# A Presentation for the Commutator Sugroups of the Braid Groups $B_n$

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**Abstract**

In [1] Gorin and Lin gave a presentation for the commutator subgroup $B_n'$ of the braid group $B_n$, $n \geq 3$, which consists of finitely many generators and relations. Here we fill in all the details of their computation.

## Introduction

The braid group on n strands, denoted $B_n$, is defined by the presentation

$$B_n = < \sigma_1, ..., \sigma_{n-1} : \quad \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \leq i \leq n-2 > . \end{array}$$

The commutator subgroup $B_n'$ of the braid group $B_n$ is the group generated by the commutators; $[\beta_1, \beta_2] = \beta_1\beta_2\beta_1^{-1}\beta_2^{-1}$, for all $\beta_1, \beta_2 \in B_n$. In [1] Gorin and Lin outlined the proof of the following theorem giving a presentation for $B_n'$.

**Theorem 1** *For every $n \geq 3$ the commutator subgroup $B_n'$ of the braid group $B_n$ is a finitely presented group. $B_3'$ is a free group with two free generators*

$$u = \sigma_2\sigma_1^{-1}, \quad v = \sigma_1\sigma_2\sigma_1^{-2}.$$

*$B_4'$ is the group generated by*

$$p_0 = \sigma_2\sigma_2^{-1}, \quad p_1 = \sigma_1\sigma_2\sigma_1^{-2}, \quad a = \sigma_3\sigma_1^{-1}, \quad b = \sigma_2\sigma_1^{-1}\sigma_3\sigma_2^{-1},$$

*with defining relations*

$$\begin{array}{rcl} b & = & p_0 a p_0^{-1} \\ p_0 b p_0^{-1} & = & b^2 a^{-1} b \\ p_1 a p_1^{-1} & = & a^{-1} b \\ p_1 b p_1^{-1} & = & (a^{-1}b)^3 a^{-2} b. \end{array}$$

*For $n \geq 5$ the group $B_n'$ is generated by*

$$p_0 = \sigma_2\sigma_2^{-1}, \quad p_1 = \sigma_1\sigma_2\sigma_1^{-2}, \quad a = \sigma_3\sigma_1^{-1}, \quad b = \sigma_2\sigma_1^{-1}\sigma_3\sigma_2^{-1}, \quad q_l = \sigma_l\sigma_1^{-1} \quad (4 \leq l \leq n-1),$$

*with defining relations*

$$\begin{array}{rcl} b & = & p_0 a p_0^{-1}, \\ p_0 b p_0^{-1} & = & b^2 a^{-1} b, \\ p_1 a p_1^{-1} & = & a^{-1} b, \\ p_1 b p_1^{-1} & = & (a^{-1}b)^3 a^{-2} b, \end{array}$$

$$\begin{array}{rl} p_0 q_i = q_i p_1 \quad (4 \leq i \leq n-1), & p_1 q_i = q_1 p_0^{-1} p_1 \quad (4 \leq i \leq n-1) \\ a q_i = q_i a \quad (5 \leq i \leq n-1), & a q_4 a = q_4 a q_4, \\ q_i q_j = q_j q_i \quad (4 \leq i < j-1 \leq n-2), & q_i q_{i+1} q_i = q_{i+1} q_i q_{i+1} \quad (4 \leq i \leq n-2). \end{array}$$

In this paper we give all the details for the proof of this theorem.

# The Presentation

In this section we fill in all the details of the proof of Theorm 1. To do this we first need to recall the Reidemeister-Schreier method for presenting a subgroup. For a complete discussion of the Reidemeister-Schreier method see [2].

Let $G$ be an arbitrary group with presentation $< a_1, \ldots, a_n : R_\mu(a_\nu), \ldots >$ and $H$ a subgroup of $G$. A system of words $\mathbf{R}$ in the generators $a_1, \ldots, a_n$ is called a *Schreier system* if (i) every right coset of $H$ in $G$ contains exactly one word of $\mathbf{R}$ (i.e. $\mathbf{R}$ forms a system of right coset representatives), (ii) for each word in $\mathbf{R}$ any initial segment is also in $\mathbf{R}$ (i.e. initial segments of right coset representatives are again right coset representatives). Such a Schreier system always exists, see for example [2]. Suppose now that we have fixed a Schreier system $\mathbf{R}$. For each word $W$ in the generators $a_1, \ldots, a_n$ we let $\overline{W}$ denote the unique representative in $\mathbf{R}$ of the right coset $HW$. Denote

