

# PROBABILISTIC EXNER SEDIMENT CONTINUITY EQUATION FOR MIXTURES WITH NO ACTIVE LAYER

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**ABSTRACT:** The Exner equation of sediment continuity is the foundation of river morphodynamics. Generalization of this equation to mixtures of grain sizes has required the introduction of an active layer (i.e., a buffer layer between the sediment moving in the water column and the immobile substrate below). The active layer is defined to be a well-mixed layer, with no vertical structure, that encompasses those grains available to exchange directly with the moving sediment. The sediment in the substrate below exchanges with the active layer only as the bed aggrades or degrades. The active layer concept is a useful one that has served the research community well for 3 decades. However, the division of the erodible bed into a discrete active layer and substrate must represent only an approximation of a more general formulation that contains no active layer and in which parameters pertaining to the entrainment from and deposition to the bed vary continuously with depth below the sediment-water interface. Here the probability density function of bed elevation is used to derive a general Exner equation of sediment continuity with no discrete layers. The formulation is applicable to both sediment mixtures and tracers in uniform sediment. Although the treatment requires more information than that of the active layer approach, it offers the prospect of a better understanding of how streams create a stratigraphic record of their activities through deposition.

## INTRODUCTION

A rather striking discordance is apparent in regard to an aspect of sediment transport research. Since at least the time of Einstein (1937, 1972) researchers have used marked tracer particles in rivers or experimental flumes to study the processes of particle movement, deposition, burial, and re-entrainment. A fairly comprehensive review of these efforts is given in Hassan and Church (1992). Of particular interest here are studies that focus on the time development of the probability distribution of tracer burial and re-exhumation as functions of depth below the mean bed [e.g., Schick et al. (1987a,b), Hassan (1988), and Hassan and Church (1994)]. Related to this are numerous studies to determine the probabilistic nature of fluctuations of bed elevation about a mean value due to the migration of dunes or bars, or even the inevitable random variations associated with bed-load transport over a lower regime plane bed [e.g., Leopold et al. (1966), Ribberink (1987), and Hubbell et al. (1985)]. Yet very little of this research has been incorporated into predictive methods for either sediment transport itself or the variations in bed elevation and depositional stratigraphy associated with differential sediment transport.

Evidently the problem is due to the lack of a conceptual framework within which this probabilistic information might be incorporated. The purpose of this paper is to outline such a conceptual framework. The central result is a general probabilistic formulation for the conservation of bed sediment that underlies existing formulations of the Exner equation.

The reader is warned in advance that the goal of this paper is the derivation of equations describing conservation of bed sediment. No applications are pursued, largely because some of the internal relations have yet to be specified with sufficient generality. These relations can surely be specified through fur-

ther field and experimental research. It is hoped that this paper helps spur such research.

## ACTIVE LAYER CONCEPT

Consider unidirectional flow in the  $x$ -direction over an erodible bed. Let  $\eta$  denote bed elevation and  $q$  denote the volume transport rate of bed material load per unit width. Here both  $\eta$  and  $q$  are interpreted to be averaged over local fluctuations associated with, for example, bed forms. The standard form of the Exner equation of bed sediment continuity can be written in the form

$$(1 - \lambda_p) \frac{\partial \eta}{\partial t} = - \frac{\partial q}{\partial x} \quad (1)$$

where  $\lambda_p$  denotes bed porosity; and  $t$  denotes time. Implicit in the above formulation and related formulations given below is the assumption that the depth-averaged volume concentration of moving sediment is sufficiently low to allow for neglect of the storage of sediment in the water column in (1). This assumption should apply to the great majority of alluvial streams.

The above equation is too simple to describe a number of problems of importance in the fields of sediment transport and morphodynamics. In rivers containing a mixture of grain sizes over a wide range, for example, a correct accounting of sediment transport, bed level variation, and development of bed stratigraphy requires the introduction of a more advanced form of sediment conservation and, in particular, one that is grain-size specific. The major advance in this regard was made by Hirano (1971). He introduced a concept that he called the "exchange layer" (what has commonly become known as the active layer in the literature in English). Here it is introduced in a somewhat expanded form based on the derivation of Parker (1991a).

The bed is divided into a surface active layer with thickness  $L_a$ , below which lies the substrate, as shown in Fig. 1. Under steady, uniform conditions, sediment in transport exchanges only with bed sediment in the active layer. The active layer is assumed to be well mixed by the exchange process so that it has no vertical structure. Particles in the substrate are assumed to have no direct interaction with sediment in transport. The substrate and active layer may exchange material, however, as the bed aggrades or degrades. The substrate may have a vertical structure. This structure can develop as the bed aggrades, thus transferring sediment to the substrate and building a stra-

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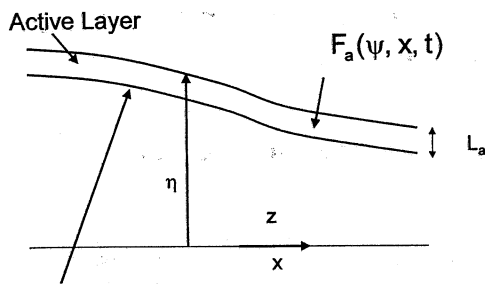


FIG. 1. Definition Diagram for Active Layer and Substrate

tigraphy. In an active layer formulation, the grain-size distribution must, in general, be assumed to display a discontinuity at the interface between the surface layer and substrate.

Here grain size is described with the logarithmic  $\psi$  scale, such that

$$\psi = \ln_2(D) \quad (2)$$

where  $D$  denotes grain size (mm). The active layer grain-size density  $F_a(\psi, x, t)$  is defined such that the mass fraction of grains in the size range  $(\psi, \psi + d\psi)$  at point  $(x, t)$  is given by  $F_a(\psi, x, t) d\psi$ . For simplicity all the sediment grains are assumed to have the same specific gravity, so mass fraction is equivalent to volume fraction. The corresponding grain-size density at the interface of the active layer and substrate is given by  $F_I(\psi, x, t)$ . Note that by definition

$$\int_{-\infty}^{\infty} F_a d\psi = \int_{-\infty}^{\infty} F_I d\psi = 1 \quad (3)$$

Assuming a constant porosity  $\lambda_p$ , (1) is extended to the form

$$(1 - \lambda_p) \left( F_I \frac{\partial \eta}{\partial t} + L_a \frac{\partial F_a}{\partial t} \right) = -\frac{\partial q_\psi}{\partial x} \quad (4)$$

where  $q_\psi$  denotes a sediment transport density such that  $q_\psi d\psi$  denotes the mass or volume fraction of the sediment transport in the size range  $(\psi, \psi + d\psi)$ . An integration of (4) over all grain sizes recovers the standard Exner formulation of (1), where

$$q = \int_{-\infty}^{\infty} q_\psi d\psi \quad (5)$$

The grain-size density of the transported bed material load  $F_I$  is thus given by

$$F_I = \frac{q_\psi}{q} \quad (6)$$

The interfacial grain-size density  $F_I$  is typically specified as equal to that of the substrate just below the active layer for the case of a degrading bed ( $\partial\eta/\partial t < 0$ ) and equal to a weighted average of  $F_a$  and  $F_I$  for an aggrading bed ( $\partial\eta/\partial t > 0$ ) (Hoey and Ferguson 1994; Toro-Escobar et al. 1996).

