-(m+1)} (a b^{-1} a^{m+2})^m a \quad \text{by IH,} \\ &= (a b^{-1} a^{m+2})^{-1} a, \\ &= a^{-(m+2)} b, \end{aligned}$$

$$\begin{aligned} p_{m+2} b p_{m+2}^{-1} &= p_m^{-1} p_{m+1} b p_{m+1}^{-1} p_m \quad \text{by (3.5),} \\ &= p_m^{-1} (a^{-(m+1)} b)^{m+3} a^{-(m+2)} b p_m \quad \text{by IH,} \\ &= ((p_m^{-1} a p_m)^{-(m+1)} (p_m^{-1} b p_m))^{m+3} (p_m^{-1} a p_m)^{-(m+2)} p_m^{-1} b p_m, \\ &= ((a b^{-1} a^{m+2})^{-(m+1)} (a b^{-1} a^{m+2})^m a)^{(m+3)} \\ &\qquad \cdot (a b^{-1} a^{m+2})^{-(m+2)} (a b^{-1} a^{m+2})^m a \quad \text{by IH,} \\ &= (a^{-(m+2)} b)^{m+3} (a b^{-1} a^{m+2})^{-2} a, \\ &= (a^{-(m+2)} b)^{m+4} a^{-(m+3)} b, \end{aligned}$$

Similarly for the other two equations. Thus, the result follows by induction.
□

3.3.2 Type A

The first presentation for the commutator subgroup $\mathfrak{B}'_{n+1} = \mathcal{A}'_{A_n}$ of the braid group $\mathfrak{B}_{n+1} = \mathcal{A}_{A_n}$ appeared in [GL69] but the details of the computation were minimal. Here we fill in the details of Gorin and Lin's computation.

The presentation of $\mathcal{A}_{A_n}$ is

$$\mathcal{A}_{A_n} = \langle a_1, ..., a_n : \quad \begin{array}{l} a_i a_j = a_j a_i \quad \text{for } |i-j| \geq 2, \\ a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} \quad \text{for } 1 \leq i \leq n-1 \,\rangle. \end{array}$$

Since $(A_n)_{odd}$ is connected then by theorem 3.1 $\mathcal{A}'_{A_n} = \mathcal{Z}_{A_n}$. To simplify notation in the following let $\mathcal{Z}_n$ denote $\mathcal{A}'_{A_n} = \mathcal{Z}_{A_n}$. Elements $U, V \in \mathcal{A}_{A_n}$ lie in the same right coset precisely when they have the same degree:

$$\begin{aligned} \mathcal{Z}_n U = \mathcal{Z}_n V &\iff UV^{-1} \in \mathcal{Z}_n \\ &\iff \deg(U) = \deg(V), \end{aligned}$$

thus a Schreier system of right coset representatives for $\mathcal{A}_{A_n}$ modulo $\mathcal{Z}_n$ is

$$\mathcal{R} = \{a_1^k : k \in \mathbb{Z}\}$$

By the Reidemeister-Schreier method, in particular equation (3.2), $\mathcal{Z}_n$ has generators $s_{a_1^k, a_j} := a_1^k a_j (\overline{a_1^k a_j})^{-1}$ with presentation

$$\langle s_{a_1^k, a_j}, \ldots : s_{a_1^m, a_\lambda}, \ldots, \tau(a_1^\ell R_i a_1^{-\ell}), \ldots, \tau(a_1^\ell T_{i,j} a_1^{-\ell}), \ldots \rangle, \tag{3.17}$$

where $j \in \{1, \ldots, n\}$, $k, \ell \in \mathbb{Z}$, and $m \in \mathbb{Z}$, $\lambda \in \{1, \ldots, n\}$ such that $a_1^m a_\lambda \approx \overline{a_1^m a_\lambda}$ ("freely equal"), and $T_{i,j}$, R_i represent the relators $a_i a_j a_i^{-1} a_j^{-1}$, $|i-j| \geq 2$, and $a_i a_{i+1} a_i a_{i+1}^{-1} a_i^{-1} a_{i+1}^{-1}$, respectively. Our goal is to clean up this presentation.

The first thing to notice is that

$$a_1^m a_\lambda \approx \overline{a_1^m a_\lambda} = a_1^{m+1} \iff \lambda = 1$$

Thus, the first type of relation in (3.17) is precisly $s_{a_1^m, a_1} = 1$, for all $m \in \mathbb{Z}$.

Next, we use the definition of the Reidemeister rewriting function (3.3) to express the second and third types of relations in (3.17) in terms of the generators $s_{a_1^k,a_j}$:

$$\tau(a_1^k T_{i,j} a_1^{-k}) = s_{a_1^k,a_i} s_{a_1^{k+1},a_j} s^{-1}_{a_1^{k+1},a_i} s^{-1}_{a_1^k,a_j} \tag{3.18}$$
$$\tau(a_1^k R_i a_1^{-k}) = s_{a_1^k,a_i} s_{a_1^{k+1},a_{i+1}} s_{a_1^{k+2},a_i} s^{-1}_{a_1^{k+2},a_{i+1}} s^{-1}_{a_1^{k+1},a_i} s^{-1}_{a_1^k,a_{i+1}} \tag{3.19}$$

Taking $i = 1$, $j \geq 3$ in (3.18) we get

$$s_{a_1^{k+1},a_j} = s_{a_1^k,a_j}$$

Thus, by induction on k,

$$s_{a_1^k,a_j} = s_{1,a_j} \tag{3.20}$$

for $j \geq 3$ and for all $k \in \mathbb{Z}$.

Therefore, $\mathcal{Z}_n$ is generated by $s_{a_1^k,a_2} = a_1^k a_2 a_1^{-(k+1)}$ and $s_{1,a_\ell} = a_\ell a_1^{-1}$, where $k \in \mathbb{Z}$, $3 \leq \ell \leq n$. To simplify notation let us rename the generators; let $p_k := a_1^k a_2 a_1^{-(k+1)}$ and $q_\ell := a_\ell a_1^{-1}$, for $k \in \mathbb{Z}$, $3 \leq \ell \leq n$. We now investigate the relations in (3.18) and (3.19).

The relations in (3.19) break up into the following three types (using 3.20):

$$p_{k+1} p^{-1}_{k+2} p^{-1}_k \qquad \text{(taking } i = 1\text{)} \tag{3.21}$$
$$p_k q_3 p_{k+2} q_3^{-1} p^{-1}_{k+1} q_3^{-1} \qquad \text{(taking } i = 2\text{)} \tag{3.22}$$
$$q_i q_{i+1} q_i q^{-1}_{i+1} q_i^{-1} q^{-1}_{i+1} \qquad \text{for } 3 \leq i \leq n-1. \tag{3.23}$$

The relations in (3.18) break up into the following two types:

$$p_k q_j p^{-1}_{k+1} q_j^{-1} \qquad \text{for } 4 \leq j \leq n, \text{ (taking } i = 2\text{)} \tag{3.24}$$
$$q_i q_j q_i^{-1} q_j^{-1} \qquad \text{for } 3 \leq i < j \leq n, |i - j| \geq 2. \tag{3.25}$$

We now have a presentation for $\mathcal{Z}_n$ consisting of the generators p_k, q_ℓ, where $k \in \mathbb{Z}$, $3 \leq \ell \leq n-1$, and defining relations (3.21) -(3.25). However, notice that relation (3.21) splits up into the two relations

$$p_{k+2} = p_k^{-1} p_{k+1} \qquad \text{for } k \geq 0, \tag{3.26}$$
$$p_k = p_{k+1} p^{-1}_{k+2} \qquad \text{for } k < 0. \tag{3.27}$$

Thus, for $k \neq 0, 1$, p_k can be expressed in terms of p_0 and p_1. It follows that $\mathcal{Z}_n$ is finitely generated. In order to show $\mathcal{Z}_n$ is finitely presented we need to be able to replace the infinitly many relations in (3.22) and (3.24) with finitely many relations. This can be done using lemmas 3.3 and 3.4, but this requires us to add a new letter b to the generating set with a new relation $b = p_0 q_3 p_0^{-1}$. Thus $\mathcal{Z}_n$ is generated by p_0, p_1, q_ℓ, b, where $3 \leq \ell \leq n-1$, with defining relations:

$$p_0 q_3 p_0^{-1} = b, \quad p_0 b p_0^{-1} = b^2 q_3^{-1} b, \quad p_1 q_3 p_1^{-1} = q_3^{-1} b, \quad p_1 b p_1^{-1} = (q_3^{-1} b)^3 q_3^{-2} b,$$
$$q_i q_{i+1} q_i q_{i+1}^{-1} q_i^{-1} q_{i+1}^{-1} \quad (3 \leq i \leq n-1),$$
$$p_0 q_j = q_j p_1 \quad (4 \leq j \leq n), \quad p_1 q_j = q_j p_0^{-1} p_1 \quad (4 \leq j \leq n).$$
$$q_i q_j q_i^{-1} q_j^{-1} \qquad (3 \leq i < j \leq n, |i-j| \geq 2).$$

Noticing that for $n = 2$ the generators q_k ($3 \leq k \leq n$), and b do not exist, and for $n = 3$ the generators q_k ($4 \leq k \leq n$) do not exist, we have proved the following theorem.