$$s_{K,a_v} = K a_v \cdot \overline{K a_v}^{-1}, \tag{1}$$

for each $K \in \mathbf{R}$ and generator $a_v$. A theorem of Reidemeister-Schreier states that $H$ has presentation

$$< s_{K,a_\nu}, \ldots : s_{M,a_\lambda}, \ldots, \tau(K R_\mu K^{-1}), \ldots > \tag{2}$$

where $K$ is an arbitrary Schreier representative, $a_v$ is an arbitrary generator and $R_\mu$ is an arbitrary defining relator in the presentation of $G$, and $M$ is a Schreier representative and $a_\lambda$ a generator such that

$$M a_\lambda \approx \overline{M a_\lambda},$$

where $\approx$ means "freely equal". The function $\tau$ is a *Reidemeister rewriting function* and is defined according to the rule

$$\tau(a_{i_1}^{\epsilon_1} \cdots a_{i_p}^{\epsilon_p}) = s_{K_{i_1},a_{i_1}}^{\epsilon_1} \cdots s_{K_{i_p},a_{i_p}}^{\epsilon_1} \tag{3}$$

where $K_{i_j} = \overline{a_{i_1}^{\epsilon_1} \cdots a_{i_{j-1}}^{\epsilon_{j-1}}}$, if $\epsilon_j = 1$, and $K_{i_j} = \overline{a_{i_1}^{\epsilon_1} \cdots a_{i_j}^{\epsilon_j}}$, if $\epsilon_j = -1$. It should be noted that computation of $\tau(U)$ can be carried out by replacing a symbol $a_v^\epsilon$ of U by the appropriate s-symbol $s_{K,a_\nu}^\epsilon$. The main property of a Reidemeister rewriting function is that for an element $U \in H$ given in terms of the generators $a_\nu$ the word $\tau(U)$ is the same element of $H$ rewritten in terms of the generators $s_{K,a_\nu}$.

Now we may begin the study of the commutator subgroups of the braid groups. Let $H_n = \{\beta \in B_n : exp(\beta) = 0\}$ , which is easily seen to be a subgroup of $B_n$. In fact, this subgroup is precisely $B_n'$.

**Lemma 2** $H_n = B_n'$

Proof: Since $B_n'$ is generated by the commutators $[\beta_1, \beta_2]$, $\beta_1, \beta_2 \in B_n$, and $exp([\beta_1, \beta_2]) = 0$ then $B_n' \leq H_n$. To prove the converse we apply the Reidemeister-Schreier method to the subgroup $H_n$ to find a generating set. Since

$$\begin{aligned} H_n\beta_1 = H_n\beta_2 \quad &\Leftrightarrow \quad \beta_1\beta_2^{-1} \in H_n \\ &\Leftrightarrow \quad exp(\beta_1) = exp(\beta_2), \end{aligned}$$

then a Schreier system of right coset representatives for $B_n$ modulo $H_n$ is

$$\mathbf{R} = \{\sigma_1^k : k \in \mathbb{Z}\}$$

The discussion above tells us that $H_n$ is generated by the s-symbols

$$\begin{aligned} s_{\sigma_1^k, \sigma_j} &= \sigma_1^k \sigma_j \overline{\sigma_1^k \sigma_j}^{-1} \\ &= \sigma_1^k \sigma_j \sigma_1^{-(k+1)}. \end{aligned}$$

For $j = 2$ we have

$$\begin{aligned} \sigma_1^k \sigma_2 \sigma_1^{-(k+1)} &= \sigma_1^k \sigma_2 \sigma_1^{-k} (\sigma_2^{-1} \sigma_2) \sigma_1^{-1} \\ &= [\sigma_1^k, \sigma_2] \sigma_2 \sigma_1^{-1} \\ &= [\sigma_1^k, \sigma_2] \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_2 \qquad \text{by} \quad \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \\ &= [\sigma_1^k, \sigma_2][\sigma_1^{-1}, \sigma_2^{-1}]. \end{aligned}$$

For $j > 2$ we have

$$\begin{aligned} \sigma_1^k \sigma_j \sigma_1^{-(k+1)} &= \sigma_j \sigma_1^k \sigma_1^{-(k+1)} \qquad \text{by} \quad \sigma_1\sigma_j = \sigma_j\sigma_1, \\ &= \sigma_j \sigma_1^{-1} \\ &= \sigma_j (\sigma_{j-1}^{-1} \sigma_{j-1}) \sigma_1^{-1} \\ &= (\sigma_j \sigma_{j-1}^{-1}) \sigma_{j-1} \sigma_1^{-1} \\ &= (\sigma_{j-1}^{-1} \sigma_j^{-1} \sigma_{j-1} \sigma_j) \sigma_{j-1} \sigma_1^{-1} \\ &= \begin{cases} [\sigma_{j-1}^{-1}, \sigma_j^{-1}][\sigma_1^{-1}, \sigma_2^{-1}] & \text{if } j = 3, \\ [\sigma_{j-1}^{-1}, \sigma_j^{-1}] \sigma_1^k \sigma_{j-1} \sigma_1^{-(k+1)} & \text{if } j \geq 4. \end{cases} \end{aligned}$$