The original form of the Exner equation of sediment mass conservation [(1)] is recovered by integrating (4) over all grain sizes and applying (3). Between (1) and (4), the following relation for the evolution of the grain-size distribution of the active layer is obtained:

$$(1 - \lambda_p) L_a \frac{\partial F_a}{\partial t} = -\frac{\partial q_\psi}{\partial x} + F_I \frac{\partial q}{\partial x} \quad (7)$$

It is interesting to note here that the active layer concept also can be used in regard to a simpler problem (i.e., the dispersal within the bed of marked tracer particles in uniform sediment). To this end, let  $f_i$  denote the fraction of tracer par-

ticles in the bed material load and  $f_a$  denote the fraction of tracer particles in the active layer. For this case the active layer formulation of tracer conservation takes the form

$$(1 - \lambda_p) \left( f_i \frac{\partial \eta}{\partial t} + L_a \frac{\partial f_a}{\partial t} \right) = -\frac{\partial q f_i}{\partial x} \quad (8)$$

where  $f_i$  denotes the fraction of tracers in the sediment at the interface between the active layer and substrate as the bed degrades (in which case  $f_i$  is the fraction of tracers in the substrate just below the interface) or aggrades (in which case  $f_i$  is likely expressed as a weighted average of  $f_a$  and  $f_i$ ). Reducing between (1) and (8), it is found that

$$(1 - \lambda_p) L_a \frac{\partial f_a}{\partial t} = -\frac{\partial q f_i}{\partial x} + f_i \frac{\partial q}{\partial x} \quad (9)$$

The active layer formulation has served the research community well over the nearly 3 decades since its introduction. It has been used to explain the evolution of static bed armoring [e.g., Ashida and Michiue (1971), Bettess and White (1981), and Parker and Sutherland (1990)], sediment sorting by dunes [e.g., Ribberink (1987)], sediment sorting in bends [e.g., Parker and Andrews (1985)], gravel sheets [e.g., Seminara et al. (1996)], roughness streaks [e.g., Colombini and Parker (1995)], bars [e.g., Lanzoni and Tubino (1999)], and patterns of longitudinal sorting in rivers [e.g., Diegaard and Fredsoe (1978), Paola and Seal (1995), Parker (1991b), Armanini (1991), Hoey and Ferguson (1994), and Cui et al. (1996)]. Yet its form hints of a more general underlying form of which it is only a special case. The basis for this supposition is explored below.

## DISPLACEMENT THICKNESS ANALOGY

Consider the diagram of Fig. 2(a) (Hassan and Church 1994). Because instantaneous bed elevation fluctuates, particles can be entrained into the bed material load from a range of depths of burial. The deeper a particle is buried, the lower is its probability of being entrained into motion per unit time. This is because, among other factors, the probability of exposure of a grain decreases with burial depth. The probability of entrainment within any specified time interval is thus illustrated in the figure as a function that continuously decreases with depth of burial.

The active layer concept approximates this continuously decreasing function as a step function, such that the probability of entrainment per unit time is a set constant within the active layer but immediately declines to zero in the substrate. This is illustrated in Fig. 2(b).

An apt analogy for this approximation can be found in the field of boundary layer fluid mechanics. Consider the steady boundary layer on a flat plate with zero pressure gradient [e.g., Schlichting (1968)]. Let  $(x, z)$  denote coordinates in the streamwise and normal directions. The actual profile of streamwise velocity  $u(x, z)$  varies continuously in the  $z$ -direc-

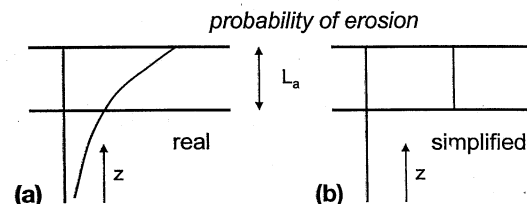


FIG. 2. Variation in Probability of Entrainment: (a) Schematization of Actual Variation in Probability of Entrainment of Particle within Given Time as Function of Depth (Variation is Seen To Be Continuous, Although Active Layer Has Been Drawn In for Reference); (b) Approximation of Variation in Probability of Entrainment within Given Time as Step Function of Depth (Constant and Greater Than Zero in Active Layer and Equal to Zero below Active Layer) Offered by Active Layer Formulation

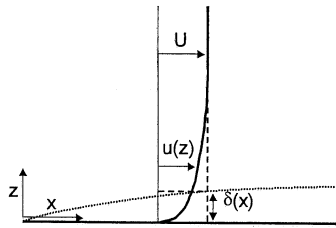


FIG. 3. Diagram Illustrating Definition of Displacement Thickness for Boundary Layer Flow

tion normal to the plate, vanishes at the plate itself, and converges to a constant outer value  $U$  far away from the plate (Fig. 3). A simplified but useful treatment of boundary layers can be obtained, however, by means of the introduction of a displacement thickness  $\delta$  defined such that

$$U\delta(x) = \int_0^{\infty} [U - u(x, z)] dz \quad (10)$$

That is,  $\delta(x)$  denotes the distance the outer flow would have to be displaced from the plate to realize the same deficit in forward flow discharge per unit plate width as actually occurs. The boundary layer equations can be integrated to yield a governing equation (i.e., the von Kármán integral equation of momentum conservation), which can then be used to solve for  $\delta(x)$ . The integral formulation, however, does not contain as much information as the boundary layer equations from which it is derived and requires approximate closure relations to complete the analysis.