Theorem 3.6 *For every $n \geq 2$ the commutator subgroup $\mathcal{A}'_{A_n}$ of the Artin group $\mathcal{A}_{A_n}$ is a finitely presented group. $\mathcal{A}'_{A_2}$ is a free group with two free generators*

$$p_0 = a_2 a_1^{-1}, \;\; p_1 = a_1 a_2 a_1^{-2}.$$

$\mathcal{A}'_{A_3}$ is the group generated by

$$p_0 = a_2 a_1^{-1}, \;\; p_1 = a_1 a_2 a_1^{-2}, \;\; q = a_3 a_1^{-1}, \;\; b = a_2 a_1^{-1} a_3 a_2^{-1},$$

with defining relations

$$b = p_0 q p_0^{-1}, \quad p_0 b p_0^{-1} = b^2 q^{-1} b,$$
$$p_1 q p_1^{-1} = q^{-1} b, \quad p_1 b p_1^{-1} = (q^{-1} b)^3 q^{-2} b.$$

For $n \geq 4$ the group $\mathcal{A}'_{A_n}$ is generated by

$$p_0 = a_2 a_1^{-1}, \;\; p_1 = a_1 a_2 a_1^{-2}, \;\; q_3 = a_3 a_1^{-1},$$
$$b = a_2 a_1^{-1} a_3 a_2^{-1}, \;\; q_\ell = a_\ell a_1^{-1} \;\; (4 \leq \ell \leq n-1),$$

with defining relations

$$b = p_0 q_3 p_0^{-1}, \quad p_0 b p_0^{-1} = b^2 q_3^{-1} b,$$
$$p_1 q_3 p_1^{-1} = q_3^{-1} b, \quad p_1 b p_1^{-1} = (q_3^{-1} b)^3 q_3^{-2} b,$$
$$p_0 q_i = q_i p_1 \ (4 \leq i \leq n), \quad p_1 q_i = q_i p_0^{-1} p_1 \ (4 \leq i \leq n)$$
$$q_3 q_i = q_i q_3 \ (5 \leq i \leq n), \quad q_3 q_4 q_3 = q_4 q_3 q_4,$$
$$q_i q_j = q_j q_i \ (4 \leq i < j-1 \leq n-1), \quad q_i q_{i+1} q_i = q_{i+1} q_i q_{i+1} \ (4 \leq i \leq n-1).$$

□

Corollary 3.7 *For $n \geq 4$ the commutator subgroup $\mathcal{A}'_{A_n}$ of the Artin group of type A_n is finitely generated and perfect (i.e. $\mathcal{A}''_{A_n} = \mathcal{A}'_{A_n}$).*

Proof. Abelianizing the presentation of $\mathcal{A}'_{A_n}$ in the theorem results in a presentation of the trivial group. Hence $\mathcal{A}''_{A_n} = \mathcal{A}'_{A_n}$. □

Now we study in greater detail the group $\mathcal{A}'_{A_3}$, the results of which will be used in section 4.2.1. From the presentation of $\mathcal{A}'_{A_3}$ given in theorem 3.6 one can easily deduce the relations:

$$p_0^{-1} q p_0 = q b^{-1} q^2, \quad p_0^{-1} b p_0 = q,$$
$$p_1^{-1} q p_1 = q b^{-1} q^3, \quad p_1^{-1} b p_1 = q b^{-1} q^4.$$

Let T be the subgroup of $\mathcal{A}'_{A_3}$ generated by q and b. The above relations and the defining relations in the presentation for $\mathcal{A}'_{A_3}$ tell us that T is a normal subgroup of $\mathcal{A}'_{A_3}$. To obtain a representation of the factor group $\mathcal{A}'_{A_3}/T$ it is sufficient to add to the defining relations in the presentation for $\mathcal{A}'_{A_3}$ the relations $q = 1$ and $b = 1$. It is easy to see this results in the presentation of the free group generated by p_0 and p_1. Thus, $\mathcal{A}'_{A_3}/T$ is a free group of rank 2, F_2. We have the exact sequence

$$1 \longrightarrow T \longrightarrow \mathcal{A}'_{A_3} \longrightarrow \mathcal{A}'_{A_3}/T \longrightarrow 1.$$

Since $\mathcal{A}'_{A_3}/T$ is free then the exact sequence is actually split so

$$\mathcal{A}'_{A_3} \simeq T \rtimes \mathcal{A}'_{A_3}/T \simeq T \rtimes F_2,$$

where the action of F_2 on T is defined by the defining relations in the presentation of $\mathcal{A}'_{A_3}$ and the relations above. In [GL69] it is shown (theorem 2.6) the group T is also free of rank 2, so we have the following theorem.

Theorem 3.8 *The commutator subgroup $\mathcal{A}'_{A_3}$ of the Artin group of type A_3 is the semidirect product of two free groups each of rank 2;*

$$\mathcal{A}'_{A_3} \simeq F_2 \rtimes F_2.$$

□

3.3.3 Type B

The presentation of $\mathcal{A}_{B_n}$ is

$$\begin{aligned}\mathcal{A}_{B_n} = \langle a_1, ..., a_n : \quad & a_i a_j = a_j a_i \quad \text{for } |i-j| \geq 2, \\ & a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} \quad \text{for } 1 \leq i \leq n-2 \\ & a_{n-1} a_n a_{n-1} a_n = a_n a_{n-1} a_n a_{n-1} \rangle.\end{aligned}$$

Let $T_{i,j}$, R_i $(1 \leq i \leq n-2)$, and R_{n-1} denote the associated relators $a_i a_j a_i^{-1} a_j^{-1}$, $a_i a_{i+1} a_i a_{i+1}^{-1} a_i^{-1} a_{i+1}^{-1}$, and $a_{n-1} a_n a_{n-1} a_n a_{n-1}^{-1} a_n^{-1} a_{n-1}^{-1} a_n^{-1}$, respectively.

As seen in figure 3.1 the graph $(B_n)_{odd}$ has two components: Γ_1 and Γ_2, where Γ_2 denotes the component containing the single vertiex a_n. Let $\deg_1$ and $\deg_2$ denote the associated degree maps, respectively, so from theorem 3.2

$$\mathcal{A}'_{B_n} = \mathcal{Z}^{(2)}_{B_n} = \{U \in \mathcal{A}_{B_n} : \deg_1(U) = 0 \text{ and } \deg_2(U) = 0\}.$$

For simplicity of notation let $\mathcal{Z}^{(2)}_{B_n}$ be denoted by $\mathcal{Z}_n$.

For elements $U, V \in \mathcal{A}_{A_n}$,

$$\begin{aligned}\mathcal{Z}_n U = \mathcal{Z}_n V \quad & \Leftrightarrow \quad UV^{-1} \in \mathcal{Z}_n \\ & \Leftrightarrow \quad \deg_1(U) = \deg_1(V), \text{ and} \\ & \qquad \deg_2(U) = \deg_2(V),\end{aligned}$$

thus a Schreier system of right coset representatives for $\mathcal{A}_{B_n}$ modulo $\mathcal{Z}_n$ is

$$\mathcal{R} = \{a_1^k a_n^\ell : k, \ell \in \mathbb{Z}\}$$

By the Reidemeister-Schreier method, in particular equation (3.2), $\mathcal{Z}_n$ is generated by

$$\begin{aligned}s_{a_1^k a_n^k, a_j} \quad & := \quad a_1^k a_n^\ell a_j (\overline{a_1^k a_n^\ell a_j})^{-1} \\ & = \quad \begin{cases} a_1^k a_n^\ell a_j a_n^{-\ell} a_1^{-(k+1)} & \text{if } j \neq n \\ 1 & \text{if } j = n. \end{cases}\end{aligned}$$

with presentation

$$\begin{aligned}\mathcal{Z}_n = \langle s_{a_1^k a_n^\ell, a_j}, \dots : s_{a_1^p a_n^q, a_\lambda}, \dots, \\ &\tau(a_1^k a_n^\ell T_{i,j} (a_1^k a_n^\ell)^{-1}), \dots, \quad (1 \le i < j \le n, |i-j| \ge 2), \\ &\tau(a_1^k a_n^\ell R_i (a_1^k a_n^\ell)^{-1}), \dots, \quad (1 \le i \le n-2), \\ &\tau(a_1^k a_n^\ell R_{n-1} (a_1^k a_n^\ell)^{-1}), \dots \rangle,\end{aligned} \tag{3.28}$$

where $p, q \in \mathbb{Z}$, $\lambda \in \{1, \dots, n-1\}$ such that $a_1^p a_n^q a_\lambda \approx \overline{a_1^p a_n^q a_\lambda}$ (”freely equal”). Again, our goal is to clean up this presentation.