It follows by induction on $j$ that the generators of $H_n$ lie in $B_n'$. $\square$

In the proof of the previous lemma we found a Schreier system for $B_n$ modulo $B_n'$, namely $\mathbf{R} = \{\sigma_1^k : k \in \mathbb{Z}\}$, thus, by (2), $B_n'$ has presentation

$$< s_{\sigma_1^k,\sigma_j}, \ldots : s_{\sigma_1^m,\sigma_\lambda}, \ldots, \tau(\sigma_1^l R_i \sigma_1^{-l}), \ldots, \tau(\sigma_1^l T_{i,j} \sigma_1^{-l}), \ldots > \tag{4}$$

where $j \in \{1, \ldots, n-1\}$, $k, l \in \mathbb{Z}$, and $m \in \mathbb{Z}$, $\lambda \in \{1, \ldots, n-1\}$ such that $\sigma_1^m \sigma_\lambda \approx \overline{\sigma_1^m \sigma_\lambda}$ ("freely equal"), and $T_{i,j}$, $R_i$ represent the braid relations $\sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1}$, $|i-j| \geq 2$, and $\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1}$, respectively. Our goal now is to clean up this presentation.

The first thing to notice is that

$$\sigma_1^m \sigma_\lambda \approx \overline{\sigma_1^m \sigma_\lambda} = \sigma_1^{m+1}$$

if and only if $\lambda = 1$. Thus, the first type of relations in (4) are precisly $s_{\sigma_1^m,\sigma_1} = 1$, for all $m \in \mathbb{Z}$.

Next, we use the definition of the Reidemeister rewriting function (3) to express the second and third types of relations in (4) in terms of the generators $s_{\sigma_1^k,\sigma_j}$:

$$\tau(\sigma_1^k T_{i,j} \sigma_1^{-k}) = s_{\sigma_1^k,\sigma_i} s_{\sigma_1^{k+1},\sigma_j} s^{-1}_{\sigma_1^{k+1},\sigma_i} s^{-1}_{\sigma_1^k,\sigma_j} \tag{5}$$

$$\tau(\sigma_1^k R_i \sigma_1^{-k}) = s_{\sigma_1^k,\sigma_i} s_{\sigma_1^{k+1},\sigma_{i+1}} s_{\sigma_1^{k+2},\sigma_i} s^{-1}_{\sigma_1^{k+2},\sigma_{i+1}} s^{-1}_{\sigma_1^{k+1},\sigma_i} s^{-1}_{\sigma_1^k,\sigma_{i+1}} \tag{6}$$

From (5) with $i = 1$, $j \geq 3$, and using the relations $s_{\sigma_1^m,\sigma_1} = 1$ we get the relation

$$s_{\sigma_1^{k+1},\sigma_j} = s_{\sigma_1^k,\sigma_j}$$

Thus, by induction on k, $s_{\sigma_1^k,\sigma_j} = s_{1,\sigma_j}$ for $j \geq 3$ and for all $k \in \mathbb{Z}$.

Therefore, $B_n'$ is generated by $s_{\sigma_1^k,\sigma_2} = \sigma_1^k \sigma_2 \sigma_1^{-(k+1)}$ and $s_{1,\sigma_l} = \sigma_l \sigma_1^{-1}$, where $k \in \mathbb{Z}$ , $3 \leq l \leq n-1$. To simplify notation let us rename the generators; let $p_k := \sigma_1^k \sigma_2 \sigma_1^{-(k+1)}$ and $q_l := \sigma_l \sigma_1^{-1}$, for $k \in \mathbb{Z}$ , $3 \leq l \leq n-1$. Now we need to investigate the relations in (5),(6).