It is suggested here that the active layer formulation is similarly an approximate integral representation of a more fundamental expression for mass conservation of bed sediment—one that contains more information and allows for a richer range of behavior. This more fundamental expression should not need any kind of active layer.

To explore the issues at hand, it is of use to consider a specific case. Leclair (1999) pursued both experimental and field studies on the stratigraphy created in a river by migrating dunes. Several of the experiments in question were conducted in the 2.71-m wide main channel of the St. Anthony Falls Laboratory, University of Minnesota, Minneapolis.

Experiment 33 of Leclair (1999) is chosen here for illustrative purposes. A view of the channel and dune-covered bed at the end of the experiment is given in Fig. 4. The grain-size distribution of the sediment used in the experiment is given in Fig. 5. Run time was 15.4 h. The energy slope, depth, and discharge associated with the flow that created the dunes in Fig. 4 were 0.0021, 0.87 m, and 1.89 m<sup>3</sup>/s, respectively. The mean flow velocity was thus 0.80 m/s. An acoustic sensor was used to obtain a continuous record of bed elevation at a point near the middle of the channel that was 15.5-m downstream of the beginning and 6.5-m upstream of the end of the erodible-bed portion of the channel.

The experiment left a rich stratigraphic record of the passage of dunes. These were recorded in peels (i.e., slices of the deposit that have been impregnated with and stabilized by rubber latex). Fig. 6 shows a peel from Run 33. It is oriented such that the short axis is directed upward from the bottom and the long axis is directed from upstream (right) to downstream (left). Waste sand has been placed on the peel to enhance the top of the deposit. The peels show the distinct pattern of cross-bedding associated with dune migration. Of particular interest is the tendency for the coarsest sediment to concentrate in the troughs of the dunes. Three lines demarcating the passage of dune troughs are evident in the photograph. The line of deepest trough shown in Fig. 6 could be thought of as an instantaneous realization of the bottom of the

active layer. This interface can be expected to move downward in time and become more diffuse as it feels the effect of rare dunes with deep troughs. The probabilistic nature of bed elevation variation is more clearly illustrated in Fig. 7, which shows the probability  $P_r(y)$  that the bed is higher than an amount  $y$  above the mean bed elevation.

It is evident from the above brief introduction that (a) the zone affected by the passage of dunes does have an internal structure in the vertical direction; (b) the bottom of this layer is difficult to define due to the probabilistic nature of the dunes; and (c) the probability density of bed elevation is intimately associated with the process of vertical sorting of grains. This last point is underlined by the tendency for coarser grains to concentrate near the base of dunes, as seen in Fig. 6. Experiments of the above type also have been conducted

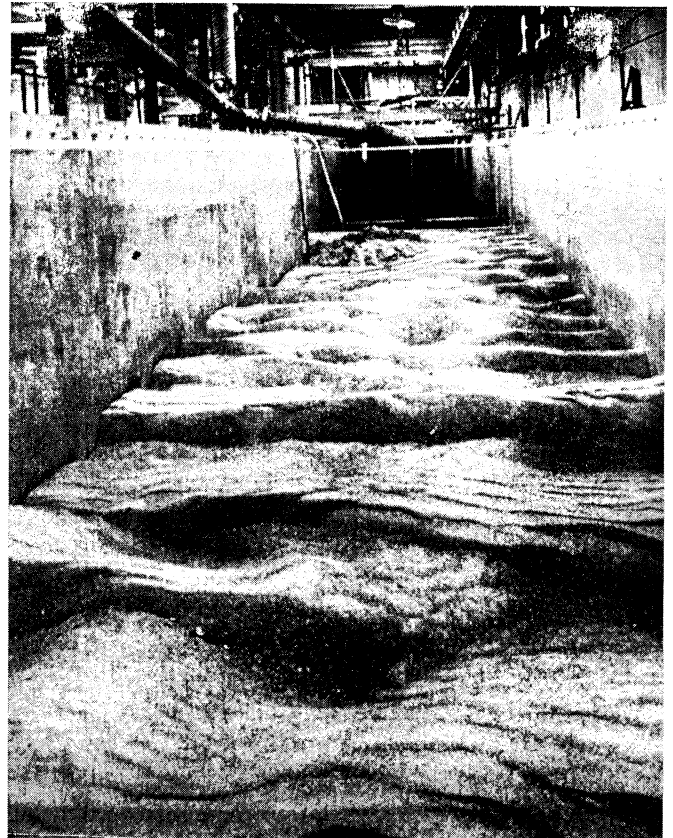


FIG. 4. Photograph of Dunes at End of Run 33 of Leclair (1999) (Flow Was from Top to Bottom; Width of Flume Is 2.7 m)