The cases $n = 2, 3$, and 4 are straightforward after one sees the computation for the general case $n \ge 5$, so we will not include the computations for these cases. The results are included in theorem 3.9. From now on it will be assumed that $n \ge 5$.

Since

$$a_1^p a_n^q a_\lambda \approx \overline{a_1^p a_n^q a_\lambda} = \begin{cases} a_1^{p+1} a_n^q & \lambda \ne n \\ a_1^p a_n^{q+1} & \lambda = n \end{cases} \quad \iff \lambda = n \text{ or; } \lambda = 1 \text{ and } q = 0,$$

the first type of relations in (3.28) are precisely

$$s_{a_1^k a_n^\ell, a_n} = 1, \text{ and } s_{a_1^k, a_1} = 1. \tag{3.29}$$

The second type of relations in (3.28), after rewriting using equation (3.3), are

$$s_{a_1^k a_n^\ell, a_i} s_{\overline{a_1^k a_n^\ell a_i}, a_j} s^{-1}_{\overline{a_1^k a_n^\ell a_i a_j a_i^{-1}}, a_i} s^{-1}_{\overline{a_1^k a_n^\ell a_i a_j a_i^{-1} a_j^{-1}}, a_j}. \tag{3.30}$$

where $1 \le i < j \le n$, $|i - j| \ge 2$. Taking $i = 1$ and $3 \le j \le n - 1$ gives: for $\ell = 0$ (using (3.29));

$$s_{a_1^{k+1}, a_j} = s_{a_1^k, a_j}, \tag{3.31}$$

so by induction on k,

$$s_{a_1^k, a_j} = s_{1, a_j} \quad \text{for } 3 \le j \le n-1, \tag{3.32}$$

and for $\ell \ne 0$;

$$s_{a_1^k a_n^\ell, a_1} s_{a_1^{k+1} a_n^\ell, a_j} s^{-1}_{a_1^{k+1} a_n^\ell, a_1} s^{-1}_{a_1^k a_n^\ell, a_j}. \tag{3.33}$$

We will come back to relation (3.33) in a bit.

Taking $i = 1$ and $j = n$ in (3.30) (and using (3.29)) gives

$$s_{a_1^k a_n^\ell, a_1} s^{-1}_{a_1^k a_n^{\ell+1}, a_1}. \tag{3.34}$$

So, by induction on ℓ (and (3.29)) we get

$$s_{a_1^k a_n^\ell, a_1} = 1 \quad \text{for } k, \ell \in \mathbb{Z}. \tag{3.35}$$

Taking $2 \leq i \leq n-2$, $i+2 \leq j \leq n$ in (3.30) gives

$$\begin{cases} s_{a_1^k a_n^\ell, a_i} s_{a_1^{k+1} a_n^\ell, a_j} s^{-1}_{a_1^{k+1} a_n^\ell, a_i} s^{-1}_{a_1^k a_n^\ell, a_j} & \text{for } j \leq n-1, \\ s_{a_1^k a_n^\ell, a_i} s^{-1}_{a_1^k a_n^{\ell+1}, a_i} & \text{for } j = n. \end{cases} \tag{3.36}$$

In the case $j = n$ induction on ℓ gives

$$s_{a_1^k a_n^\ell, a_i} = s_{a_1^k, a_i} \quad (2 \leq i \leq n-2). \tag{3.37}$$

So from (3.32) it follows

$$s_{a_1^k a_n^\ell, a_i} = \begin{cases} s_{1, a_i} & 3 \leq i \leq n-2 \\ s_{a_1^k, a_2} & i = 2. \end{cases} \tag{3.38}$$

We come back to the case $j \leq n-1$ later.

Returning now to (3.33), we can use (3.35) to get

$$s_{a_1^{k+1} a_n^\ell, a_j} = s_{a_1^k a_n^\ell . a_j} \quad (3 \leq j \leq n-1).$$

Thus, by induction on k

$$s_{a_1^k a_n^\ell, a_j} = s_{a_n^\ell, a_j} \quad (3 \leq j \leq n-1). \tag{3.39}$$

For $3 \leq j \leq n-2$ we already know this (equation (3.38)), so the only new information we get from (3.33) is

$$s_{a_1^k a_n^\ell, a_{n-1}} = s_{a_n^\ell, a_{n-1}} \quad (k \in \mathbb{Z}). \tag{3.40}$$

Collecting all the information we have obtained from $\tau(a_1^k a_n^\ell T_{i,j}(a_1^k a_n^\ell)^{-1})$, $1 \leq i < j \leq n, |i-j| \geq 2$, we get:

$$\begin{aligned} s_{a_1^k a_n^\ell, a_1} &= 1 \quad (k, \ell \in \mathbb{Z}), \\ s_{a_1^k a_n^\ell, a_i} &= \begin{cases} s_{1,a_i} & 3 \leq i \leq n-2, \\ s_{a_1^k, a_2} & i = 2, \end{cases} \\ s_{a_1^k a_n^\ell, a_{n-1}} &= s_{a_n^\ell, a_{n-1}}, \end{aligned} \tag{3.41}$$

and (from (3.36)), for $2 \leq i \leq n-3$ and $i+2 \leq j \leq n-1$,

$$s_{a_1^k a_n^\ell, a_i} s_{a_1^{k+1} a_n^\ell, a_j} s^{-1}_{a_1^{k+1} a_n^\ell, a_i} s^{-1}_{a_1^k a_n^\ell, a_j}. \tag{3.42}$$

This relation breaks up into the following cases (using (3.41))

$$\begin{cases} s_{a_1^k, a_2} s_{1,a_j} s^{-1}_{a_1^{k+1}, a_2} s^{-1}_{1,a_j} & \text{for } i = 2,\ 4 \leq j \leq n-2, \\ s_{a_1^k, a_2} s_{a_n^\ell, a_{n-1}} s^{-1}_{a_1^{k+1}, a_2} s^{-1}_{a_n^\ell, a_{n-1}} & \text{for } i = 2,\ j = n-1, \\ s_{1,a_i} s_{1,a_j} s^{-1}_{1,a_i} s^{-1}_{1,a_j} & \text{for } 3 \leq i \leq n-3,\ i+2 \leq j \leq n-2, \\ s_{1,a_i} s_{a_n^\ell, a_{n-1}} s^{-1}_{1,a_i} s^{-1}_{a_n^\ell, a_{n-1}} & \text{for } 3 \leq i \leq n-3,\ j = n-1, \end{cases} \tag{3.43}$$

The third type of relations in (3.28); $\tau(a_1^k a_n^\ell R_i (a_1^k a_n^\ell)^{-1})$, after rewriting using equation (3.3), are

$$s_{a_1^k a_n^\ell, a_i} s_{a_1^{k+1} a_n^\ell, a_{i+1}} s_{a_1^{k+2} a_n^\ell, a_i} s^{-1}_{a_1^{k+2} a_n^\ell, a_{i+1}} s^{-1}_{a_1^{k+1} a_n^\ell, a_i} s^{-1}_{a_1^k a_n^\ell, a_{i+1}}, \tag{3.44}$$

which break down as follows (using (3.41)):

$$\begin{cases} s_{a_1^{k+1}, a_2} s^{-1}_{a_1^{k+2}, a_2} s^{-1}_{a_1^k, a_2} & (i = 1), \\ s_{a_1^k, a_2} s_{1,a_3} s_{a_1^{k+2}, a_2} s^{-1}_{1,a_3} s^{-1}_{a_1^{k+1}, a_2} s^{-1}_{1,a_3} & (i = 2), \\ s_{1,a_i} s_{1,a_{i+1}} s_{1,a_i} s^{-1}_{1,a_{i+1}} s^{-1}_{1,a_i} s^{-1}_{1,a_{i+1}}, & \text{for } 3 \leq i \leq n-3, \\ s_{1,a_{n-2}} s_{a_n^\ell, a_{n-1}} s_{1,a_{n-2}} s^{-1}_{a_n^\ell, a_{n-1}} s^{-1}_{1,a_{n-2}} s^{-1}_{a_n^\ell, a_{n-1}}, & (i = n-2), \end{cases} \tag{3.45}$$

The fourth type of relations in (3.28); $\tau(a_1^k a_n^\ell R_{n-1}(a_1^k a_n^\ell)^{-1})$, after rewriting using equation (3.3), is

$$s_{a_n^\ell, a_{n-1}} s_{a_n^{\ell+1}, a_{n-1}} s^{-1}_{a_n^{\ell+2}, a_{n-1}} s^{-1}_{a_n^{\ell+1}, a_{n-1}}, \tag{3.46}$$

where we have made extensive use of the relations (3.41).