The relations in (6) break up into the following three types (using the relations $s_{\sigma_1^k,\sigma_j} = s_{\sigma_1,\sigma_j} = q_j$ for $j \geq 3$):

$$p_{k+1} p_{k+2}^{-1} p_k^{-1} \qquad (i = 1) \tag{7}$$

$$p_k q_3 p_{k+2} q_3^{-1} p_{k+1}^{-1} q_3^{-1} \qquad (i = 2) \tag{8}$$

$$q_i q_{i+1} q_i q_{i+1}^{-1} q_i^{-1} q_{i+1}^{-1} \qquad \text{for } 3 \leq i \leq n-2. \tag{9}$$

The relations in (5) break up into the following two types

$$p_k q_j p_{k+1}^{-1} q_j^{-1} \qquad \text{for } 4 \leq j \leq n-1 \ (i = 2) \tag{10}$$

$$q_i q_j q_i^{-1} q_j^{-1} \qquad \text{for } 3 \leq i < j \leq n-1,\ |i-j| \geq 2. \tag{11}$$

We now have a presentation for $B_n'$ consisting of the generators $p_k, q_l$, where $k \in \mathbb{Z}$, $3 \leq l \leq n-1$, and defining relations (7)-(11). However, notice that relation (7) splits up into the two relations

$$p_{k+2} = p_k^{-1} p_{k+1} \qquad \text{for } \mathrm{k} \geq 0 \tag{12}$$
$$p_k = p_{k+1} p_{k+2}^{-1} \qquad \text{for } \mathrm{k} < 0. \tag{13}$$

Thus, for $k \neq 0, 1$, $p_k$ can be expressed in terms of $p_0$ and $p_1$. From this it follows that $B_n'$ is finitely generated. In fact, the presentation for $B_n'$ with generators $p_k, q_l$, where $k \in \mathbb{Z}, 3 \leq l \leq n-1$ and defining relations (8)-(13) can be Tietze transformed into a presentation which consists of finitely many generators and relations. We do this for the cases $n = 3$, $n = 4$, and $n = 5$ first and then proceed to the general case.

## The Case $n = 3$

For $n = 3$ the presentation above reduces to $< p_k, (k \in \mathbb{Z}) : (12), (13) > \; = \; < p_0, p_1 : \; > \; = \; \mathbb{Z} * \mathbb{Z}$. Thus, $B_3'$ is a free group with two free generators $u = p_0 = \sigma_2 \sigma_1^{-1}$ and $v = p_1 = \sigma_1 \sigma_2 \sigma_1^{-2}$.

## The Case $n = 4$

Let $n = 4$. The set of generators is $p_k$,$k \in \mathbb{Z}$ and $a := q_3$, and the set of defining relations (8)-(13) reduces to

$$p_{k+2} = p_k^{-1} p_{k+1} \quad \text{for } \mathrm{k} \geq 0 \qquad p_k = p_{k+1} p_{k+2}^{-1} \quad \text{for } \mathrm{k} < 0 \tag{14}$$
$$p_k a p_{k+2} a^{-1} p_{k+1}^{-1} a^{-1} \;=\; 1. \tag{15}$$

We know that for $k \neq 0, 1$, $p_k$ can be expressed in terms of $p_0$ and $p_1$ but we can't delete these generators from the generating set since they appear in the relation (15). What we want to do is to replace the infinte set of relations in (15) with a finite set of equivalent relations. We do this as follows. Introduce to the generators $p_k$, $a$ of the group $B_4'$ a new generator $b$ and to the relations (14), (15) a new relation

$$b = p_0 a p_0^{-1}. \tag{16}$$

By a theorem of Tietze this gives an equivalent representation (see [2]). Now we show that in the system of relations (14)-(16) we can replace (15) by

$$p_0 b p_0^{-1} \;=\; b^2 a^{-1} b \tag{17}$$
$$p_1 a p_1^{-1} \;=\; a^{-1} b \tag{18}$$
$$p_1 b p_1^{-1} \;=\; (a^{-1} b)^3 a^{-2} b, \tag{19}$$

and obtain a system of relations equivalent to the original relations. We do this in two steps: first we show (14),(16)-(19) follow from (14)-(16), then we show the converse.

Taking $k = 0$ in (15) we get the relation

$$p_0 a p_2 a^{-1} p_1^{-1} a^{-1} = 1,$$

and, using the relations $p_2 = p_0^{-1} p_1$ and $b = p_0 a p_0^{-1}$, (18) easily follows. Taking $k = 1$ in (15) we get the relation

$$p_1 a p_3 a^{-1} p_2^{-1} a^{-1} = 1.$$

Using the relations $p_3 = p_1^{-1} p_2$ and $p_2 = p_0^{-1} p_1$ this becomes

$$p_1 a p_1^{-1} p_0^{-1} p_1 a^{-1} p_1^{-1} p_0 a^{-1} = 1.$$

But $p_1 a p_1^{-1} = a^{-1} b$ (by (18)) so this reduces to

$$a^{-1} b p_0^{-1} b^{-1} a p_0 a^{-1} = 1.$$

Isolating $b p_0^{-1}$ on one side of the equation gives

$$b p_0^{-1} = a^2 p_0^{-1} a^{-1} b.$$

Multiplying both sides on the left by $p_0$ and using the relation $p_0 a p_0^{-1} = b$ it easily follows $p_0 b p_0^{-1} = b^2 a^{-1} b$, which is (17). Taking $k = 2$ in (15) we get the relation