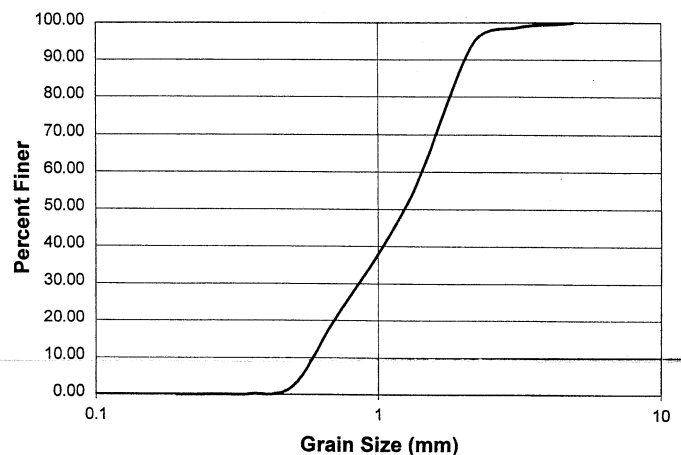


FIG. 5. Grain-Size Distribution of Sediment Used in Run 33

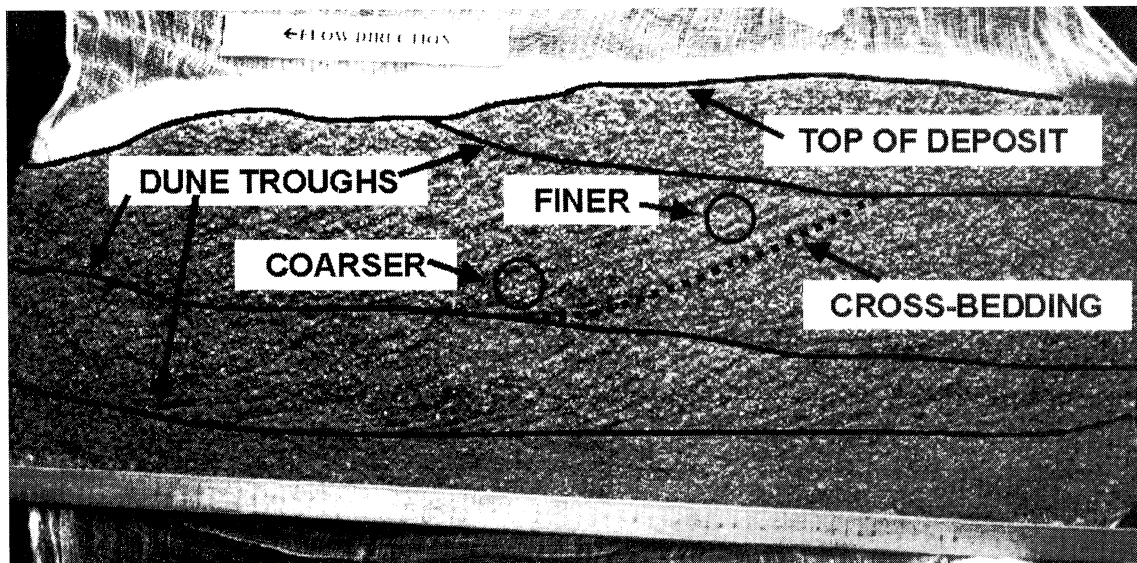


FIG. 6. Photograph of Peel of Stratigraphy Created in Run 33—Flow Was from Right to Left; White Waste Sand Was Placed on Peel to Enhance Top of Deposit; Rich Pattern of Grain Sorting Is Apparent in Photograph, Including Dune Cross Bedding, Tendency for Coarse Sediment to Concentrate at Base of Dunes, and Multiple Tracks Denoting Passage of Dune Troughs; Peel Is Approximately 30-cm Thick

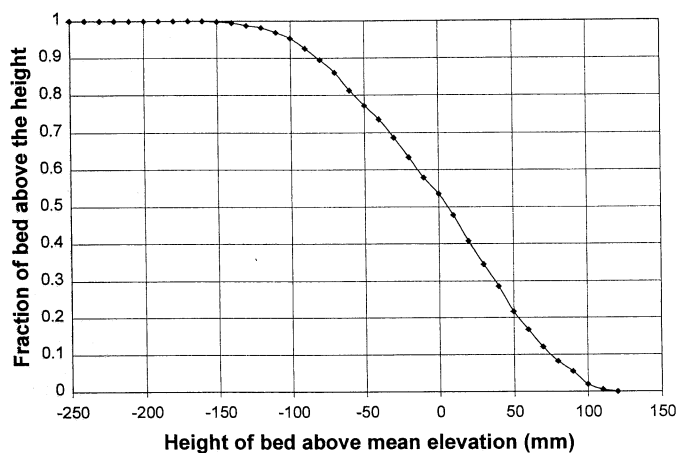


FIG. 7. Function  $P_s(y)$  Denoting Fraction of Bed Record Measured at Point That Was above Level  $y$ , Where  $y$  Is Measured Relative to Mean Bed Level, as Determined for Run 33

by Ribberink (1987), Nino and Aracena (1999), and Blom and Ribberink (1999).

The search for a more general expression for the mass conservation of bed sediment appears to have begun with the work of Ribberink (1987), which clearly points in the direction of the analysis presented here. Ribberink emphasized the role of the probability distribution of bed elevation in the vertical sorting process. He also demonstrated that this sorting has a continuous structure in the vertical direction. Finally, he proposed the idea of using multiple layers to obtain better resolution of this structure. Armanini and Di Silvio (1988), Di Silvio (1991), and Ashida et al. (1989) have further pursued the concept of multiple layers. Armanini (1995) has gone a step farther and introduced a diffusional model that replaces the discrete active layer(s) with a vertical continuum. These insightful analyses have helped motivate the formulation presented here.

#### ENTRAINMENT FORMULATION OF SEDIMENT CONTINUITY

Sediment of uniform size is again considered. Eq. (1) expresses the Exner equation of bed sediment conservation in what may be termed divergence form; bed elevation changes

in accordance with the divergence of the sediment transport rate. The present analysis is somewhat better expressed in terms of a completely equivalent entrainment formulation; that is (1) is replaced with the form

$$(1 - \lambda_p) \frac{\partial \eta}{\partial t} = D - E \quad (11)$$

where  $D$  denotes the volume rate of deposition of bed material load per unit area per unit time onto the bed; and  $E$  denotes the corresponding volume rate of entrainment of bed sediment into transport per unit area per unit time. Here a particle is considered as deposited only when it comes to rest on the bed; e.g., a saltating particle that strikes the bed and is immediately ejected again is not considered to have been deposited. The entrainment formulation has been steadfastly advocated by Tsujimoto in the analysis of morphodynamics, both generally [e.g., Tsujimoto (1978)] and specifically in regard to sediment sorting [e.g., Tsujimoto (1991)]. The same formulation also can be found in the treatments of grain sorting due to Ribberink (1987) and Ashida et al. (1989).

The fundamental equivalency of (1) and (11) is easily demonstrated by means of the intermediary of the function  $f_p(x)$  denoting the probability density that a particle, once entrained, will travel a distance  $x$  before being deposited again. The deposition rate  $D$  can thus be related to the entrainment rate  $E$

$$D(x) = \int_{-\infty}^x E(y) f_p(x - y) dy \quad (12)$$

The volume transport rate of sediment per unit width  $q(x)$  can be computed in terms of the quantity of particles entrained upstream of point  $x$  that travel at least as far as  $x$  before depositing

$$q(x) = \int_{-\infty}^x E(y) \int_{x-y}^{\infty} f_p(y') dy' dy \quad (13)$$

Among (1), (12), and (13), it is quickly demonstrated that

$$\frac{\partial q}{\partial x} = E - D \quad (14)$$

thus demonstrating the equivalency of the divergence and entrainment formulations, (1) and (11), respectively, of sediment

continuity. The above analysis is based on that presented in Tsujimoto (1978).