From (3.41) it follows that $\mathcal{Z}_n$ is generated by $s_{a_1^k,a_2}$, s_{1,a_i}, and $s_{a_n^\ell,a_{n-1}}$ for $k,\ell \in \mathbb{Z}$ and $3 \leq i \leq n-2$. For simplicity of notation let these generators be denoted by p_k, q_i, and r_ℓ, respectively. Thus, we have shown that the following is a set of defining relations for $\mathcal{Z}_n$:

$$\begin{aligned}
p_k q_j &= q_j p_{k+1} && (4 \leq j \leq n-2,\, k \in \mathbb{Z}),\\
p_k r_\ell &= r_\ell p_{k+1} && (k,\ell \in \mathbb{Z}),\\
q_i q_j &= q_j q_i && (3 \leq i < j \leq n-2,\, |i = j| \geq 2),\\
q_i r_\ell &= r_\ell q_i && (3 \leq i \leq n-3),\\
&p_{k+1} p_{k+2}^{-1} p_k^{-1} && (k \in \mathbb{Z}),\\
&p_k q_3 p_{k+2} q_3^{-1} p_{k+1}^{-1} q_3^{-1} && (k \in Z),\\
q_i q_{i+1} q_i &= q_{i+1} q_i q_{i+1} && (3 \leq i \leq n-3),\\
q_{n-2} r_\ell q_{n-2} &= r_\ell q_{n-2} r_\ell && (\ell \in \mathbb{Z}),\\
&r_\ell r_{\ell+1} r_{\ell+2}^{-1} r_{\ell+1}^{-1} && (\ell \in \mathbb{Z}),
\end{aligned} \tag{3.47}$$

The first four relations are from (3.43), the next four are from (3.45), and the last one is from (3.46).

The fifth relation tells us that for $k \neq 0, 1$, p_k can be expressed in terms of p_0 and p_1. Similarly the last relation tells us that for $\ell \neq 0, 1$, r_ℓ can be expressed in terms of r_0 and r_1. From this it follows that $\mathcal{Z}_n$ is finitely generated. Using lemmas 3.3 and 3.4 to replace the first, second and sixth relations, assuming we have added a new generator b and relation $b = p_0 q_3 p_0^{-1}$, we arrive at the following theorem.

Theorem 3.9 *For every $n \geq 3$ the commutator subgroup $\mathcal{A}'_{B_n}$ of the Artin group $\mathcal{A}_{B_n}$ is a finitely generated group. Presentations for $\mathcal{A}'_{B_n}$, $n \geq 2$ are as follows:*
$\mathcal{A}'_{B_2}$ is a free group on countably many generators:

$$[a_2^\ell, a_1] \;\; (\ell \in \mathbb{Z} \setminus \{0, \pm 1\}), \quad [a_1^k a_2, a_1] \;\; (k \in \mathbb{Z} \setminus \{0\}).$$

$\mathcal{A}'_{B_3}$ is a free group on four generators:

$$[a_1^{-1}, a_2^{-1}], \quad [a_3, a_2][a_1^{-1}, a_2^{-1}], \quad [a_1, a_2][a_1^{-1}, a_2^{-1}], \quad [a_1 a_3, a_2][a_1^{-1}, a_2^{-1}].$$

$\mathcal{A}'_{B_4}$ is the group generated by

$$p_k = a_1^k a_2 a_1^{-(k+1)} = [a_1^k, a_2][a_1^{-1}, a_2^{-1}], \quad (k \in \mathbb{Z})$$
$$q_\ell = a_4^\ell a_3 (a_1 a_4^\ell)^{-1} = [a_4^\ell, a_3][a_2^{-1}, a_3^{-1}][a_1^{-1}, a_2^{-1}], \quad (\ell \in \mathbb{Z}),$$

with defining relations

$$p_{k+1} p_{k+2}^{-1} p_k^{-1} \quad (k \in \mathbb{Z}),$$
$$p_k q_\ell p_{k+2} = q_\ell p_{k+1} q_\ell \quad (k, \ell \in \mathbb{Z}),$$
$$q_\ell q_{\ell+1} = q_{\ell+1} q_{\ell+2} \quad (3 \leq i \leq n-3).$$

For $n \geq 5$ the group $\mathcal{A}'_{B_n}$ is generated by

$$p_0 = a_2 a_1^{-1}, \;\; p_1 = a_1 a_2 a_1^{-2}, \;\; q_3 = a_3 a_1^{-1}, \;\; r_\ell = a_n^\ell a_{n-1} (a_1 a_n^\ell)^{-1} \;\; (\ell \in \mathbb{Z}),$$
$$b = a_2 a_1^{-1} a_3 a_2^{-1}, \;\; q_i = a_i a_1^{-1} \;\; (4 \leq i \leq n-2),$$

with defining relations

$$p_0 q_j = q_j p_1, \quad p_1 q_j = q_j p_o^{-1} p_1 \quad (4 \leq j \leq n-2),$$
$$p_0 r_\ell = r_\ell p_1, \quad p_1 r_\ell = r_\ell p_0^{-1} p_1 \quad (\ell \in \mathbb{Z}),$$
$$q_i q_j = q_j q_i \quad (3 \leq i < j \leq n-2, |i = j| \geq 2),$$
$$q_i r_\ell = r_\ell q_i \quad (3 \leq i \leq n-3),$$
$$p_0 q_3 p_0^{-1} = b, \quad p_0 b p_0^{-1} = b^2 q_3^{-1} b,$$
$$p_1 q_3 p_1^{-1} = q_3^{-1} b, \quad p_1 b p_1^{-1} = (q_3^{-1} b)^3 q_3^{-2} b,$$
$$q_i q_{i+1} q_i = q_{i+1} q_i q_{i+1} \quad (3 \leq i \leq n-3),$$
$$q_{n-2} r_\ell q_{n-2} = r_\ell q_{n-2} r_\ell \quad (\ell \in \mathbb{Z}),$$
$$r_\ell r_{\ell+1} r_{\ell+2}^{-1} r_{\ell+1}^{-1} \quad (\ell \in \mathbb{Z}),$$

□

Corollary 3.10 *For $n \geq 5$ the commutator subgroup $\mathcal{A}'_{B_n}$ of the Artin group of type B_n is finitely generated and perfect.*

Proof. Abelianizing the presentation of $\mathcal{A}'_{B_n}$ in the theorem results in a presentation of the trivial group. Hence $\mathcal{A}''_{B_n} = \mathcal{A}'_{B_n}$. □

3.3.4 Type D

The presentation of $\mathcal{A}_{D_n}$ is

$$\begin{aligned} \mathcal{A}_{D_n} = \langle a_1, ..., a_n : \quad & a_i a_j = a_j a_i \quad \text{for } 1 \le i < j \le n-1, |i-j| \ge 2, \\ & a_n a_j = a_j a_n \quad \text{for } j \ne n-2, \\ & a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} \quad \text{for } 1 \le i \le n-2 \\ & a_{n-2} a_n a_{n-2} = a_n a_{n-2} a_n \rangle. \end{aligned}$$

As seen in figure 3.1 the graph $(D_n)_{odd}$ is connected. So by theorem 3.1

$$\mathcal{A}'_{D_n} = \mathcal{Z}_{D_n} = \{U \in \mathcal{A}_{D_n} : \deg(U) = 0\}.$$

The computation of the presentation of $\mathcal{A}'_{D_n}$ is similar to that of $\mathcal{A}'_{A_n}$, so we will not include it.