$$p_2 a p_4 a^{-1} p_3^{-1} a^{-1} = 1.$$

Using the relation $p_4 = p_2^{-1} p_3$ this becomes

$$p_2 a p_2^{-1} p_3 a^{-1} p_3^{-1} a^{-1} = 1. \tag{20}$$

Note that

$$\begin{aligned} p_2 a p_2^{-1} &= p_0^{-1} p_1 a p_1^{-1} p_0 \qquad \text{by } \mathrm{p}_2 = \mathrm{p}_0^{-1}\mathrm{p}_1 \\ &= p_0^{-1} a^{-1} b p_0 \qquad \text{by (18)} \\ &= a^{-2} b a^{-1} a \qquad \text{by (16) and (17)} \\ &= a^{-2} b \end{aligned}$$

and

$$\begin{aligned} p_3 a p_3^{-1} &= p_1^{-1} p_2 a p_2^{-1} p_1 \qquad \text{by } \mathrm{p}_3 = \mathrm{p}_1^{-1}\mathrm{p}_2 \\ &= p_1^{-1} a^{-2} b p_1, \end{aligned}$$

where the second equality follows from the previous statement. Thus, (20) becomes

$$a^{-2} b p_1^{-1} b^{-1} a^2 p_1 a^{-1} = 1$$

Isolating $b p_1^{-1}$ on one side of the equation and using the relation (18) we easily get the relation (19). Therefore we have that relations (14),(16)-(19) follow from relations (14)-(16). Next we show the converse holds.

Let $\mathcal{R}$ denote the set of relations (14),(16)-(19). We wish to show relations (14)-(16) follow from $\mathcal{R}$, in particular (15) follows from $\mathcal{R}$. To do this we use the following lemma.

**Lemma 3** *The relations*

$$\begin{aligned}
p_k a p_k^{-1} &= a^k b && (21)\\
p_k b p_k^{-1} &= (a^{-k} b)^{k+2} a^{-(k+1)} b && (22)\\
p_k^{-1} a p_k &= a b^{-1} a^{k+2} && (23)\\
p_k^{-1} b p_k &= (a b^{-1} a^{k+2})^k a && (24)
\end{aligned}$$

*follow from $\mathcal{R}$.*

Proof: We will use induction to prove the result for nonnegative indices k, the result for negative indices k is similar. Clearly this holds for $k = 0, 1$. For $k = m + 2$ we have

$$\begin{aligned}
p_{m+2} a p_{m+2}^{-1} &= p_m^{-1} p_{m+1} a p_{m+1}^{-1} p_m \quad \text{by } \mathcal{R},\\
&= p_m^{-1} a^{-(m+1)} b p_m \quad \text{by induction hypothesis (IH)},\\
&= (p_m^{-1} a^{-(m+1)} p_m)(p_m^{-1} b p_m),\\
&= (p_m^{-1} a p_m)^{-(m+1)} (p_m^{-1} b p_m),\\
&= (a b^{-1} a^{m+2})^{-(m+1)} (a b^{-1} a^{m+2})^m a \quad \text{by IH},\\
&= (a b^{-1} a^{m+2})^{-1} a,\\
&= a^{-(m+2)} b,
\end{aligned}$$

$$\begin{aligned}
p_{m+2} b p_{m+2}^{-1} &= p_m^{-1} p_{m+1} b p_{m+1}^{-1} p_m \quad \text{by } \mathcal{R},\\
&= p_m^{-1} (a^{-(m+1)} b)^{m+3} a^{-(m+2)} b p_m \quad \text{by IH},\\
&= ((p_m^{-1} a p_m)^{-(m+1)} (p_m^{-1} b p_m))^{m+3} (p_m^{-1} a p_m)^{-(m+2)} p_m^{-1} b p_m,\\
&= ((a b^{-1} a^{m+2})^{-(m+1)} (a b^{-1} a^{m+2})^m a)^{(m+3)} (a b^{-1} a^{m+2})^{-(m+2)} (a b^{-1} a^{m+2})^m a,\\
&= (a^{-(m+2)} b)^{m+3} (a b^{-1} a^{m+2})^{-2} a,\\
&= (a^{-(m+2)} b)^{m+4} a^{-(m+3)} b,
\end{aligned}$$