## PROBABILISTIC FORMULATION FOR UNIFORM SEDIMENT

An instantaneous realization of a bed profile is shown in Fig. 8. The bed variations are assumed to be statistically uniform in space and time over scales that are large compared to those associated with the fluctuations but small compared to those of variation of the mean bed. Let  $z$  denote a coordinate that is oriented upward normal to the local mean bed elevation, and  $P_s(z)$  denote the mean fraction of a line at elevation  $z$  perpendicular to the  $z$ -coordinate that falls within the sediment bed rather than the flowing water. The parameter  $P_s(z)$  should approach unity for  $z \rightarrow -\infty$  (deep in the deposit) and zero for  $z \rightarrow \infty$  (in the water column well above the deposit). It can be interpreted as the probability that the instantaneous bed is higher than elevation  $z$ . An example of its form is given in Fig. 7. Although that example pertains to a sand bed covered with dunes, the concept should work equally well for gravel beds with no bed forms as long as the parameter  $P_s(z)$  is defined so as to exclude the ambient porosity of the gravel deposit, so  $P_s(z)$  does, indeed, approach unity for  $z \rightarrow -\infty$ .

The probability density that the instantaneous bed elevation is at level  $z$  is given by  $p_e(z)$ , where

$$p_e(z) = -\frac{\partial P_s}{\partial z} \quad (15)$$

Thus  $p_e(z) dz$  denotes the probability that the instantaneous bed elevation is in the range  $(z, z + dz)$ . By definition, then

$$\int_{-\infty}^{\infty} p_e dz = 1 \quad (16)$$

In general  $p_e$  is a function of  $x$  and  $t$  as well as  $z$ , where the spatial and temporal scales of variation are large compared to those associated with instantaneous bed fluctuations. The bed elevation  $\eta(x, t)$  averaged over these fluctuations can be expressed as a moment of  $p_e$

$$\eta = \int_{-\infty}^{\infty} zp_e(z) dz \quad (17)$$

Let  $L(z)$  denote the average thickness of the bed above elevation  $z$ . This can be computed

$$L(z) = \int_z^{\infty} (z' - z)p_e(z') dz' \quad (18)$$

Introducing (15) into (18) and integrating by parts, it is found that

$$L(z) = \int_z^{\infty} P_s dz \quad (19)$$

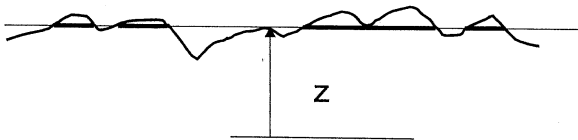


FIG. 8. Definition Diagram for Bed with Probabilistic Structure to Variation in Bed Elevation

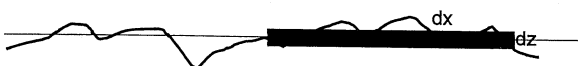


FIG. 9. Definition Diagram for Derivation of Probabilistic Forms of Exner Equation of Sediment Continuity

In addition to the example shown in Fig. 7, functional forms for  $P_s$  have been determined experimentally by, for example, Ribberink (1987). Hubbell et al. (1985) showed that similar probability distributions for the bed-load transport rate determined in streams with well-developed dunes could be approximately identified with the associated probability distribution of bed levels.

In the analysis below, a probabilistic formulation for sediment conservation is pursued. Implicit in the analysis is a dichotomy between "short" time and length scales characterizing fluctuations of the instantaneous bed and "long" time and length scales in which the statistical characteristics of the bed vary, including the mean bed elevation  $\eta$  and the elevation probability distribution  $P_s$  itself.

Now consider the control volume illustrated in Fig. 9, in which the distance  $dx$  is long compared to the characteristic short scale of bed fluctuations but short compared to the characteristic long scale of variation in mean bed level due to aggradation or degradation. Mass conservation of bed sediment in the volume control can be written

$$\frac{\partial}{\partial t} (c_b P_s dx dz) = (D_e - E_e) dx dz \quad (20a)$$

where  $c_b$  denotes the volume fraction of the bed that is solid and is related to the porosity  $\lambda_p$

$$c_b = 1 - \lambda_p \quad (20b)$$

and  $E_e$  and  $D_e$  denote elevation-specific densities of the entrainment and deposition rates defined such that, for example,  $E_e dz$  denotes the rate of entrainment of bed sediment from the elevation range  $(z, z + dz)$ . By definition, then

$$D = \int_{-\infty}^{\infty} D_e dz; \quad E = \int_{-\infty}^{\infty} E_e dz \quad (20c,d)$$

For uniform sediment,  $c_b$  can safely be taken to be constant, so (20a) reduces to

$$c_b \frac{\partial P_s}{\partial t} = D_e - E_e \quad (21)$$

At this point an assumption that is crucial to the analysis is introduced. Rather than  $P_s$  being a direct function of  $z$ , it is assumed that

$$P_s = P_s(y); \quad y = z - \eta \quad (22a,b)$$

That is, the probability distribution of bed elevation is specified not in terms of absolute bed elevation, but rather the deviation of the bed elevation about the mean value. It then follows from the chain rule and (15) that

$$\frac{\partial P_s}{\partial t} = -\frac{\partial P_s}{\partial y} \frac{\partial \eta}{\partial t} = p_e \frac{\partial \eta}{\partial t} \quad (23)$$

so that (21) becomes

$$c_b p_e \frac{\partial \eta}{\partial t} = D_e - E_e \quad (24)$$

The above equation represents the probabilistic formulation that underlies the deterministic formulation [(11)] of the Exner equation of sediment continuity. The deterministic formulation is realized as a special case obtained by integrating (24) over all bed elevations and applying (16), (20c), and (20d).

Note that (24) includes no active layer in this formulation. Although general forms for  $D_e$  and  $E_e$  have yet to be determined, it can be surmised that both decline smoothly toward zero as  $y \rightarrow -\infty$ . That is, particles at sufficient depth of burial should not be entrained, nor should particles be deposited at

such a depth because the probability of the bed scouring to that depth is negligibly small. This expected behavior can at least loosely be modeled by taking  $D_e$  and  $E_e$  as proportional to  $p_e$ . In both cases, however, a bias can be expected based on bed shape, variation of hydrodynamic forces with elevation, and other factors. With this in mind, the following general forms for  $D_e$  and  $E_e$  are proposed here:

$$D_e = D\beta_D(y)p_e(y); \quad E_e = E\beta_E(y)p_e(y) \quad (25a,b)$$

where  $\beta_D$  and  $\beta_E$  are as yet undetermined bias functions that are seen from (20c) and (20d) to satisfy the conditions

$$\int_{-\infty}^{\infty} \beta_D p_e dy = 1; \quad \int_{-\infty}^{\infty} \beta_E p_e dy = 1 \quad (25c,d)$$