Theorem 3.11 *For every $n \ge 4$ the commutator subgroup $\mathcal{A}'_{D_n}$ of the Artin group $\mathcal{A}_{D_n}$ is a finitely presented group. $\mathcal{A}'_{D_4}$ is the group generated by*

$$p_0 = a_2 a_1^{-1}, \;\; p_1 = a_1 a_2 a_1^{-2}, \;\; q_3 = a_3 a_1^{-1},$$
$$q_4 = a_4 a_1^{-1}, \;\; b = a_2 a_1^{-1} a_3 a_2^{-1}, c = a_2 a_1^{-1} a_4 a_2^{-1},$$

with defining relations

$$\begin{gathered} b = p_0 q_3 p_0^{-1}, \quad p_0 b p_0^{-1} = b^2 q_3^{-1} b, \\ p_1 q_3 p_1^{-1} = q_3^{-1} b, \quad p_1 b p_1^{-1} = (q_3^{-1} b)^3 q_3^{-2} b, \\ c = p_0 q_4 p_0^{-1}, \quad p_0 c p_0^{-1} = c^2 q_4^{-1} c, \\ p_1 q_4 p_1^{-1} = q_4^{-1} c, \quad p_1 c p_1^{-1} = (q_4^{-1} c)^3 q_4^{-2} c, \\ q_3 q_4 = q_4 q_3. \end{gathered}$$

For $n \ge 5$ the group $\mathcal{A}'_{D_n}$ is generated by

$$p_0 = a_2 a_1^{-1}, \quad p_1 = a_1 a_2 a_1^{-2},$$
$$q_\ell = a_\ell a_1^{-1} \;\; (3 \le \ell \le n), \quad b = a_2 a_1^{-1} a_3 a_2^{-1},$$

with defining relations

$$b = p_0 q_3 p_0^{-1}, \quad p_0 b p_0^{-1} = b^2 q_3^{-1} b,$$
$$p_1 q_3 p_1^{-1} = q_3^{-1} b, \quad p_1 b p_1^{-1} = (q_3^{-1} b)^3 q_3^{-2} b,$$
$$p_0 q_j = q_j p_1, \quad p_1 q_j = q_j p_0^{-1} p_1 \quad (4 \leq j \leq n),$$
$$q_i q_{i+1} q_i = q_{i+1} q_i q_{i+1} \ (3 \leq i \leq n-2),$$
$$q_n q_{n-2} q_n = q_{n-2} q_n q_{n-2},$$
$$q_i q_j = q_j q_i \quad (3 \leq i < j \leq n-1, |i-j| \geq 2),$$
$$q_n q_j = q_j q_n \quad (j \neq n-2).$$

□

Corollary 3.12 *For $n \geq 5$ the commutator subgroup $\mathcal{A}'_{D_n}$ of the Artin group of type D_n is finitely presented and perfect.* □

3.3.5 Type E

The presentation of $\mathcal{A}_{E_n}$, $n = 6, 7$, or 8, is

$$\mathcal{A}_{E_n} = \langle a_1, ..., a_n : \quad \begin{array}{l} a_i a_j = a_j a_i \ \text{ for } 1 \leq i < j \leq n-1, |i-j| \geq 2, \\ a_i a_n = a_n a_i \ \text{ for } i \neq 3, \\ a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} \ \text{ for } 1 \leq i \leq n-2 \\ a_3 a_n a_3 = a_n a_3 a_n \rangle. \end{array}$$

As seen in figure 3.1 the graph $(E_n)_{odd}$ is connected. So by theorem 3.1

$$\mathcal{A}'_{E_n} = \mathcal{Z}_{E_n} = \{U \in \mathcal{A}_{E_n} : \deg(U) = 0\}.$$

The computation of the presentation of $\mathcal{A}'_{E_n}$ is similar to that of $\mathcal{A}'_{A_n}$.

Theorem 3.13 *For $n = 6, 7$, or 8 the commutator subgroup $\mathcal{A}'_{E_n}$ of the Artin group $\mathcal{A}_{E_n}$ is a finitely presented group. $\mathcal{A}'_{E_n}$ is the group generated by*

$$p_0 = a_2 a_1^{-1}, \quad p_1 = a_1 a_2 a_1^{-2}, \quad q_\ell = a_\ell a_1^{-1} \ (3 \leq \ell \leq n), \quad b = a_2 a_1^{-1} a_3 a_2^{-1},$$

with defining relations

$$\begin{gathered}
b = p_0 q_3 p_0^{-1}, \quad p_0 b p_0^{-1} = b^2 q_3^{-1} b, \\
p_1 q_3 p_1^{-1} = q_3^{-1} b, \quad p_1 b p_1^{-1} = (q_3^{-1} b)^3 q_3^{-2} b, \\
p_0 q_j = q_j p_1, \quad p_1 q_j = q_j p_0^{-1} p_1 \quad (4 \leq j \leq n), \\
q_i q_{i+1} q_i = q_{i+1} q_i q_{i+1} \quad (3 \leq i \leq n-2), \\
q_n q_3 q_n = q_3 q_n q_3, \\
q_i q_j = q_j q_i \quad (3 \leq i < j \leq n-1, |i-j| \geq 2), \\
q_i q_n = q_n q_i \quad (4 \leq i \leq n-1).
\end{gathered}$$

□

Corollary 3.14 *For $n = 6, 7,$ or 8 the commutator subgroup $\mathcal{A}'_{E_n}$ of the Artin group of type E_n is finitely presented and perfect.* □

3.3.6 Type F

The presentation of $\mathcal{A}_{F_4}$ is

$$\begin{aligned}
\mathcal{A}_{F_n} = \langle a_1, a_2, a_3, a_4 : \qquad & a_i a_j = a_j a_i \quad \text{for } |i-j| \geq 2, \\
& a_1 a_2 a_1 = a_2 a_1 a_2, \\
& a_2 a_3 a_2 a_3 = a_3 a_2 a_3 a_2, \\
& a_3 a_4 a_3 = a_4 a_3 a_4 \rangle.
\end{aligned}$$

As seen in figure 3.1 the graph $(E_n)_{odd}$ has two components: Γ_1 and Γ_2, where Γ_1 denotes the component containing the vertices a_1, a_2, and Γ_2 the component containing the vertices a_3, a_4. Let $\deg_1$ and $\deg_2$ denote the associated degree maps, respectively, so from theorem 3.2

$$\mathcal{A}'_{F_4} = \mathcal{Z}^{(2)}_{F_4} = \{U \in \mathcal{A}_{F_4} : \deg_1(U) = 0 \text{ and } \deg_2(U) = 0\}.$$

By a computation similar to that of B_n we get the following.

Theorem 3.15 *The commutator subgroup $\mathcal{A}'_{F_4}$ of the Artin group of type F_4 is the group generated by*

$$\begin{aligned}
p_k &= a_1^k a_2 a_1^{-(k+1)} = [a_1^k, a_2][a_1^{-1}, a_2^{-1}] \quad (k \in \mathbb{Z}), \\
q_\ell &= a_4^\ell a_3 a_4^{-(\ell+1)} = [a_4^\ell, a_3][a_4^{-1}, a_3^{-1}] \quad (\ell \in \mathbb{Z}),
\end{aligned}$$

with defining relations

$$p_{k+1}p_{k+2}^{-1}p_k^{-1} \ (k \in \mathbb{Z}), \quad q_{\ell+1}q_{\ell+2}^{-1}q_\ell^{-1} \ (\ell \in \mathbb{Z}),$$
$$p_k q_\ell p_{k+1} q_{\ell+1} = q_\ell p_k q_{\ell+1} p_{k+1} \ (k, \ell \in \mathbb{Z}).$$

The first two types of relations in the above presentation tell us that for $k \neq 0, 1$, p_k can be expressed in terms of p_0 and p_1, and similarly for q_ℓ. Thus $\mathcal{A}'_{F_4}$ is finitely generated. However, $\mathcal{A}'_{F_4}$ is not perfect since abelianizing the above presentation gives $\mathcal{A}'_{F_4}/\mathcal{A}''_{F_4} \simeq \mathbb{Z}^4$.

3.3.7 Type H

The presentation of $\mathcal{A}_{H_n}$, $n = 3$ or 4, is

$$\begin{aligned}\mathcal{A}_{H_n} = \langle a_1, ..., a_n : \quad & a_i a_j = a_j a_i \quad \text{for } |i-j| \geq 2,\\ & a_1a_2a_1a_2a_1 = a_2a_1a_2a_1a_2,\\ & a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} \quad \text{for } 2 \leq i \leq n-1 \,\rangle.\end{aligned}$$

As seen in figure 3.1 the graph $(H_n)_{odd}$ is connected. So by theorem 3.1

$$\mathcal{A}'_{H_n} = \mathcal{Z}_{H_n} = \{U \in \mathcal{A}_{H_n} : \deg(U) = 0\}.$$

The computation of the presentation of $\mathcal{A}'_{H_n}$ is similar to that of $\mathcal{A}'_{A_n}$.