$$\begin{aligned}
p_{m+2}^{-1} a p_{m+2} &= p_{m+1}^{-1} p_m a p_m^{-1} p_{m+1} \quad \text{by } \mathcal{R},\\
&= p_{m+1}^{-1} a^{-m} b p_{m+1} \quad \text{by IH},\\
&= (p_{m+1}^{-1} a p_{m+1})^{-m} (p_{m+1}^{-1} b p_{m+1}),\\
&= (a b^{-1} a^{m+3})^{-m} (a b^{-1} a^{m+3})^{m+1} a \quad \text{by IH},\\
&= a b^{-1} a^{m+3} a,\\
&= a b^{-1} a^{m+4},
\end{aligned}$$

and

$$\begin{aligned}
p_{m+2}^{-1}bp_{m+2} &= p_{m+1}^{-1}p_m b p_m^{-1}p_{m+1} \qquad \text{by } \mathcal{R}, \\
&= p_{m+1}^{-1}(a^{-m}b)^{m+2}a^{-(m+1)}bp_{m+1} \qquad \text{by IH}, \\
&= ((p_{m+1}^{-1}ap_{m+1})^{-m}(p_{m+1}^{-1}bp_{m+1}))^{m+2}(p_{m+1}^{-1}ap_{m+1})^{-(m+1)}p_{m+1}^{-1}bp_{m+1}, \\
&= ((ab^{-1}a^{m+3})^{-m}(ab^{-1}a^{m+3})^{m+1}a)^{(m+2)}(ab^{-1}a^{m+3})^{-(m+1)}(ab^{-1}a^{m+3})^{m+1}a, \\
&= (ab^{-1}a^{m+4})^{m+2}a.
\end{aligned}$$

Thus, the result follows by induction. $\square$

From the relations (21)-(24) we obtain

$$p_{k+1}ap_{k+1}^{-1} = a^{-(k+1)}b = a^{-1}\cdot a^{-k}b = a^{-1}p_k a p_k^{-1}, \tag{25}$$

and

$$p_{k+1}^{-1}ap_{k+1} = ab^{-1}a^{k+2} = ab^{-1}a^{k+1}a = p_k^{-1}ap_k a. \tag{26}$$

Now we are in a position to show that (15) follows from $\mathcal{R}$. For $k \geq 0$

$$\begin{aligned}
p_k a p_{k+2}a^{-1}p_{k+1}^{-1}a^{-1} &= p_k a p_k^{-1}\underbrace{p_{k+1}a^{-1}p_{k+1}^{-1}}a^{-1} \qquad \text{by } (14) \\
&= p_k a p_k^{-1}(a^{-1}p_k a p_k^{-1})^{-1}a^{-1} \qquad \text{by } (25) \\
&= 1.
\end{aligned}$$

and for $k < 0$

$$\begin{aligned}
p_k a p_{k+2}a^{-1}p_{k+1}^{-1}a^{-1} &= p_{k+1}\underbrace{p_{k+2}^{-1}ap_{k+2}}a^{-1}p_{k+1}^{-1}a^{-1} \qquad \text{by } (14) \\
&= p_{k+1}(p_{k+1}^{-1}ap_{k+1}a)a^{-1}p_{k+1}^{-1}a^{-1} \qquad \text{by } (26) \\
&= 1.
\end{aligned}$$

Therefore, the relations

$$p_k a p_{k+2}a^{-1}p_{k+1}^{-1}a^{-1} = 1, \qquad k \in \mathbb{Z}$$

follows from the relations in $\mathcal{R}$. Thus, we have that $B_4'$ is generated by $p_k, a, b$, $k \in \mathbb{Z}$ with the set of defining relations

$$\begin{aligned}
p_{k+2} &= p_k^{-1}p_{k+1} \qquad (\text{k} \geq 0) \\
p_k &= p_{k+1}p_{k+2}^{-1} \qquad (\text{k} < 0) \\
b &= p_0 a p_0^{-1} \\
p_0 b p_0^{-1} &= b^2 a^{-1} b \\
p_1 a p_1^{-1} &= a^{-1}b \\
p_1 b p_1^{-1} &= (a^{-1}b)^3 a^{-2} b.
\end{aligned}$$

Since the generators $p_k$, $k \neq 0, 1$ appear in only the first two relations we have proved the following theorem.