The parameters  $\beta_D$  and  $\beta_E$  cannot be selected independently. Substituting (11) into (24) and reducing with the aid of (20b) and (25), it is found that

$$D - E = \beta_D D - \beta_E E \quad (25e)$$

It is possible to solve (25e) for, for example,  $\beta_D$  as a function of  $\beta_E$ ,  $D$ , and  $E$ . In doing so, it is quickly found that the only possible specifications of  $\beta_D$  and  $\beta_E$  that are independent of  $D$  and  $E$  and satisfy (25c) and (25d) are

$$\beta_D = \beta_E = 1 \quad (25f)$$

The form of (24) is adequate only when bed variation is accompanied by only modest change in the probability distribution  $P_s$  of elevation. This distribution may change with altered flow conditions, in some cases radically. For example, a river in the lower regime plane bed may make a transition to the dune regime, thus greatly altering the structure of  $P_s$ . This degree of freedom may be included by generalizing (22a) to

$$P_s = P_s(y, X_k) \quad (26)$$

where  $X_k$  = hydraulic parameter controlling the shape of the probability distribution ( $k = 1, \dots, N$ ). Hydraulic parameters may include bed slope, Froude number, Shields stress, and other parameters, all of which can vary at the long scales of  $x$  and  $t$ . Substitution of (26) into (21) and reduction leads to the form

$$c_b p_e \frac{\partial \eta}{\partial t} = D_e - E_e - c_b \frac{\partial P_s}{\partial X_k} \frac{\partial X_k}{\partial t} \quad (27)$$

where the last term on the right-hand side of (27) describes the effect of temporal changes in the bed elevation probability distribution. Integrating (27) over all elevations leads to the generalization of (11) of the form

$$(1 - \lambda_p) \frac{\partial \eta}{\partial t} = D - E - (1 - \lambda_p) A_k \frac{\partial X_k}{\partial t} \quad (28a)$$

$$A_k = \int_{-\infty}^{\infty} \frac{\partial P_s}{\partial X_k} dy \quad (28b)$$

Here the last term on the right-hand side of (27) describes the effect of the changing bed elevation probability distribution on the mean bed elevation. The convergence of the integrals of (28b) is implied (but not proved) by the fact that  $P_s \rightarrow 0$  as  $y \rightarrow \infty$  and  $P_s \rightarrow 1$  as  $y \rightarrow -\infty$ , so  $\partial P_s / \partial X_k \rightarrow 0$  as  $y \rightarrow \pm\infty$ . It can be easily demonstrated that, if the elevation probability density  $p_e$  is always symmetrical about  $y = 0$ , then the coefficients  $A_k$  vanish identically.

In fact, the algebraic specification of the bed elevation probability distribution as a function of hydraulic parameters given by (26) may still be insufficient when the time required for the adjustment on the bed forms to changed flow is at least as long as the characteristic time for aggradation or degradation

of the mean bed. In such cases (26) must be replaced by a time-differential formulation.

The derivations given below the terms  $X_k$  in (26) are neglected for simplicity. In each case they can be added in a straightforward manner analogous to that described above.

## PROBABILISTIC FORMULATION FOR TRACERS IN UNIFORM SEDIMENT

The case of tracer grains in uniform sediment is now re-considered from a probabilistic point of view. Let  $f(y, x, t)$  denote the fraction of the sediment at level  $y$  consisting of marked tracers. Following the analysis above, the general form of the conservation equation for tracers is found to be

$$c_b \frac{\partial}{\partial t} (f P_s) = p_e (D \beta_D f_t - E \beta_E f) \quad (29)$$

Reducing in accordance with (23) yields the result

$$c_b \left( p_e f \frac{\partial \eta}{\partial t} + P_s \frac{\partial f}{\partial t} \right) = p_e (D \beta_D f_t - E \beta_E f) \quad (30a)$$

Reducing (30a) with the aid of (11), (20b), and (25e), it is quickly found that

$$c_b P_s \frac{\partial f}{\partial t} = p_e D \beta_D (f_t - f) \quad (30b)$$

The above equation represents the probabilistic formulation of conservation of mass of tracer particles.

The probabilistic formulation can be reduced to the active layer formulation upon manipulation with the aid of several assumptions. In particular, the first term on the left-hand side of (30a) can be manipulated using (15) to yield

$$p_e f \frac{\partial \eta}{\partial t} = \left( -\frac{\partial}{\partial y} (P_s f) + P_s \frac{\partial f}{\partial y} \right) \frac{\partial \eta}{\partial t} \quad (31)$$

Eq. (30) is now integrated from a point  $y = -L_{na}$  corresponding to a nominal lowest elevation of scour to  $y = \infty$  under the assumptions that (a) within this range  $f$  takes the value  $f_a$  independently of  $y$ , and  $\beta_E$  and  $\beta_D$  take the value unity; and (b) that  $p_e(y)$  vanishes and  $P_s(y)$  takes the value unity for  $y < -L_{na}$ . Manipulating with the aid of (19), (20b), and (31) and the connection that  $P_s \rightarrow 0$  as  $y \rightarrow \infty$ , it is found that

$$(1 - \lambda_p) \left( f_t \frac{\partial \eta}{\partial t} + L_a \frac{\partial f_a}{\partial t} \right) = D f_t - E f_a \quad (32a)$$

where

$$f_t = f|_{y=-L_{na}}; \quad L_a = \int_{-L_{na}}^{\infty} P_s dy \quad (32b)$$

Eq. (32a) is the entrainment form of the active layer formulation for conservation of tracer mass for uniform sediment, in precise analogy to the divergence form [(8)]. It is further seen from (19) that the active layer thickness  $L_a$  corresponds to the mean thickness of sediment above a point  $y = -L_{na}$  or  $z = \eta - L_{na}$ .

## PROBABILISTIC FORMULATION FOR SEDIMENT MIXTURES

Let  $F(y, x, t)$  denote the grain-size density at level  $y$ , and let  $D_{e\psi}$  and  $E_{e\psi}$  denote elevation-specific and grain-size-specific densities of deposition and entrainment rates such that, for example,  $D_{e\psi} dz d\psi$  denotes the volume rate per unit bed area per unit time of the deposition of particles in the size range  $(\psi, \psi + d\psi)$  within the elevation range  $(y, y + dy)$ .