Theorem 3.16 *For $n = 3$ or 4 the commutator subgroup $\mathcal{A}'_{H_n}$ of the Artin group $\mathcal{A}_{H_n}$ is the group generated by*

$$p_k = a_1^k a_2 a_1^{-(k+1)} \quad (k \in \mathbb{Z}), \quad q_\ell = a_\ell a_1^{-\ell} \quad (3 \leq \ell \leq n),$$

with defining relations

$$\begin{gathered} p_k q_j = q_j p_{k+1} \ (4 \leq j \leq n),\\ p_{k+1}p_{k+3}p_{k+4}^{-1}p_{k+2}^{-1}p_k^{-1} \ (k \in \mathbb{Z}),\\ p_k q_3 p_{k+2} q_3^{-1} p_{k+1}^{-1} q_3^{-1}\\ q_i q_{i+1} q_i = q_{i+1} q_i q_{i+1} \quad (3 \leq i \leq n-1). \end{gathered}$$

□

The second relation tells us that for $k \neq 0, 1, 2, 3$, p_k can be expressed in terms of p_0, p_1, p_2, and p_3. Thus, $\mathcal{A}'_{H_n}$ is finitely generated. Abelianizing the above presentation results in the trivial group. Thus, we have the following.

Corollary 3.17 *For $n = 3$ or 4 the commutator subgroup $\mathcal{A}'_{H_n}$ of the Artin group of type H_n is finitely generated and perfect.* □

3.3.8 Type I

The presentation of $I_2(m)$, $m \geq 5$, is

$$\mathcal{A}_{I_2(m)} = \langle a_1, a_2 : \langle a_1 a_2 \rangle^m = \langle a_2 a_1 \rangle^m \rangle.$$

In figure 3.1 the graph $(I_2(m))_{odd}$ is connected for m odd and disconnected for m even. Thus, different computations must be done for these two cases. We have the following.

Theorem 3.18 *The commutator subgroup $\mathcal{A}'_{I_2(m)}$ of the Artin group of type $I_2(m)$, $m \geq 5$, is the free group generated by the $(m-1)$-generators*

$$a_1^k a_2 a_1^{-(k+1)} \quad (k \in \{0, 1, 2, \ldots, m-2\}),$$

when m is odd, and is the free group with countably many generators

$$[a_2^\ell, a_1] \;\; (\ell \in \mathbb{Z} \setminus \{-(m/2-1)\}), \quad [a_1^j a_2^\ell, a_1] \;\; (\ell \in \mathbb{Z}, \;\; j = 1, 2, \ldots, m/2-3),$$
$$[a_1^{m/2-2} a_2^\ell, a_1] \;\; (\ell \in \mathbb{Z} \setminus \{m/2-1\}), \quad [a_1^k a_2, a_1] \;\; (k \in \mathbb{Z}).$$

when m is even.

3.3.9 Summary of Results

Table 3.1 summarizes the results in this section. The question marks (?) in the table indicate that it is unknown whether the commutator subgroup is finitely presented. However, we do know that for these cases the group is finitely generated. If one finds more general relation equivalences along the lines of lemmas 3.3 and 3.4 then we may be able to show that these groups are indeed finitely presented.

Type Γ	finitely generated/presented	perfect
A_n	yes/yes	$n = 1, 2, 3$: no, $n \geq 4$: yes
B_n	$n = 2$: no, $n \geq 3$: yes / $n = 3$: yes, $n \geq 3$: ?	$n = 2, 3, 4$: no, $n \geq 5$: yes
D_n	yes/yes	$n = 4$: no, $n \geq 5$: yes
E_n	yes/yes	yes
F_4	yes/?	no
H_n	yes/?	yes
$I_2(m)$ (m even) (m odd)	no/no yes/yes	no no

Table 3.1: Properties of the commutator subgroups

Chapter 4

Local Indicability of Finite-Type Artin Groups

Locally indicable groups first appeared in Higman's thesis [Hig40a] on group rings. He showed that if G is a locally indicable group and R an integral domain then the group ring RG has no zero divisors, no idempotents other than 0 and 1, and no units other than those of the form ug (u a unit in R, $g \in G$). Higman's results have subsequently been extended to larger classes of groups, for example *right-orderable groups*. Our primary interest in local indicability is its application to the theory of right-orderability which is the topic of chapter 5.

4.1 Definitions

A group G is **indicable** if there exists a *nontrivial* homomorphism $G \longrightarrow \mathbb{Z}$ (called an *indexing function*). A group G is **locally indicable** if every nontrivial, finitely generated subgroup is indicable. Notice, finite groups cannot be indicable, so locally indicable groups must be torsion-free.

Every free group is locally indicable. Indeed, it is well known that every subgroup of a free group is itself free, and since free groups are clearly indicable the result follows.

Local indicability is clearly inherited by subgroups. The following simple theorem shows that the category of locally indicable groups is preseved under *extensions*.

Theorem 4.1 *If K, H and G are groups such that K and H are locally indicable and* *fit into a short exact sequence*

$$1 \longrightarrow K \xrightarrow{\ \phi\ } G \xrightarrow{\ \psi\ } H \longrightarrow 1,$$

then G is locally indicable.

Proof. Let $g_1, \ldots, g_n \in G$, and let $\langle g_1, \ldots, g_n \rangle$ denote the subgroup of G which they generate. If $\psi(\langle g_1, \ldots, g_n \rangle) \neq \{1\}$ then by the local indicability of H there exists a nontrivial homomorphism $f : \psi(\langle g_1, \ldots, g_n \rangle) \longrightarrow \mathbb{Z}$. Thus, the map

$$f \circ \psi : \langle g_1, \ldots, g_n \rangle \longrightarrow \mathbb{Z}$$

is nontrivial. Else, if $\psi(\langle g_1, \ldots, g_n \rangle) = \{1\}$ then $g_1, \ldots, g_n \in \ker\psi = \mathrm{Im}\phi$ (by exactness), so there exist $k_1, \ldots, k_n \in K$ such that $\phi(k_i) = g_i$, for all i. Since ϕ is one-to-one (short exact sequence) then $\phi : \langle k_1, \ldots, k_n \rangle \longrightarrow \langle g_1, \ldots, g_n \rangle$ is an isomorphism. By the local indicability of K there exists a nontrivial homomorphism $h : \langle k_1, \ldots, k_n \rangle \longrightarrow \mathbb{Z}$, therefore the map

$$h \circ \phi^{-1} : \langle g_1, \ldots, g_n \rangle \longrightarrow \mathbb{Z}$$

is nontrivial. □

Corollary 4.2 *If G and H are locally indicable then so is $G \oplus H$.*

Proof. The sequence

$$1 \longrightarrow H \xrightarrow{\ \phi\ } G \oplus H \xrightarrow{\ \psi\ } G \longrightarrow 1$$

where $\phi(h) = (1, h)$ and $\psi(g, h) = g$ is exact, so the theorem applies. □

If G and H are groups and $\phi : G \longrightarrow \mathrm{Aut}(H)$. The **semidirect product** of G and H is defined to be the set $H \times G$ with binary operation

$$(h_1, g_1) \cdot (h_2, g_2) = (h_1 \cdot g_1 * h_2, g_1 g_2)$$

where $g * h$ denotes the action of G on H determined by ϕ, i.e. $g * h := \phi(g)(h) \in H$. This group is denoted by $H \rtimes_\phi G$.

Corollary 4.3 *If G and H are locally indicable then so is $H \rtimes_\phi G$.*

Proof. If $\psi : H \rtimes_\phi G \longrightarrow G$ denotes the map $(h, g) \longmapsto g$ then $\ker\psi = \mathrm{H}$ and the groups fit into the exact sequence

$$1 \longrightarrow H \xrightarrow{\text{incl.}} H \rtimes_\phi G \xrightarrow{\psi} G \longrightarrow 1$$

□

The following theorem of Brodskii [Bro80], [Bro84], which was discovered independently by Howie [How82], [How00], tells us that the class of torsion-free 1-relator groups lies inside the class of locally indicable groups. Also, for 1-relator groups: locally indicable $\Leftrightarrow$ torsion free.

Theorem 4.4 *A torsion-free* 1*-relator group is locally indicable.*

To show a group is not locally indicable we need to show there exists a finitely generated subgroup in which the only homomorphism into $\mathbb{Z}$ is the trivial homomorphism.