**Theorem 4** *The commutator subgroup $B_4'$ of the braid group $B_4$ is generated by*

$$p_0 = \sigma_2\sigma_2^{-1}, \quad p_1 = \sigma_1\sigma_2\sigma_1^{-2}, \quad a = \sigma_3\sigma_1^{-1}, \quad b = \sigma_2\sigma_1^{-1}\sigma_3\sigma_2^{-1},$$

*with defining relations*

$$\begin{aligned} b &= p_0 a p_0^{-1} \\ p_0 b p_0^{-1} &= b^2 a^{-1} b \\ p_1 a p_1^{-1} &= a^{-1} b \\ p_1 b p_1^{-1} &= (a^{-1}b)^3 a^{-2} b. \end{aligned}$$

□

## The Case $n = 5$

Let $n = 5$. We have already shown that $B_5'$ is generated by $p_k$,$q_3$,$q_4$, $k \in \mathbb{Z}$ with defining relations (8)-(13), which in the case for $n = 5$ become

$$\begin{aligned} p_{k+2} = p_k^{-1} p_{k+1} \ \ (\mathrm{k} \geq 0), &\qquad p_k = p_{k+1} p_{k+2}^{-1} \ \ (\mathrm{k} < 0). \\ p_k q_4 &= q_4 p_{k+1} \\ p_k q_3 p_{k+2} &= q_3 p_{k+1} q_3 \\ q_3 q_4 q_3 &= q_4 q_3 q_4, \end{aligned}$$

for $k \in \mathbb{Z}$. Let us denote the generator $q_3$ by $a$. Then the relations can be be written as

$$\begin{aligned} p_{k+2} = p_k^{-1} p_{k+1} \ \ (\mathrm{k} \geq 0), &\qquad p_k = p_{k+1} p_{k+2}^{-1} \ \ (\mathrm{k} < 0). && (27) \\ p_k a p_{k+2} &= a p_{k+1} a && (28) \\ p_k q_4 &= q_4 p_{k+1} && (29) \\ a q_4 a &= q_4 a q_4. && (30) \end{aligned}$$

As was done in the case for $n = 4$ we add to the generators $p_k$,$q_3$,$q_4$, $k \in \mathbb{Z}$ of $B_5'$ a new generator b, and to the relations (27)-(30) a new relation

$$b = p_0 a p_0^{-1}.$$

Notice the relations (27)-(28), and $b = p_0 a p_0^{-1}$ are exactly those that occured in the case $n = 4$, and we showed that they are equivalent to the relations

$$\begin{aligned} p_{k+2} &= p_k^{-1} p_{k+1} \ \ (\mathrm{k} \geq 0) && (31) \\ p_k &= p_{k+1} p_{k+2}^{-1} \ \ (\mathrm{k} < 0) && (32) \\ b &= p_0 a p_0^{-1} && (33) \\ p_0 b p_0^{-1} &= b^2 a^{-1} b && (34) \\ p_1 a p_1^{-1} &= a^{-1} b && (35) \\ p_1 b p_1^{-1} &= (a^{-1}b)^3 a^{-2} b. && (36) \end{aligned}$$

So $B_5'$ is generated by $p_k$,$a$,$b$,$q_4$, $k \in \mathbb{Z}$ with eight defining relations (29)-(36). Relations (31),(32) tell us that $p_k$, $k \neq 0, 1$ can be expressed in terms of $p_0$ and $p_1$, so the only relations on $p_k$, $k \neq 0, 1$ are (29); $p_k q_4 = q_4 p_k$, $k \in \mathbb{Z}$. It is these relations that we want to try to get rid of by replacing them with finitely many other relations involving only the generators $p_0$,$p_1$,$a$,$b$,$q_4$.

Taking $k = 0$ and $k = 1$ in (29) we get the two relations

$$p_0 q_4 = q_4 p_1 \qquad p_1 q_4 = q_4 p_0^{-1} p_1.$$

The following lemma tells us that these two relations can replace the relations (29).

**Lemma 5** *The set of relations:*

$$\begin{array}{rcl} p_{k+2} = p_k^{-1} p_{k+1} \quad (\mathrm{k} \geq 0), & & p_k = p_{k+1} p_{k+2}^{-1} \quad (\mathrm{k} < 0), \\ p_k q_4 & = & q_4 p_{k+1} \quad (\mathrm{k} \in \mathbb{Z}), \end{array}$$

*is equivalent to the set*

$$\begin{array}{cc} p_{k+2} = p_k^{-1} p_{k+1} \quad (\mathrm{k} \geq 0), & p_k = p_{k+1} p_{k+2}^{-1} \quad (\mathrm{k} < 0), \\ p_0 q_4 = q_4 p_1 & p_1 q_4 = q_4 p_0^{-1} p_1. \end{array}$$