Note that  $F$ ,  $D_{e\psi}$ , and  $E_{e\psi}$  must satisfy the following integral constraints:

$$\int_{-\infty}^{\infty} F d\psi = 1; \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D_{e\psi} dy d\psi = D \quad (33a,b)$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_{e\psi} dy d\psi = E \quad (33c)$$

Following the analysis leading to (21), the probabilistic formulation for the sediment conservation of sediment size mixtures is quickly found to be

$$\frac{\partial}{\partial t} c_b F P_s = D_{e\psi} - E_{e\psi} \quad (34)$$

In the present case  $c_b$  must be allowed to be at least a function of  $z$ , because porosity tends to decrease with increasing standard deviation of the size mixture. Manipulating as before, it is found that

$$c_b \left( p_e F \frac{\partial \eta}{\partial t} + P_s \frac{\partial F}{\partial t} \right) = p_e (D \beta_{\psi D} F_t - E \beta_{\psi E} F) \quad (35)$$

where  $\beta_{\psi D}$  and  $\beta_{\psi E}$  are bias functions of  $y$  and  $\psi$  defines such that

$$D_{e\psi} = D \beta_{\psi D}(y, \psi) p_e(y) F_t \quad (36a)$$

$$E_{e\psi} = E \beta_{\psi E}(y, \psi) p_e(y) F \quad (36b)$$

and where the following integral constraints are satisfied:

$$\int_{-\infty}^{\infty} \beta_{\psi D} F_t d\psi = \beta_D; \quad \int_{-\infty}^{\infty} \beta_{\psi E} F d\psi = \beta_E \quad (36c,d)$$

$$\int_{-\infty}^{\infty} \beta_{\psi D} p_e F_t dy d\psi = 1; \quad \int_{-\infty}^{\infty} \beta_{\psi E} p_e F dy d\psi = 1 \quad (36e,f)$$

Eq. (35) represents the probabilistic formulation of the conservation of mass of sediment mixtures.

Integration of (35) over all sizes and then over all elevations precisely recovers (11) for total sediment mass balance, where  $\lambda_p$  is an average porosity given by the expressions

$$\lambda_p = 1 - \bar{c}_b; \quad \bar{c}_b = \int_{-\infty}^{\infty} c_b p_e dy \quad (37a,b)$$

Further reduction of (35) with the aid of (11), (36), (37a) and (37b) yields the relations

$$c_b P_s \frac{\partial F}{\partial t} = p_e \left[ D \left( \beta_{\psi D} F_t - \frac{c_b}{\bar{c}_b} F \right) - E \left( \beta_{\psi E} - \frac{c_b}{\bar{c}_b} \right) F \right] \quad (37c)$$

$$\frac{c_b}{\bar{c}_b} (D - E) = \beta_D D - \beta_E E \quad (37d)$$

where (37d) is the generalization of (25e) to mixtures. In analogy to (25f), the only specification of  $\beta_D$  and  $\beta_E$  that renders them independent of each other as well as  $D$  and  $E$  is

$$\beta_D = \beta_E = \frac{c_b}{\bar{c}_b} \quad (37e)$$

The steps by which (35) can be approximated to the active layer formulation are strictly analogous to those used for the case of tracers in uniform sediment. The first term in (35) is rewritten

$$p_e F \frac{\partial \eta}{\partial t} = \left( -\frac{\partial}{\partial y} (P_s F) + P_s \frac{\partial F}{\partial y} \right) \frac{\partial \eta}{\partial t} \quad (38)$$

Eq. (35) is now integrated from a point  $y = -L_{na}$  corresponding to a nominal lowest elevation of scour to  $y = \infty$  under the assumptions that (a) within this range  $F$  and  $c_b$  take the value  $F_a$  and  $c_{ba}$  independently of  $y$ , and  $\beta_E$  and  $\beta_D$  are equal to unity; and (b)  $p_e(y)$  vanishes and  $P_s(y)$  takes the value unity for  $y < -L_{na}$ . Manipulating with the aid of (19), (20b), and (38) and the condition that  $P_s \rightarrow 0$  as  $y \rightarrow \infty$ , it is found that

$$(1 - \lambda_p) \left( F_t \frac{\partial \eta}{\partial t} + L_a \frac{\partial F_a}{\partial t} \right) = D F_t - E F_a \quad (39a)$$

where

$$\lambda_p = 1 - c_{ba}; \quad F_t = F|_{y=-L_{na}}; \quad L_a = \int_{-L_{na}}^{\infty} P_s dy \quad (39b-d)$$

Eq. (39a) is the entrainment form of the active layer formulation for mass conservation of sediment mixtures. It is precisely analogous to the divergence form below, which is obtained by reducing (4) with (6)

$$(1 - \lambda_p) \left( F_t \frac{\partial \eta}{\partial t} + L_a \frac{\partial F_a}{\partial t} \right) = -\frac{\partial q F_t}{\partial x} \quad (40)$$

It is again seen from (19) that the active layer thickness  $L_a$  corresponds to the mean thickness of sediment above a point  $y = -L_{na}$  or  $z = \eta - L_{na}$ .

## DISCUSSION

It is not possible at present to implement the probabilistic formulations for mass conservation of sediment proposed here. This is principally because general predictors for the probability distribution of bed elevation and elevation-specific densities for erosion and deposition (grain-size specific or otherwise) of sediment have not yet been developed. A considerable amount of progress has been made in regard to the former [e.g., Hubell et al (1985), Ribberink (1987), and Leclair (1999)]. Rather less progress has been made in regard to the latter, but the work of Marion and Fraccarollo (1997) and Marion et al. (1997) represents a useful step forward. The tracer studies referred to at the beginning of this paper can play a crucial role in future experimental work designed to evaluate the densities for erosion and deposition. It is the hope of the writers that this paper will help spur the necessary experimental and field research.

The proposed probabilistic formulations are not meant to "overthrow" the active layer formulation, which will remain the method of choice for broad-brush representations of sediment sorting. They do, however, provide the theoretical underpinning for the active layer formulation. In addition, when the issue at hand is the vertical structure of the stratigraphy (averaged over a horizontal scale including many individual bed forms) left by aggrading streams, they offer an avenue toward predictive models at a level of sophistication that has been previously unavailable.

Any numerical implementation of the probabilistic formulations given here will require discretization of the bed into a finite number of vertical layers. The detail provided by such an analysis can potentially provide useful information about, for example, change in the composition of spawning gravels as a function of varying hydrology and sediment supply.