Theorem 4.5 *If G contains a finitely generated perfect sugroup then G is not locally indicable.*

Proof. The image of a commutator $[a, b] := aba^{-1}b^{-1}$ under a homomorphism into $\mathbb{Z}$ is 0, thus the image of a perfect group is trivial. □

4.2 The Local Indicability of Finite-Type Artin Groups

Since finite-type Artin groups are torsion-free (see section 2.8), theorem 4.4 implies that the Artin groups of type A_2, B_2, and $I_2(m)$ ($m \geq 5$) are locally indicable. In this section we determine the local indicability of *all*[1] irreducible finite-type Artin groups.

It is of interest to note that the discussion in section 3.2, in particular theorem 3.2, shows that an Artin group $\mathcal{A}_\Gamma$ and its commutator subgroup $\mathcal{A}'_\Gamma$ fit into a short exact sequence:

$$1 \longrightarrow \mathcal{A}'_\Gamma \longrightarrow \mathcal{A}_\Gamma \xrightarrow{\phi} \mathbb{Z}^m \longrightarrow 1,$$

[1]with the exception of type F_4 which at this time remains undetermined.

where m is the number of connected components in Γ_{odd}, and ϕ can be identified with the abelianization map. Thus, the local indicability of an Artin group $\mathcal{A}_\Gamma$ is completely determined by the local indicability of its commutator subgroup $\mathcal{A}'_\Gamma$ (by theorem 4.1). In other words,

$$\mathcal{A}_\Gamma \text{ is locally indicable} \Longleftrightarrow \mathcal{A}'_\Gamma \text{ is locally indicable.}$$

This gives another proof that the Artin groups of type A_2, B_2, and $I_2(m)$ ($m \geq 5$) are locally indicable, since their corresponding commutator subgroups are free groups as shown in Chapter 3.

4.2.1 Type A

A_{A_1} is clearly locally indicable since $A_{A_1} \simeq \mathbb{Z}$, and, as noted above, $\mathcal{A}_{A_2}$ is also locally indicable.

For $\mathcal{A}_{A_3}$, theorem 3.8 tells us $\mathcal{A}'_{A_3}$ is the semidirect product of two free groups, thus $\mathcal{A}'_{A_3}$ is locally indicable. It follows from our remarks above that $\mathcal{A}_{A_3}$ is also locally indicable.

As for $\mathcal{A}_{A_n}$, $n \geq 4$, corollary 3.7 and theorem 4.5 imply that $\mathcal{A}_{A_n}$ is not locally indicable.

Thus, we have the following theorem.

Theorem 4.6 *$\mathcal{A}_{A_n}$ is locally indicable if and only if $n = 1, 2,$ or 3.*

4.2.2 Type B

We saw above $\mathcal{A}_{B_2}$ is locally indicable. For $n = 3$ and 4 we argue as follows.

Let $\mathcal{P}^{n+1}_{n+1}$ denote the $(n+1)$-pure braids in $\mathfrak{B}_{n+1} = \mathcal{A}_{A_n}$, that is the braids which only permute the first n-strings. Letting $b_1, \ldots, b_n$ denote the generators of $\mathcal{A}_{B_n}$ a theorem of Crisp [Cri99] states

Theorem 4.7 *The map*

$$\phi : \mathcal{A}_{B_n} \longrightarrow \mathcal{A}_{A_n}$$

defined by

$$b_i \longmapsto a_i, \quad b_n \longmapsto a_n^2$$

is an injective homomorphism onto $\mathcal{P}^{n+1}_{n+1}$. That is, $\mathcal{A}_{B_n} \simeq \mathcal{P}^{n+1}_{n+1} < \mathfrak{B}_{n+1} = \mathcal{A}_{A_n}$.

By "forgetting the n^{th}-strand" we get a homomorphism $f : \mathcal{P}_{n+1}^{n+1} \longrightarrow \mathfrak{B}_n$ which fits into the short exact sequence

$$1 \longrightarrow K \longrightarrow \mathcal{P}_{n+1}^{n+1} \xrightarrow{\ f\ } \mathfrak{B}_n \longrightarrow 1,$$

where $K = \ker \mathrm{f} = \{\beta \in \mathcal{P}_{\mathrm{n+1}}^{\mathrm{n+1}}$: the first n strings of β are trivial$\}$. It is known that $K \simeq F_n$, the free group of rank n. Since F_n is locally indicable and $\mathfrak{B}_n$ $(n = 3, 4)$ is locally indicable then so is $\mathcal{A}_{B_n}$, for $n = 3, 4$. Futhermore, the above exact sequence is actually a split exact sequence so $\mathcal{A}_{B_n} \simeq \mathcal{P}_{n+1}^{n+1} \simeq F_n \rtimes \mathfrak{B}_n$.

As for $\mathcal{A}_{B_n}$, $n \geq 5$, corollary 3.10 and theorem 4.5 imply that $\mathcal{A}_{B_n}$ is not locally indicable, for $n \geq 5$.

Thus, we have the following theorem.

Theorem 4.8 $\mathcal{A}_{B_n}$ *is locally indicable if and only if* $n = 2, 3,$ *or* 4.

4.2.3 Type D

It follows corollary 3.12 and 4.5 that $\mathcal{A}_{D_n}$ is not locally indicable for $n \geq 5$. As for $\mathcal{A}_{D_4}$, we will show it is locally indicable as follows.

A theorem of Crisp and Paris [CP02] says:

Theorem 4.9 *Let* F_{n-1} *denote the free group of rank* $n-1$. *There is an action* $\rho : \mathcal{A}_{A_{n-1}} \longrightarrow \mathrm{Aut}(F_{n-1})$ *such that* $\mathcal{A}_{D_n} \simeq F_{n-1} \rtimes \mathcal{A}_{A_{n-1}}$ *and* ρ *is faithful.*

Since $\mathcal{A}_{A_3}$ and F_3 are locally indicable, then so is $\mathcal{A}_{D_4}$. Thus, we have the following theorem.

Theorem 4.10 $\mathcal{A}_{D_n}$ *is locally indicable if and only if* $n = 4$.

4.2.4 Type E

Since the commutator subgroups of $\mathcal{A}_{E_n}$, $n = 6, 7, 8$, are finitely generated and perfect (corollary 3.14) then $\mathcal{A}_{E_n}$ is not locally indicable.

4.2.5 Type F

Unfortunately, we have yet to determine the local indicability of the Artin group $\mathcal{A}_{F_4}$.

4.2.6 Type H

Since the commutator subgroups of $\mathcal{A}_{H_n}$, $n = 3, 4$, are finitely generated and perfect (corollary 3.17) then $\mathcal{A}_{H_n}$ is not locally indicable.

4.2.7 Type I

As noted above, since the commutator subgroup $\mathcal{A}'_{I_2(m)}$ of $\mathcal{A}_{I_2(m)}$ $(m \geq 5)$ is a free group (theorem 3.18) then $\mathcal{A}'_{I_2(m)}$ is locally indicable and therefore so is $\mathcal{A}_{I_2(m)}$. One could also apply theorem 4.4 to conclude the same result.

Chapter 5
Open Questions: Orderability

In this chapter we discuss the connection between the theory of *orderable groups* and the theory of locally indicable groups. Then we discuss the current state of the orderability of the irreducible finite-type Artin groups.

5.1 Orderable Groups

A group or monoid G is **right-orderable** if there exists a strict linear ordering $<$ of its elements which is right-invariant: $g < h$ implies $gk < hk$ for all g, h, k in G. If there is an ordering of G which is invariant under multiplication on both sides, we say that G is **orderable** or for emphasis **bi-orderable** .

Theorem 5.1 *G is right-orderable if and only if there exists a subset $\mathcal{P} \subset G$ such that:*

$$\begin{gathered}\mathcal{P} \cdot \mathcal{P} \subset \mathcal{P} \text{ (subsemigroup),} \\ G \setminus \{1\} = \mathcal{P} \sqcup \mathcal{P}^{-1}.\end{gathered}$$

Proof. Given $\mathcal{P}$ define $<$ by: $g < h$ iff $hg^{-1} \in \mathcal{P}$. Given $<$ take $\mathcal{P} = \{g \in G : 1 < g\}$. □

In addition, the ordering is a bi-ordering if and only if also

$$g\mathcal{P}g^{-1} \subset \mathcal{P}, \quad \forall g \in G.$$

The set $\mathcal{P} \subset G$ in the previous theorem is called the **positive cone** with respect to the ordering $<$.

The class of right-orderable groups is closed under: subgroups, direct products, free products, semidirect products, and extension. The class of orderable groups is closed under: subgroups, direct products, free products, but not necessarily extensions. Both left-orderability and bi-orderability are local properties: a group has the property if and only if every finitely-generated subgroup has it.