Proof: Clearly the second set of relations follows from the first set of relations. To prove the converse we first prove that $p_k q_4 = q_4 p_{k+1}$, $k \geq 0$, follows from the second set of relations by induction on $k$. It is easy to see then that the same is true for $k < 0$. For $k = 0, 1$ the result clearly holds. Now, for $k = m + 2$;

$$\begin{array}{rcl} p_{m+2} q_4 p_{m+3}^{-1} q_4^{-1} & = & p_{m+2} q_4 p_{m+2}^{-1} p_{m+1} q_4^{-1}, \\ & = & p_{m+2} (p_{m+1}^{-1} q_4) p_{m+1} q_4^{-1} \quad \text{by IH } (k = m+1), \\ & = & p_{m+2} p_{m+1}^{-1} (q_4 p_{m+1}) q_4^{-1}, \\ & = & p_{m+2} p_{m+1}^{-1} (p_m q_4) q_4^{-1} \quad \text{by IH } (k = m), \\ & = & p_{m+2} p_{m+1}^{-1} p_m, \\ & = & 1. \end{array}$$

□

It follows that $B_5'$ is generated by $p_k$,$a$,$b$,$q_4$, $k \in \mathbb{Z}$ with defining relations

$$\begin{array}{rcl} p_{k+2} = p_k^{-1} p_{k+1} \quad (\mathrm{k} \geq 0), & & p_k = p_{k+1} p_{k+2}^{-1} \quad (\mathrm{k} < 0), \\ b & = & p_0 a p_0^{-1}, \\ p_0 b p_0^{-1} & = & b^2 a^{-1} b, \\ p_1 a p_1^{-1} & = & a^{-1} b, \\ p_1 b p_1^{-1} & = & (a^{-1} b)^3 a^{-2} b, \\ a q_4 a & = & q_4 a q_4, \\ p_0 q_4 = q_4 p_1, & & p_1 q_4 = q_4 p_0^{-1} p_1. \end{array}$$

Since the generators $p_k$, $k \neq 0, 1$ now appear in only the first two relations we have proved the following theorem.

**Theorem 6** *The commutator subgroup $B_5'$ of the braid group $B_5$ is generated by*

$$p_0 = \sigma_2\sigma_2^{-1}, \quad p_1 = \sigma_1\sigma_2\sigma_1^{-2}, \quad a = \sigma_3\sigma_1^{-1}, \quad b = \sigma_2\sigma_1^{-1}\sigma_3\sigma_2^{-1}, \quad q = \sigma_4\sigma_1^{-1},$$

*with defining relations*

$$\begin{array}{rcl} b & = & p_0 a p_0^{-1}, \\ p_0 b p_0^{-1} & = & b^2 a^{-1} b, \\ p_1 a p_1^{-1} & = & a^{-1} b, \\ p_1 b p_1^{-1} & = & (a^{-1}b)^3 a^{-2} b, \\ aqa & = & qaq, \\ p_0 q = q p_1, & & p_1 q = q p_0^{-1} p_1. \end{array}$$

□

## The General Case; $n > 5$

Having done most of the work in the case $n = 5$ it is relatively easy to check the following theorem.

**Theorem 7** *The commutator subgroup $B_n'$ of the braid group $B_n$, $n \geq 5$, is generated by*

$$p_0 = \sigma_2\sigma_2^{-1}, \quad p_1 = \sigma_1\sigma_2\sigma_1^{-2}, \quad a = \sigma_3\sigma_1^{-1}, \quad b = \sigma_2\sigma_1^{-1}\sigma_3\sigma_2^{-1}, \quad q_l = \sigma_l\sigma_1^{-1} \quad (4 \leq l \leq n-1),$$

*with defining relations*

$$\begin{array}{rcl} b & = & p_0 a p_0^{-1}, \\ p_0 b p_0^{-1} & = & b^2 a^{-1} b, \\ p_1 a p_1^{-1} & = & a^{-1} b, \\ p_1 b p_1^{-1} & = & (a^{-1}b)^3 a^{-2} b, \\ p_0 q_i = q_i p_1 \quad (4 \leq i \leq n-1), & & p_1 q_i = q_1 p_0^{-1} p_1 \quad (4 \leq i \leq n-1) \\ a q_i = q_i a \quad (5 \leq i \leq n-1), & & a q_4 a = q_4 a q_4, \\ q_i q_j = q_j q_i \quad (4 \leq i < j-1 \leq n-2), & & q_i q_{i+1} q_i = q_{i+1} q_i q_{i+1} \quad (4 \leq i \leq n-2). \end{array}$$

□

# References


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