Rivers contain bed variations at multiple scales, including those at the level of the size of the grains themselves, ripples, dunes, bars, and bends. The precise probabilistic formulation to be used thus becomes a function of the scale of the phenomenon of interest. Although the present formulation is not specifically designed for multiple scales, it can be adapted to include them.

A limitation of the present model is the assumption that the

instantaneous river bed level also specifies the instantaneous depth at which particles may be deposited and entrained. This is not always the case, particularly when the particles in question are the finest particles in a sediment mix. This is because fine particles can both settle down and be leached upward through a sufficiently porous coarse layer. The present model would have to be modified to include these effects.

## CONCLUSIONS

The major conclusions of this paper are three probabilistic forms for mass conservation of bed sediment that bear repeating. The first of these pertains to uniform sediment and takes the form of the following three relations:

$$c_b \frac{\partial \eta}{\partial t} = D - E; \quad c_b p_e \frac{\partial \eta}{\partial t} = p_e (D_{\beta D} - E \beta_E) \quad (41a, b)$$

$$D - E = \beta_D D - \beta_E E \quad (41c)$$

In (41c), the only possible specifications for  $\beta_D$  and  $\beta_E$  that renders them independent of each other and  $D$  and  $E$  are

$$\beta_D = \beta_E = 1 \quad (42)$$

The second of these pertains to the conservation of tracers within uniform sediment and takes the form of (41a) and (41c)

$$c_b p_s \frac{\partial f}{\partial t} = p_e D \beta_D (f_t - f) \quad (43)$$

and applies to the conservation of marked tracers within uniform sediment. The third and most important result applies to the conservation of mixtures of sediment sizes, and takes the form of the following three relations:

$$\bar{c}_b \frac{\partial \eta}{\partial t} = D - E \quad (44a)$$

$$c_b p_s \frac{\partial F}{\partial t} = p_e \left[ D \left( \beta_{\psi D} F_t - \frac{c_b}{\bar{c}_b} F \right) - E \left( \beta_{\psi E} - \frac{c_b}{\bar{c}_b} \right) F \right] \quad (44b)$$

$$\frac{c_b}{\bar{c}_b} (D - E) = \beta_D D - \beta_E E \quad (44c)$$

where the only possible specifications for  $\beta_D$  and  $\beta_E$  that renders them independent of each other and  $D$  and  $E$  are given by

$$\beta_D = \beta_E = \frac{c_b}{\bar{c}_b} \quad (45)$$

The concept of the active layer appears in none of these formulations. It is thus seen to be necessary neither for the treatment of tracers in uniform sediment or mixtures of sediment sizes. The active layer formulation is recovered from (30a) in the case of tracers in uniform sediment and (35) in the case of sediment mixtures, however, through an approximate layer integration.

An implementation of the probabilistic formulations in a prediction of riverbed variation with the development of vertical bed stratigraphy requires the specification of functional relations for the probability distribution of bed elevation and elevation-specific densities for sediment entrainment and deposition. At present such relations are not generally available. Their pursuit, however, promises to be an exciting avenue for future research. The probabilistic formulations also open a window toward the development of more sophisticated models of the vertical structure of the stratigraphy left by aggrading streams.

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## APPENDIX II. NOTATION

The following symbols are used in this paper:

- $c_b = 1 - \lambda_p$ , volume concentration of sediment within bed;  
 $\bar{c}_b$  = vertically averaged volume concentration of sediment;  
 $D$  = grain size (mm);  
 $D$  = volume rate of sediment deposition per unit timer per unit bed area;  
 $D_e$  = elevation-specific density of volume rate of deposition per unit time per unit bed area;  
 $D_{e\psi}$  = elevation-specific and grain-size-specific density of volume rate of deposition per unit time per unit bed area;  
 $E$  = volume rate of sediment entrainment per unit time per unit bed area;  
 $E_e$  = elevation-specific density of volume rate of entrainment per unit time per unit bed area;  
 $E_{e\psi}$  = elevation-specific and grain-size-specific density of volume rate of entrainment per unit time per unit bed area;
- $F$  = probability density of grain size  $\psi$  at elevation  $z$  or distance above bed  $y = z - \eta$ ;  
 $F_a$  = volume probability density of grain size  $\psi$  within active layer;  
 $F_i$  = volume probability density of grain size  $\psi$  at interface between active layer and substrate;  
 $F_t$  = volume probability density of grain size  $\psi$  in transport;  
 $f$  = volume fraction content of tracer particles at elevation  $z$  or distance above mean bed  $y = z - \eta$ ;  
 $f_a$  = volume fraction content of tracer particles in active layer;  
 $f_i$  = volume fraction content of tracer particles at interface between active layer and substrate;  
 $f_t$  = volume fraction content of tracer particles in transport;  
 $L(z)$  = average thickness of bed above elevation  $z$ ;  
 $L_a$  = thickness of active layer;  
 $L_{na}$  = nominal distance below mean bed where interface between active layer and substrate is located;  
 $P_s$  = probability distribution such that bed elevation is higher than level  $z$  or variation of bed elevation about mean level is higher than level  $y = z - \eta$ ;  
 $p_e$  = probability density of bed elevation  $z$  or variation of bed elevation about mean level  $y = z - \eta$ ;  
 $q$  = volume sediment load per unit width;  
 $q_\psi$  = grain-size-specific density of volume sediment load per unit width;  
 $t$  = time;  
 $U$  = streamwise velocity in outer flow beyond boundary layer;  
 $u$  = streamwise velocity in boundary layer;  
 $X_k$  = various hydraulic parameters ( $k = 1, 2, 3, \dots$ ) on which  $P_s$  depends;  
 $x$  = streamwise coordinate;  
 $y = z - \eta$ , variation of bed elevation about mean level;  
 $z$  = upward coordinate normal to mean bed level;  
 $\beta_D$  = bias function associated with elevation-specific density of deposition;  
 $\beta_E$  = bias function associated with elevation-specific density of entrainment;  
 $\beta_{\psi D}$  = bias function associated with elevation-specific and grain-size-specific density of deposition;  
 $\beta_{\psi E}$  = bias function associated with elevation-specific and grain-size-specific density of entrainment;  
 $\delta$  = displacement thickness of boundary layer;  
 $\eta$  = mean bed elevation;  
 $\lambda_p$  = bed porosity; and  
 $\psi = \ln_2(D)$ ; grain size on psi scale such that  $D = 2^\psi$ .