Knowing a group is right-orderable or bi-orderable provides useful information about the internal structure of the group. For example, if G is right-orderable then it must be torsion-free: for $1 < g$ implies $g < g^2 < g^3 < \cdots < g^n < \cdots$. Moreover, if G is bi-orderable then G has no *generalised torsion* (products of conjugates of a nontrivial element being trivial), G has unique roots: $g^n = h^n \Rightarrow g = h$, and if $[g^n, h] = 1$ in G then $[g, h] = 1$. Further consequences of orderablility are as follows. For any group G, let $\mathbb{Z}G$ denote the *group ring* of formal linear combinations $n_1 g_1 + \cdots n_k g_k$.

Theorem 5.2 *If G is right-orderable, then $\mathbb{Z}G$ has no zero divisors.*

Theorem 5.3 *(Malcev, Neumann) If G is bi-orderable, then $\mathbb{Z}G$ embeds in a division ring.*

Theorem 5.4 *(LaGrange, Rhemtulla) If G is right orderable and H is any grooup, then $\mathbb{Z}G \simeq \mathbb{Z}H$ implies $G \simeq H$*

It may be of interest of note that theorem 5.2 has been conjectured to hold for a more general class of groups: the class of torsion-free groups. This is known as the **Zero Divisor Conjecture**. At this time the Zero Divisor Conjecture is still an open question.

The theory of orderable groups is well over a hundred years old. For a general exposition on the theory of orderable groups see [MR77] or [KK74].

Conrad [Con59] investigated the structure of arbitrary right-orderable groups, and defined a useful concept which lies between right-invariance and bi-invariance. A right-ordered group $(G, <)$ is said to be of **Conrad type** if for all $a, b \in G$, with $1 < a$, $1 < b$ there exists a positive integer N such that $a < a^N b$. The following theorems gives the connection between orderable groups and locally indicable groups (see [RR02]).

Theorem 5.5 *For a group G we have* *bi-orderable* $\Rightarrow$ *locally indicable* $\Rightarrow$ *right-orderable.*

Theorem 5.6 *A group is locally indicable if and only if it admits a right-ordering of Conrad type.*

One final connection between local indicability and right-orderability was given by Rhemtulla and Rolfsen [RR02].

Theorem 5.7 *(Rhemtulla, Rolfsen) Suppose* $(G, <)$ *is right-ordered and there is a finite-index subgroup* H *of* G *such that* $(H, <)$ *is a bi-ordered group. Then* G *is locally indicable.*

An application of this theorem is as follows. It is known that the braid groups $\mathfrak{B}_n = \mathcal{A}_{A_{n-1}}$ are right orderable [DDRW02] and that the pure braids $\mathcal{P}_n$ are bi-orderable [KR02]. However, theorem 4.6 tells us that $\mathfrak{B}_n$ is not locally indicable for $n \geq 5$ therefore, by theorem 5.7, the bi-ordering on $\mathcal{P}_n$ and the right-ordering on $\mathfrak{B}_n$ are incompatible for $n \geq 5$.

5.2 Finite-Type Artin Groups

The first proof the that braid groups $\mathfrak{B}_n$ enjoy a right-invariant total ordering was given in [Deh92], [Deh94]. Since then several quite different approaches have been applied to understand this phenomenon.[1] However, it is unknown whether *all* the irreducible finite-type Artin groups are right-orderable. For a few cases one can use theorem 5.6 along with the results of chapter 4 to conclude right-orderability.

One approach is to reduce the problem to showing that the positive Artin monoid is right-orderable.

5.2.1 Ordering the Monoid is Sufficient

We will show that for a Coxeter graph Γ of finite-type the Artin group $\mathcal{A}_\Gamma$ is right-orderable (resp. bi-orderable) if and only if the Artin monoid $\mathcal{A}_\Gamma^+$ is right-orderable (resp. bi-orderable). One direction is of course trivial.

[1]For a wonderful look at this problem and all the differents approaches used to understand it see the book [DDRW02].

Let $\mathcal{A}_\Gamma$ be an Artin group of finite-type. Recall that theorems 2.17 and 2.18 tell us that:

For each $U \in \mathcal{A}_\Gamma$ there exist $U_1, U_2 \in \mathcal{A}_\Gamma^+$, where U_2 is central in $\mathcal{A}_\Gamma$ such that

$$U = U_1 U_2^{-1}.$$

All decompositions of elements of $\mathcal{A}_\Gamma$ in this section are assumed to be of this form.

Suppose $\mathcal{A}_\Gamma^+$ is right-orderable, let $<^+$ be such a right-invariant linear ordering. We wish to prove that $\mathcal{A}_\Gamma$ is right-orderable.

The following lemma indicates how we should extend the ordering on the monoid to the entire group.

Lemma 5.8 *If $U \in \mathcal{A}_\Gamma$ has two decompositions;*

$$U = U_1 U_2^{-1} = \overline{U}_1 \overline{U}_2^{-1},$$

where $U_i, \overline{U}_i \in \mathcal{A}_\Gamma^+$ and $U_2, \overline{U}_2$ central in $\mathcal{A}_\Gamma$, then

$$U_1 <^+ U_2 \Longleftrightarrow \overline{U}_1 <^+ \overline{U}_2.$$

Proof. $U = U_1 U_2^{-1} = \overline{U}_1 \overline{U}_2^{-1}$ implies $U_1 \overline{U}_2 =_p \overline{U}_1 U_2$, since $U_2, \overline{U}_2$ central and $\mathcal{A}_\Gamma^+$ canonically injects in $\mathcal{A}_\Gamma$.

If $U_1 <^+ U_2$ then

$$\begin{aligned}
&\Rightarrow U_1 \overline{U}_2 <^+ U_2 \overline{U}_2 && \text{since } <^+ \text{ right-invariant,}\\
&\Rightarrow U_1 \overline{U}_2 <^+ \overline{U}_2 U_2 && \text{since } U_2 \text{ central,}\\
&\Rightarrow \overline{U}_1 U_2 <^+ \overline{U}_2 U_2 && \text{since } U_1 \overline{U}_2 =_p \overline{U}_1 U_2,\\
&\Rightarrow \overline{U}_1 <^+ \overline{U}_2,
\end{aligned}$$

where the last implication follows from the fact that if $\overline{U}_1 \ {}^+\!\geq \overline{U}_2$ then either: (i) $\overline{U}_1 = \overline{U}_2$, in which case $U = 1$ and so $U_1 = U_2$. Contradiction. (ii) $\overline{U}_1 \ {}^+\!\geq \overline{U}_2$, in which case $\overline{U}_1 U_2 \ {}^+\!\geq \overline{U}_2 U_2$. Again, a contradiction.

The reverse implication follows by symmetry. □

This lemma shows that the following set is well defined:

$$\mathcal{P} = \{U \in \mathcal{A}_\Gamma : \ U \text{ has decomposition } U = U_1 U_2^{-1} \text{ where } U_2 <^+ U_1\}.$$

It is an easy exercise to check that $\mathcal{P}$ is a positive cone in $\mathcal{A}_\Gamma$ which contains $\mathcal{P}^+$: the positive cone in $\mathcal{A}_\Gamma^+$ with respect to the order $<^+$. Thus, the right-invariant order $<^+$ on $\mathcal{A}_\Gamma^+$ extends to a right-invariant order $<$ on $\mathcal{A}_\Gamma$. Furthermore, one can check that if $<^+$ is a bi-invariant order on $\mathcal{A}_\Gamma^+$ then $\mathcal{P}$ satisfies:

$$U\mathcal{P}U^{-1} \subset \mathcal{P}, \quad \forall U \in \mathcal{A}_\Gamma.$$

Thus, the bi-invariant order $<^+$ on $\mathcal{A}_\Gamma^+$ extends to a bi-invariant order $<$ on $\mathcal{A}_\Gamma$.

Open question. Determine the orderability of the finite-type Artin monoids by giving an explicit order condition.

5.2.2 Reduction to Type E_8

Table 2.1 shows that every irreducible finite-type Artin group injects into one type A, D, or E. Thus, if Artin groups of these three types are right-orderable then every finite-type Artin group is right-orderable. It is know that Artin groups of type A, i.e. the braid groups, are right orderable. Also, by theorem 4.9, and the fact that free groups are right-orderable, it follows that $\mathcal{A}_{D_n}$ is right-orderable. Finally, the Artin group of types E_6 and E_7 naturally live inside $\mathcal{A}_{E_8}$, so it suffices to show $\mathcal{A}_{E_8}$ is right-orderable. At this point in time it is unknown whether $\mathcal{A}_{E_8}$ is right-orderable. As section 5.2.1 indicates it is enough to decide whether the Artin monoid $\mathcal{A}_{E_8}^+$ is right-orderable.

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